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Generalized harmonic maps and applications in image processing

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Abstract. This study explores the application of tension fields and harmonic maps in image processing, highlighting their utility in enhancing computational techniques. A significant focus is placed on the integration of harmonic maps into image processing tasks, such as edge detection, image segmentation, and boundary analysis. The notion of $(\alpha, f)-$harmonic maps, encompassing both α-harmonic and $f-$harmonic maps, further broadens their applicability. Practical methodologies for implementing harmonic maps in image enhancement are discussed, showcasing their effectiveness in improving image clarity and structure. Additionally, the study proposes a Liouville-type theorem for $(\alpha, f)-$harmonic maps, contributing to the theoretical framework of this field.

Keywords: Image processing, Riemannian geometry, Liouville-type theorem, Harmonic maps, α-harmonic maps, tension field.

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1. **Introduction**

Sacks-Uhlenbeck $\alpha-$ harmonic maps, as an extension of harmonic maps, minimizes $\alpha-$energy functional $E_\alpha(\phi) = \int_M (1+ \mid d\phi \mid^2)^\alpha dV_g$, for $\alpha > 1$. These maps satisfies the corresponding Euler-Lagrange equation associated to E_α as follows

$$\tau_\alpha(\phi) := 2\alpha(1+ \mid d\phi \mid^2)^{\alpha-1}\tau(\phi) + 2\alpha grad_M(d\phi((1+ \mid d\phi \mid^2)^{\alpha-1})) = 0, \quad (1.1)$$

where $\tau(\phi)$ is the tension field of ϕ defined by $\tau(\phi) = trace_g \nabla d\phi$.

In physics, α-harmonic maps significantly contribute to the theory of gauge fields, serving as a generalization of classical electromagnetic fields, which form the foundation of the standard model of particle physics, [14, 11]. An interesting observation is that for any sequence of Sacks-Uhlenbeck $\alpha-$harmonic maps from a compact Riemannian surface M to a sphere $\mathbb{S}^{k-1}$, there is no energy loss during the blow-up process as α approaches 1. Additionally, it has been noted that the image of the weak limit maps and bubbles is a connected set [12]. In 2019, Karen Uhlenbeck made history by becoming the first woman to win the prestigious Abel Prize for her remarkable contributions to the study of α-harmonic maps and their applications in physics.

In view of physics, $\alpha-$harmonic maps play an important role in the theory of gauge fields, as a generalization of the theory of classical electromagnetic fields, that underpin the standard model of particle physics, [14, 11], For instance, there is no energy loss for any sequence of Sacks-Uhlenbeck $\alpha-$harmonic maps from a compact Riemannian surface M to a sphere $\mathbb{S}^{k-1}$ during the blow up process as $\alpha \searrow 1$. Moreover, the image of the weak limit maps and bubbles is a connected set, [12]. In 2019, Karen Uhlenbeck , as the first woman, won prestigious Abel prize for her prominent works on α- harmonic maps and their physical applications.

Recently, $\alpha-$harmonic maps were investigated by many scholars. In [9], The stability and existence of $\alpha-$harmonic maps are investigated while the researchers in [19] examined the instability of nonconstant α-harmonic maps in relation to the Ricci curvature criterion of their target space. Additionally, they calculated the Morse index to quantify the degree of instability of certain specific α-harmonic maps. In the paper [18], the concept of sacks-Uhlenbeck α-harmonic maps has been expanded to Finsler spaces and studied in depth. Additionally, the conditions have been identified under which any non-constant α-harmonic maps from a compact Finsler manifold to a standard unit sphere S^n (where $n > 2$) are shown to be unstable. In [13], the authors examined the energy identity and necklessness for a sequence of $\alpha-$harmonic maps as they undergo blowing up, specifically when their codomain is a sphere S^{k-1}. Additionally, they demonstrated that the energy identity can be utilized to present an alternative proof of Perelman's result [15] that the Ricci flow from a compact orientable prime non-aspherical 3-dimensional manifold becomes extinct in finite time. This is in contrast to the findings in [10].

