Journal of Finsler Geometry and its Applications
Vol. 6, No. 1 (2025), pp 13-22
https://doi.org/10.22098/jfga.2024.16307.1142

Lagrange spaces with changed $(\alpha, \beta)-$ metric with Shen's square Randers metric

Shiv Kumar Tiwari[a], Swati Srivastava[b*] and Chandra Prakash Maurya[c]

[a*] Department of Mathematics
K.S.Saket P.G. College, Ayodhya, India
[b] Department of Mathematics
K.S.Saket P.G. College, Ayodhya, India
[c] Adarsh inter college Saltauwa, India

E-mail: sktiwarisaket@yahoo.com
E-mail: swatisri931@gmail.com
E-mail: chandraprakashmaurya74@gmail.com

Abstract. The aim of the present paper is to study the Lagrange spaces due to changed $(\alpha, \beta)-$ metric with Z. Shen square- Randers metric $\bar{L} = (\alpha+\beta)^2/\alpha+\beta$ and obtained fundamental tensor fields for these space. Further, we studied about the variational problem with fixed endpoints for the Lagrange spaces due to above change.

Keywords: Lagrange space, Z. Shen square metric, Randers metric, Euler - Lagrange equation, metric tensor.

1. Introduction

The notion of (α, β)- metric was introduced by Matsumoto [7] as generalization of Randers metric $L = \alpha + \beta$ where $\alpha = \sqrt{a_{ij}(x)y^i y^j}$ was a regular

*Corresponding Author
AMS 2020 Mathematics Subject Classification: 53B40, 53C30

Riemannian metric and $\beta = b_i(x)y^i$ is one - form metric. Rather than Randers metric, there are several important (α, β)- metric such as Kropina metric $L = \alpha^2/\beta$, Matsumoto metric $L = \alpha^2/\alpha - \beta$, generalized Kropina metric $L = \alpha^{n+1}/\beta^n$, Shen's square metric $L = (\alpha + \beta)^2/\alpha$ etc. Z. Shen square metric [11],[15] was also very interesting because it was constructed from the Berwald metric by using suitable α and β, and it was projectively flat on unit ball with constant flag curvature and Randers metric was also introduced by Berwald in connection with a two dimensional Finsler space with rectilinear extremal and was investigated by Randers.

Matsumoto [6] also introduced the transformations of Finsler metric which was given by,

$$L^{'}(x, y) = \alpha(x, y) + \beta(x, y)$$

$$L^{''2}(x, y) = \alpha^2(x, y) + \beta^2(x, y)$$

where, $\beta = b_i(x)y^i, b_i(x)$ are components of covariant vector which was a function of position alone. Further he had obtained the relationship between the imbedding class numbers of $(M^n, L^{'}), (M^n, L^{''})$ and (M^n, L). Generalizing above transformations, Shibata [3] had studied the properties of Finsler space $(M^n, \bar{L})$ whose fundamental metric function $\bar{L}(x, y)$ was obtained from L by

$$\bar{L}(x, y) = f(\alpha, \beta)$$

where $f = f(\alpha, \beta)$ is a positively homogeneous function of degree one in L and β.

Now the geometry of a Lagrange space over a real, finite - dimensional manifold M had been introduced and studied as a sub - geometry of the geometry of the tangent bundle TM by R. Miron [10]. This geometry was developed together with his collaborators in [8], [9], [10]. Compared to Finsler geometry, when the assumption of homogeneity was relaxed then a new geometry arose which was known as Lagrange geometry, i.e. the Finsler geometry is a particular case of Lagrange geometry where the fundamental function is homogeneous.

Now we state some examples of Lagrange spaces which are reducible to Finsler spaces.

Example 1.1. *Every Riemannian space $(M, g_{ij}(x))$ determines a Finsler space $F^n = (M, F(x, y))$ and consequently a Lagrange space $L^n = (M, F^2(x, y))$, where*

$$F(x, y) = \sqrt{g_{ij}(x)y^iy^j}.$$

The fundamental tensor of this Finsler space coincides to the metric tensor $g_{ij}(x)$ of the Riemannian manifold $(M, g_{ij}(x))$.

Example 1.2. *Let us consider the function*

$$\sqrt[4]{(y^1)^4 + (y^2)^4 + ... + (y^n)^4},$$

defined in a preferential local system of coordinates on $\widetilde{TM}$. *The pair* $F^n = (M, F(x,y))$, *with* F *defined in above is a Finsler space. The fundamental tensor field* g_{ij} *can be easily calculated. This was the first example of Finsler space which was given from the lecturer of Riemann in 1854.*

Now we give some example of Lagrange spaces, which are not reducible to Finsler spaces

Example 1.3. *The following Lagrangian from electrodynamics*

$$L(x,y) = mc\gamma_{ij}(x)y^i y^j + \frac{2e}{m}A_i(x)y^i + U(x),$$

where $\gamma_{ij}(x)$ *is a pseudo - Riemannian metric,* $A_i(x)$ *a covector field and* $U(x)$ *a smooth function ,* m, c, e *are the well - known constants from physics, determine a Lagrange space* L^n.