In 1970, A. Lichnerowicz first explored f-harmonic maps as an extension of harmonic maps, geodesics, and minimal surfaces, [6]. More recently, N. Course [5] delved into the study of the f-harmonic flow on surfaces, while Y. Ou [7] analyzed f-harmonic morphisms as a specific type of harmonic maps that pull back harmonic functions to f-harmonic functions.

Let $f \in C^{\infty}(M)$ be a smooth positive function. A smooth map $\phi : (M, g) \longrightarrow (N, h)$ is said to be $f-$harmonic if it is a critical point the $f-$ energy functional $E_f(\phi) := \int_M f \mid d\phi \mid^2 dV_g$. It is notable that f-harmonic maps can be viewed as the fixed solutions of the inhomogeneous Heisenberg spin system. This means that they play a significant role in understanding the behavior of this system and its stationary states[7].

The goal of this paper is to study the $(\alpha, f)-$harmonic maps as an extension of $\alpha-$harmonic maps and f-harmonic maps. For this purpose we introduce $(\alpha, f)-$energy functional and calculate its variational formulas. Then we give a Liouville type thorem for this type of harmonic maps.

This manuscript is organized as follows:
Section 2 focuses on exploring the practical applications of tension field and harmonic maps in the field of image processing, providing in-depth details. In section 3, the $(\alpha, f)-$energy functional is introduced, and its variational formulas are derived. The key findings of this paper are presented in section 4, where a Liouville theorem is provided for $(\alpha, f)-$harmonic maps.

2. Applications of Tension Field and Harmonic Maps in the Image Processing

Image processing refers to a group of methods and techniques whose purpose is to make appropriate changes in the image for a specific application. It can be said that all the manipulations and analysis on the image are done with two goals: enhancing the image quality for a specific application or understanding and interpretation of the image by the computer. Most of the strategies utilized in image processing are based on the fundamental concepts of diverse field of mathematics, [1, 3]. For instance the theory of harmonic maps plays an important role in edge discovery, distinguishing boundaries and locales, image segmentation, etc, [2].

The concept of harmonic maps was originally introduced by Eells and Sampson in 1964 [8]. According to the variational characterization, a harmonic mapping of a Riemannian manifold $\psi : (M, g) \longrightarrow (N, h)$ is considered harmonic if its energy functional $E(\psi) = \frac{1}{2} \int_M \mid d\psi \mid^2 dV_g$, remains stable to first order of variations. In terms of the Euler-Lagrange equation, ψ is harmonic if it satisfies the following second order nonlinear PDE:

$$\tau(\psi) := trace_g \nabla d\psi = 0, \tag{2.1}$$

where, $\tau(\psi) \in \Gamma(\psi^{-1}TN)$ represents the *tension field* of ψ, and ∇ denotes the induced connection on the pull-back bundle $\psi^{-1}TN$. By (2.1), it can be seen that the tension field of any function $f(x,y) \in C^\infty(\mathbb{R}^2)$ is considered as follows

$$\tau(f) = \Delta f, \tag{2.2}$$

where Δf is the Laplacian operator of the function f which is defined as follows

$$\Delta f = \frac{\partial^2 f}{\partial x^2} - \frac{\partial^2 f}{\partial y^2}. \tag{2.3}$$

By (2.1)-(2.3), it can be seen that f is harmonic if $\Delta f = 0$.

Let $f(x,y)$ be a 2D image. Applying the numerical calculation methods for approximating the second order derivation of $f(x,y)$, we have

$$\frac{\partial^2 f}{\partial x^2} = f(x+1,y) - 2f(x,y) + f(x-1,y), \tag{2.4}$$

Similarly

$$\frac{\partial^2 f}{\partial y^2} = f(1,y+1) - 2f(x,y) + f(x,y-1). \tag{2.5}$$

Substituring (2.4) and (2.5) in (2.3), we get

$$\Delta f = f(x+1,y) - 4f(x,y) + f(x-1,y) + f(1,y+1) - 2f(x,y) + f(x,y-1). \tag{2.6}$$

By(2.6), the Laplacian filter in a 3x3 configurationis can be given as follows:

$$\Delta := \frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & -4 & 1 \\ 0 & 1 & 0 \end{bmatrix}. \tag{2.7}$$

For more details see [1].

Laplacian filters are commonly used in image processing for edge detection and sharpening. They are a type of linear filter that highlights regions of rapid intensity change in an image. By convolving the filter with an image, areas of high frequency and edges are emphasized, making them useful for various computer vision tasks such as feature extraction and image enhancement. Particularly, Laplacian filters highlight pixels that have a higher difference in intensity with their neighbors, effectively emphasizing edges. The size of the kernel affects the level of detail detected; smaller kernels detect finer details and sharper edges, while larger kernels can smooth out finer details, highlighting broader edges but potentially introducing more noise.Additionally, the effect of iteration-applying the Laplacian filter repeatedly to an image-enhances the edges, making them sharper and more prominent, while removing information from smooth areas, leading to a loss of detail. Repeated iterations also amplify noise. Consequently, after many iterations, the image may become more abstract, dominated by sharp transitions, lines, or boundaries, with most smooth areas disappearing, and the overall appearance becoming noisier. [1].

We now employ the Laplacian filter to process an image and share its corresponding pseudo code in MATLAB. Let the function $f(x, y)$ represent the famous "Cameraman" image, commonly used as a standard benchmark in image processing.

FIGURE 1. The input image, widely recognized as the "Cameraman" benchmark in image processing contexts.

When the Laplacian filter, $\Delta f(x, y)$, is applied to the image in Figure 1, the resulting image demonstrates enhanced edges.

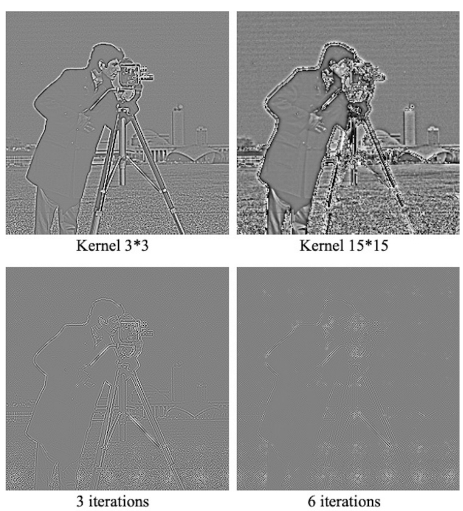

FIGURE 2. The behavior of the Laplacian filter from the perspective of the kernel size and the number of iterations

Let $g(x,y)$ denotes the output image which is presented by

$$g(x,y) = f(x,y) - \Delta f(x,y), \tag{2.8}$$

which is equivalent with the following kernel

$$g(x,y) = \begin{bmatrix} 0 & -1 & 0 \\ -1 & 5 & -1 \\ 0 & -1 & 0 \end{bmatrix}, \tag{2.9}$$

is given by by the following figure

FIGURE 3. The output image which is obtained by equation (2.8)

After applying the Laplacian filter, it is evident that the output image, Figure 3, is significantly sharper and of better quality compared to the original input image, Figure 1.

The harmonicity of f which is equivalent to $\Delta f = 0$, indicates that the image has no rapid intensity change. In other words, it indicates that the image lacks clear boundaries. This can occur in images with consistent color, images with smooth transitions, perfectly balanced content, heavily blurred images, [2].

The pseudocode of applaying Laplacian filter on input image can be given as follows:

```
# Step 1: Input Image
# Step 2: Define the Laplacian kernel
            K = [[0, -1, 0],  [-1, 4, -1], [0, -1, 0]]
# Step 3: Initialize the output image
# Step 4: Apply the Laplacian filter (Loop Through Each Pixel)
             For   i = 1 to M-1:
                For   j = 1 to N-1:
                    Sum = 0
                  For m = -1 to 1:
                     For n = -1 to 1:
                        Sum += G[i+m, j+n] * K[m+1, n+1]
                        L[i, j] = Sum
# Step 5: Handle border pixels
# Step 6: Output Image
```

FIGURE 4. The pseudocode of applaying Laplacian filter in images

Noting that the Laplacian filter is often used in conjunction with other techniques, such as Gaussian smoothing, to reduce noise before edge detection.