Example 1.4. *Consider the Lagrangian function*

$$L(x,y) = F^2(x,y) + A_i(x)y^i + U(x),$$

where $F(x,y)$ *is the fundamental function of a Finsler space,* $A_i(x)$ *are the component of a covector field and* $U(x)$ *a smooth function gives rise to a remarkable Lagrange space, called the Almost Finsler- Lagrange space (shortly AFL- space).*

Geometric Problems derived from the variational problem of a Lagrangian were studied by J Kern [4] in detail. He said that the variational problem can be formulated for differentiable Lagrangians and can be solved in cases when we consider the parameterized curves, even if the integral of action depends on the parameterization of considered the curve.

In the year 2001, B. Nicolaescu [2] studied the Lagranges spaces with $(\alpha, \beta)-$ metric and variational problem with fixed endpoints in the year 2004 [1], and in 2011, Pandey and Chaubey [13] considered this problem for the $(\gamma, \beta)-$ metric, where $\gamma^3 = a_{ijk}(x)y^i y^j y^k$ was a cubic metric and $\beta = b_i(x)y^i$ is a one form metric on TM. Further, in 2023 Tripathi, Chandak and Chaubey [14] considered the problem of Lagrange space with Z. Shen change in $\widetilde{TM}$.

In the present paper, we transform the Z. Shen square Randers metric as

$$L = \frac{(\alpha + \beta)^2}{\alpha} + \beta$$

and studied Lagrange space due to this transformation. The above generalization is very interesting because it enhances our understanding and geometric meaning of non - Riemannian quantities. we obtained fundamental tensor fields for these space and also studied about the variational problem with fixed endpoints of Lagrange spaces due to this change.

2. **Lagrange metrics**

In this section we give the definitions of regular, differentiable Lagrangian over the tangent manifolds TM and $\widehat{TM}$, where M is a differentiable, real manifold of dimension n. Let (TM, τ, M) be the tangent bundle of a C^∞-differentiable real n- dimensional manifold M. If (U, ϕ) is a local chart on M, then the coordinates of a point $u = (x, y) \in \tau^{-1}(U) \subset TM$ will be denoted by (x, y). R. Miron [10] given following definitions :

Definition 2.1. *A differentiable Lagrangian on TM is a mapping $L : (x, y) \in TM \to L(x, y) \in R, \forall u = (x, y) \in TM$, which is of class C^∞ on $\widehat{TM} = TM \backslash (0)$ and is continuous on the null section of the projection $\tau : TM \to M$, such that*

$$g_{ij} = \frac{1}{2}\frac{\partial^2 L(x,y)}{\partial y^i \partial y^j}, \tag{2.1}$$

is a (0,2)- type symmetric d- tensor field on TM.

Definition 2.2. *A differential Lagrangian L on TM is said to be regular if*

$$rank \parallel g_{ij}(x,y) \parallel = n, \qquad \forall (x,y) \in \widehat{TM}.$$

For the Lagrange space $L^n = (M, L(x, y))$ we say that $L(x, y)$ is the fundamental function and $g_{ij}(x, y)$ is the fundamental (or metric) tensor. We will denote by g^{ij} the inverse matrix of g_{ij}. This means that

$$g^{ik} g_{jk} = \delta^i_j.$$

Now the definition of a Lagrange space was given by

Definition 2.3. *A Lagrange space is a pair $L^n = (M, L)$ formed by a smooth, real n- dimensional manifold M and a regular differentiable Lagrangian L on M, for which the d- tensor field g_{ij} from (2.1) has a constant signature on $\widehat{TM}$.*

Now, Let $L : TM \to R$ be a differentiable Lagrangian on the manifold M, which was not necessarily regular. A curve $c : t \in [0, 1] \to (x^i(t)) \in U \subset M$ having the image in a domain of a chart U of M, has the extension to $\widehat{TM}$ given by

$$c^* : t \in [0,1] \to \left(x^i(t), \frac{dx^i(t)}{dt} \right) \in \tau^{-1}(U).$$