This combination, known as the Laplacian of Gaussian (LoG), is particularly effective in edge detection applications, [3].

3. (α, f)-Harmonic Maps

Throughout this study, it is assumed that $\phi : (M^m, g) \longrightarrow (N^n, h)$ is a smooth map from an $m-$dimensional compact Riemannian manifold (M^m, g) to an $n-$dimensional Riemannian manifold (N^n, h). Denote by ∇^M and ∇^N the Levi-Civita connections on M and N, respectively. In addition, the induced connection on the pullback bundle $\phi^{-1}TN$ is denoted by ∇ and defined by

$$\nabla_X \omega = \nabla^N_{d\phi(X)} \omega$$

for any section $\omega \in \Gamma(\phi^{-1}TN)$ and $X \in \chi(M)$.

For $\alpha > 1$, setting

$$\mathcal{A}_\alpha(\phi) := 2\alpha(1+ \mid d\phi \mid^2)^{\alpha-1}, \tag{3.1}$$

and

$$B_\alpha(\phi) := 4\alpha(\alpha-1)(1+ \mid d\phi \mid^2)^{\alpha-2}. \tag{3.2}$$

Definition 3.1. *Let α be a real number greater than 1 and f be a smooth function on M. The $(\alpha, f)-$ energy functional of $\phi : (M, g) \longrightarrow (N, h)$ is defined as follows*

$$E_{\alpha,f}(\phi) := \int_M f(1+ \mid d\phi \mid^2)^\alpha dV_g. \tag{3.3}$$

The critical points of $E_{\alpha,f}$ are called $\alpha-$harmonic maps.

The $(\alpha, f)-$ tension field of ϕ is defined by

$$\tau_{(\alpha,f)}(\psi) := f\tau_\alpha(\phi) + \mathcal{A}_\alpha(\phi)d\phi(grad_M f), \tag{3.4}$$

where $\tau_\alpha(\phi)$ and $\mathcal{A}_\alpha(\phi)$ are defined by (3.4) and (3.1), respectively.

Theorem 3.2. *Any smooth map $\phi : (M, g) \longrightarrow (N, h)$ is $(\alpha - f)$harmonic if and only if $\tau_{(\alpha,f)}(\phi) \equiv 0$.*

Proof. Assume that $\{\phi_s\}$ be a smooth variation of ϕ such that $\phi_0 = \phi$. Let $\Phi : M \times (-\zeta, \zeta) \longrightarrow N$ be defined by

$$\Phi(p, s) = \phi_s(p),$$

where $M \times (-\zeta, \zeta)$ is equipped with the product metric. We extend a vector field Z on M and $\dfrac{\partial}{\partial s}$ naturally on $M \times (-\zeta, \zeta)$ and denote those also by Z and $\dfrac{\partial}{\partial s}$, respectively. Setting

$$\Theta := d\Phi(\frac{\partial}{\partial s}) \mid_{s=0} . \tag{3.5}$$

The same notations ∇ and ∇^M shall be used for the induced connection on $\Phi^{-1}TN$ and the Levi-Civita connection on the $M \times (-\zeta, \zeta)$. Let $\{\bar{e}_i\}$ be an orthonormal frame with respect to g on M such that

$$\nabla^M_{\bar{e}_i}\bar{e}_j = 0$$

at $p \in M$ for all $i, j = 1, \cdots, m(m = dimM)$. By (3.1), we have

$$\begin{aligned} f\frac{\partial}{\partial s}(1+ \mid d\phi_s \mid^2)^\alpha &= \alpha f(1+ \mid d\phi_s \mid^2)^{\alpha-1}\frac{\partial}{\partial s} \mid d\phi_s \mid^2 \\ &= f\mathcal{A}_\alpha(\phi_s)\sum_{j=1}^m h(\nabla_{\frac{\partial}{\partial s}} d\Phi(\bar{e}_j), d\Phi(\bar{e}_j)) \\ &= f\mathcal{A}_\alpha(\phi_s)\sum_{j=1}^m h(\nabla_{\bar{e}_j} d\Phi(\frac{\partial}{\partial s}), d\Phi(\bar{e}_j)) \\ &= f\mathcal{A}_\alpha(\phi_s)\sum_{j=1}^m \Big\{\bar{e}_j.h\Big(d\Phi(\frac{\partial}{\partial s}), d\Phi(\bar{e}_j)) \\ &- h(d\Phi(\frac{\partial}{\partial s}), \nabla_{\bar{e}_j} d\Phi(\bar{e}_j)\Big)\Big\} \end{aligned} \tag{3.6}$$

where we use that

$$\nabla_{\frac{\partial}{\partial s}} d\Phi(\bar{e}_j) - \nabla_{\bar{e}_j} d\Phi(\frac{\partial}{\partial s}) = d\Phi[\frac{\partial}{\partial s}, \bar{e}_j] = 0, \tag{3.7}$$

for the third equality.

Let X_s be a smooth vector field on M such that

$$g(X_s, Z) = h(d\Phi(\frac{\partial}{\partial s}), d\Phi(Z)), \tag{3.8}$$

for any vector field Z on M. Utilizing (3.6) and (3.8), we get

$$\begin{aligned} f\frac{\partial}{\partial s}(1+ \mid d\psi_s \mid^2)^\alpha &= f\mathcal{A}_\alpha(\phi_s)\sum_{j=1}^m \Big\{\bar{e}_j.g(X_t, \bar{e}_j) - h(d\Psi(\frac{\partial}{\partial s}), \nabla_{\bar{e}_j} d\Phi(\bar{e}_j))\Big\} \\ &= f\mathcal{A}_\alpha(\phi_s)\sum_{j=1}^m \Big\{g(\nabla^M_{\bar{e}_i} X_t, \bar{e}_j) + g(X_t, \nabla^M_{\bar{e}_j}\bar{e}_j)\Big\} \\ &- f\mathcal{A}_\alpha(\phi_s)\sum_{i=1}^m \Big\{h(d\Phi(\frac{\partial}{\partial s}), \nabla_{\bar{e}_j} d\Phi(\bar{e}_j))\Big\} \\ &= f\mathcal{A}_\alpha(\phi_s)div(X_t) \\ &- f\mathcal{A}_\alpha(\phi_s)\sum_{j=1}^m h\Big(d\Phi(\frac{\partial}{\partial s}), \nabla_{\bar{e}_j} d\Phi(\bar{e}_j) - d\Phi(\nabla^M_{\bar{e}_j}\bar{e}_j)\Big) \\ &= div\Big(f\mathcal{A}_\alpha(\phi_s)X_s\Big) - g\Big(X_s, fgrad(\mathcal{A}_\alpha(\phi_s))\Big) \end{aligned}$$

$$
\begin{aligned}
&- g\Big(X_s, \mathcal{A}_\alpha(\phi_s) grad(f)\Big) \\
&- h(d\Phi(\frac{\partial}{\partial s}), f\mathcal{A}_\alpha(\phi_s) \sum_{j=1}^{m} \{\nabla_{\bar{e}_j} d\Phi(\bar{e}_j) - d\Phi(\nabla^M_{\bar{e}_j} \bar{e}_j)\}) \\
&= div\Big(f\mathcal{A}_\alpha(\phi_s) X_s\Big) - h\Big(d\Phi(\frac{\partial}{\partial s}), f d\Phi(grad(\mathcal{A}_\alpha(\phi_s))) \\
&+ \mathcal{A}_\alpha(\phi_s) d\Phi(gradf) \\
&+ \sum_{j=1}^{m} f\mathcal{A}_\alpha(\phi_s) \Big[\nabla_{\bar{e}_j} d\Phi(\bar{e}_j) - d\Phi(\nabla^M_{\bar{e}_j} \bar{e}_j)\Big]\Big) \qquad (3.9)
\end{aligned}
$$

Applying (3.4), (3.9) and Green's theorem, we have

$$
\begin{aligned}
\frac{d}{ds} E_{(\alpha,f)}(\phi_s) \mid_{s=0} &= -\int_M f \frac{\partial}{\partial s} (1+ \mid d\phi_s \mid^2)^\alpha \mid_{s=0} dV_g \\
&= -\int_M h\Big(\Theta, f\tau_\alpha(\phi) + \mathcal{A}_\alpha(\phi) d\phi(grad_M f)\Big) dV_g \\
&= -\int_M h(\Theta, \tau_{(\alpha,f)}(\phi)) dV_g. \qquad (3.10)
\end{aligned}
$$

The equation (3.10) implies that any smooth map $\phi : M \longrightarrow N$ is an $(\alpha, f)-$harmonic map if and only if the $(\alpha, f)-$tension field

$$\tau_{(\alpha,f)}(\phi) \equiv 0.$$

This completes the proof. □

4. A Liouville Type Theorem for (α, f)-Harmonic Maps

This section is devoted to study a Liouville type theorem for $(\alpha, f)-$ harmonic maps. The Liouville type theorem play a key role in the field of harmonic maps. The Liouville theorems investigate the conditions under which any harmonic map from a Riemannian manifold to another with finite energy, must be a constant. The theorem is named after Joseph Liouville, a French mathematician recognized for his groundbreaking work in 19th-century complex analysis, proved that every bounded entire function must be constant. This type of theorem provides a powerful tool for understanding the behavior of entire functions and has applications in various areas of mathematics and physics[19].

Many scholars have investigated Liouville type theorems for harmonic maps between complete smooth Riemannian manifolds. For instance, In [9] it is shown that the Liouville-type theorem for α-harmonic maps from a Riemannian manifold to a Riemannian manifold with non-negative Ricci curvature, while Rimoldi and Veronelli [16] also established the Liouville-type theorem for f-harmonic maps.

In this section, we aim to present a proof of the Liouville type theorem for (α, f)-harmonic maps from a complete non-compact Riemannian manifold

(M,g) with positive Ricci curvature into a Riemannian manifold (N,h) with non-positive sectional curvature.

Proposition 4.1. *Let $\phi : (M,g) \longrightarrow (N,h)$ be a smooth map between Riemannian manifolds and $\{\mathcal{E}_i\}$ be a local orthonormal frame on (M,g). The Bochner formula for ϕ is given as follows*

$$\begin{aligned}\frac{1}{2}\Delta^M\langle d\phi, d\phi\rangle = h(d\phi(\mathcal{E}_i), d\phi(Ricci^M\mathcal{E}_i)) + h(\nabla_{\mathcal{E}_i}\tau(\phi), d\phi(\mathcal{E}_i)) + h(\nabla d\phi, \nabla d\phi)\\ - h(R^N(d\phi(\mathcal{E}_i), d\phi(\mathcal{E}_j))d\phi(\mathcal{E}_j), d\phi(\mathcal{E}_i)). \qquad (4.1)\end{aligned}$$

Lemma 4.2. *Suppose that $f \in C^\infty(M)$ and $\phi : (M,g) \longrightarrow (N,h)$ be a smooth map between Riemannian manifolds. Then*

$$\langle\nabla d\phi(grad_M f), d\phi\rangle = \langle d\phi(\nabla^M grad_M f), d\phi\rangle + \frac{1}{2}\langle d\phi, d\phi\rangle grad_M f. \qquad (4.2)$$

where $\langle,\rangle$ is the inner product on the vector bundle $\phi^{-1}TN \otimes T^*M$.

By using Proposition 4.1 and Lemma 4.2, the following Theorem is given.

Theorem 4.3. *Let $f \in C^\infty(M)$ be a smooth positive function and $\phi : (M,g) \longrightarrow (N,h)$ be a $(\alpha,f)-$harmonic map from a non-compact complete Riemannian manifold with non-negative Ricci curvature to a Riemannian manifold with non-positive sectional curvature with $\alpha > 2$. Assume that*

$$\int_M f dV_g = +\infty, \qquad E_{\alpha,f}(\phi) < \infty, \qquad Hess f \leq 0, \qquad |\,\tau_\alpha(\phi)\,| < \infty$$

where $\tau_\alpha(\phi)$defined by (1.1). Then ϕ is constant.

Proof. By (1.1) and (3.4) and considering that ϕ is $(\alpha,f)-$harmonic, one can obtain

$$f\mathcal{A}_\alpha(\phi)\tau(\phi) + f\,|\,d\phi\,|\,\mathcal{B}_\alpha(\phi)d\phi(grad_M\,|\,d\phi\,|) + \mathcal{A}_\alpha(\phi)d\phi(grad_M f) = 0. \qquad (4.3)$$

Setting

$$\begin{aligned}\omega_1(Y) &= \mathcal{A}_\alpha(\phi)h(d\phi(grad_M f), d\phi(Y)),\\ \omega_2(Y) &= h(\tau(\phi), f\mathcal{A}_\alpha(\phi)d\phi(Y)),\\ \omega_3(Y) &= f\,|\,d\phi\,|\,\mathcal{B}_\alpha(\phi)h(d\phi(grad_M\,|\,d\phi\,|), d\phi(Y)), \qquad (4.4)\end{aligned}$$

for any $Y \in \chi(M)$. By calculating the divergence of ω_1 and ω_2, we get

$$\begin{aligned}div\,\omega_1 &= \mathcal{A}_\alpha(\phi)\langle\nabla d\phi(grad_M f), d\phi\rangle - \frac{1}{f}\mathcal{A}_\alpha(\phi)\,|\,d\phi(grad_M f)\,|^2,\\ div\,\omega_2 &= f\mathcal{A}_\alpha(\phi)\langle\nabla\tau(\phi), d\phi\rangle. \qquad (4.5)\end{aligned}$$

Applying (4.3), It can be seen that

$$\omega_1 + \omega_2 + \omega_3 = 0 \qquad (4.6)$$

Using (4.5) and (4.6), we have

$$div\,\omega_3 = -\mathcal{A}_\alpha(\phi)\langle d\phi(\nabla^M grad_M f), d\phi\rangle - \frac{1}{2}\mathcal{A}_\alpha(\phi)(grad_M f)(\langle d\phi, d\phi\rangle) + \frac{1}{f}\mathcal{A}_\alpha(\phi) \mid d\phi(grad_M f) \mid^2 - f\mathcal{A}_\alpha(\phi)\langle\nabla\tau(\phi), d\phi\rangle, \tag{4.7}$$

By applying Proposition (4.1) and the assumptions of Theorem 4.3, we get

$$div\,\omega_3 \geq f\mathcal{A}_\alpha(\phi) \mid \nabla d\phi \mid^2 - \frac{1}{2}f\mathcal{A}_\alpha(\phi)(grad_M f)(\mid d\phi \mid^2) - \frac{1}{2}\mathcal{A}_\alpha(\phi)\Delta^M \mid d\phi \mid^2 . \tag{4.8}$$

where Δ^M is the Laplacian operator on the Riemannian manifold (M, g). Setting

$$\bar{\Delta} \mid d\phi \mid = div\,(f\,grad_M(\mid d\phi \mid)). \tag{4.9}$$

Using (4.8) and (4.9), and applying Kato's inequality $\mid \nabla d\phi \mid^2 \geq \mid grad_M \mid d\phi \mid\mid^2$ we get

$$div\,\omega_3 \geq - \mid d\phi \mid \mathcal{A}_\alpha(\phi)\bar{\Delta} \mid d\phi \mid . \tag{4.10}$$

Assume that $\mu \in C^\infty(M)$ and multiplying (4.10) by μ^2, we get

$$\begin{aligned} & div(\mu^2\omega_3) - 2\mu f \mid d\phi \mid B_\alpha(\phi)h(d\phi(grad_M\mu), d\phi(grad_M(\mid d\phi \mid))) \\ & \geq 2\mu f \mid d\phi \mid \mathcal{A}_\alpha(\phi)g(grad_M\mu, grad_M \mid d\phi \mid) \\ & + \frac{1}{2}\mu^2 f\mathcal{B}_\alpha(\phi) \mid grad_M \mid d\phi \mid\mid^2 - div(\mu^2 fA_\alpha(\phi) \mid d\phi \mid grad_M \mid d\phi \mid) \end{aligned} \tag{4.11}$$

By the Young inequality we have

$$\begin{aligned} & - 2\mu f \mid d\phi \mid B_\alpha(\phi)h(d\phi(grad_M\mu), d\phi(grad_M(\mid d\phi \mid))) \\ & \leq \xi_1\mu^2 f\mathcal{B}_\alpha(\psi) \mid grad_M \mid d\phi \mid\mid^2 + \frac{1}{\xi_1}f(1+ \mid d\phi \mid^2)^\alpha \mid grad_M\mu \mid^2 \end{aligned} \tag{4.12}$$

and

$$\begin{aligned} & - 2\mu f \mid d\phi \mid \mathcal{A}_\alpha(\phi)g(grad_M\mu, grad_M \mid d\phi \mid) \\ & \leq \xi_2\mu^2 f\mathcal{B}_\alpha(\phi) \mid grad_M \mid d\phi \mid\mid^2 + \frac{1}{\xi_2}f(1+ \mid d\phi \mid^2)^\alpha \mid grad_M\mu \mid^2 \end{aligned} \tag{4.13}$$

for any ξ_1 and ξ_1. Substituting (4.12) and (4.13) in (4.11), we get

$$\begin{aligned} & div(\mu^2\omega_3) + (\frac{1}{\xi_1} + \frac{1}{\xi_2})f(1+ \mid d\phi \mid^2)^\alpha \mid grad_M\mu \mid^2 \\ & \geq -div(\mu^2 f\mathcal{A}_\alpha(\phi) \mid d\phi \mid grad_M \mid d\phi \mid) \\ & + (\xi_1 + \xi_2 + \frac{1}{2})\mu^2 f\mathcal{B}_\alpha(\phi) \mid grad_M \mid d\phi \mid\mid^2 . \end{aligned} \tag{4.14}$$

Applying the divergence theorem with $\xi_1 = \xi_2 = \frac{1}{2}$, we obtain

$$\int_M f(1+ \mid d\phi \mid^2)^\alpha \mid grad_M\mu \mid^2 dV_g \geq \frac{3}{2}\int_M f\mu^2\mathcal{B}_\alpha(\phi) \mid grad_M \mid d\phi \mid\mid^2 dV_g. \tag{4.15}$$

Choose the smooth cut-off $\mu = \mu_r$ on M, i.e. $\mu \leq 1$ on M, $\mu = 1$ on the geodesic ball $B(x,r)$, $\mu = 0$ on $M - B(x,3r)$, and $\mid grad_M \mu \mid \leq \frac{3}{r}$ where $x \in M$. Replacing $\mu = \mu_r$ in (4.15), it is obtained that

$$\frac{9}{r^2} \int_{B(x,3r)} f(1+ \mid d\phi \mid^2)^\alpha dV_g \geq \frac{3}{2} \int_{B(x,3r)} f\mathcal{B}_\alpha(\phi) \mid grad_M \mid d\phi \mid\mid^2 dV_g. \tag{4.16}$$

Due to the fact that $E_{(\alpha,f)} < \infty$ when $r \longrightarrow \infty$ we get

$$\int_M f\mathcal{B}_\alpha(\phi) \mid grad_M \mid d\phi \mid\mid^2 dV_g = 0. \tag{4.17}$$

Therefore, if $\mid d\phi \mid \neq 0$ on M we get

$$\mid grad_M \mid d\phi \mid\mid^2 = 0. \tag{4.18}$$

This implies that $\mid d\phi \mid$ is positive constant on M. Therefore

$$E_{\alpha,f}(\phi) = (1+ \mid d\phi \mid^2)^\alpha \int_M f dV_g < \infty. \tag{4.19}$$

This is in contradiction with $\int_M f dV_g = \infty$. Therefore ϕ is constant and hence completes the proof. □

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