The integral of action of the Lagrangian L on the curve c is given by the functional

$$I(c) = \int_0^1 L(x(t), \frac{dx}{dt}) dt. \tag{2.2}$$

Consider the curve $c_\epsilon : t \in [0,1] \to (x^i(t) + \epsilon v^i(t)) \in M$, which have the same endpoints $x^i(0), x^i(1)$ as the curve c, $v^i(0) = v^i(1) = 0$ and ϵ is a real number, sufficiently small in absolute value, such that $Imc_\epsilon \in U$. The extension of the curve c_ϵ to TM is

$$c^*_\epsilon : t \in [0,1] \to \left(x^i(t) + \epsilon v^i(t), \frac{dx^i}{dt} + \epsilon \frac{dv^i}{dt} \right) \in \tau^{-1}(U)$$

The integral of action of the Lagrangian L on the curve c_ϵ is,

$$I(c_\epsilon) = \int_0^1 L\left(x + \epsilon v, \frac{dx}{dt} + \epsilon \frac{dv}{dt} \right) dt.$$

A necessary condition for $I(c)$ to be an extremal value $I(c_\epsilon)$ is

$$\frac{dI(c_\epsilon)}{d\epsilon}|_{\epsilon=0} = 0.$$

In order that the functional $I(c)$ be an extremal value of $I(c_\epsilon)$, it is necessary that c be the solution of the Euler- Lagrange equations,

$$E_i(L) = \frac{\partial L}{\partial x^i} - \frac{d}{dt}(\frac{\partial L}{\partial y^i}) = 0, \quad y^i = \frac{dx^i}{dt}.$$

3. The fundamental tensor of a Lagrange space with changed Z. Shen square Randers metric

In general, we know that the component b_i is the electromagnetic potential of L^n and the tensor $F_{ij} = \partial_j b_i - \partial_i b_j$ is the electromagnetic tensor field in Lagrange spaces. Now we define the changed Z. Shen square Randers metrics as follows :

Definition 3.1. *A Lagrange space $\bar{L}^n = (M, \bar{L}(x,y))$ is known as Z. Shen square Randers metric if $\bar{L}$, depends only on $\alpha(x,y)$ and $\beta(x,y)$,*

$$\bar{L}\{\alpha(x,y), \beta(x,y)\} = \frac{(\alpha+\beta)^2}{\alpha} + \beta$$

Here, we shall use the following notations throughout the paper,

$$\dot{\partial}_i\alpha = \frac{\partial\alpha}{\partial y^i}, \quad \dot{\partial}_i\beta = \frac{\partial\beta}{\partial y^i}, \quad \dot{\partial}_i\dot{\partial}_j\alpha = \frac{\partial^2\alpha}{\partial y^i \partial y^j}, \quad \bar{L}_\alpha = \frac{\alpha^2 - \beta^2}{\alpha^2}$$

$$\bar{L}_\beta = \frac{3\alpha + 2\beta}{\alpha}, \quad \bar{L}_{\alpha\alpha} = \frac{2\beta^2}{\alpha^3}, \quad \bar{L}_{\alpha\beta} = -\frac{2\beta}{\alpha^2}, \quad \bar{L}_{\beta\beta} = \frac{2}{\alpha}$$

Now we have

Proposition 3.2. *For the Lagrange space L^n, the following relations hold good:*

$$\dot{\partial}_i\alpha = \alpha^{-1}y^i, \quad \dot{\partial}_i\beta = b_i(x), \quad \dot{\partial}_i\dot{\partial}_j\alpha = 2\dot{\partial}_j y_i - \alpha^{-3}y_i y_j, \quad \dot{\partial}_i\dot{\partial}_j\beta = 0, \quad (3.1)$$

where $y_i = g_{ij}y^j$.

Now, we introduce the moments of the Lagrangian $\bar{L}\{\alpha(x,y),\beta(x,y)\} = \frac{(\alpha+\beta)^2}{\alpha} + \beta$,

$$p_i = \frac{1}{2}\frac{\partial \bar{L}}{\partial y^i} = \frac{1}{2\alpha^2}\left\{(\alpha^2-\beta^2)\dot{\partial}_i\alpha + \alpha(3\alpha+2\beta)\dot{\partial}_i\beta\right\}.$$

Thus we have

Proposition 3.3. *The moments of the Lagrangian $\bar{L}(x,y)$ with changed Z.Shen square Randers metric is given by*

$$p_i = \rho y_i + \rho_1 b_i, \tag{3.2}$$

where

$$\rho = \frac{1}{2}\frac{(\alpha^2-\beta^2)}{\alpha^3}, \qquad \rho_1 = \frac{(3\alpha+2\beta)}{2\alpha}.$$

The two scalar functions defined in (3.2) are called the principal invariants of the Lagrange space $\bar{L}^n$.

Proposition 3.4. *The derivatives of principal invariants of the Lagrange space $\bar{L}^n$ due to changed Z. Shen square Randers metric are given by*

$$\dot{\partial}_i\rho = \rho_{-2}y_i + \rho_{-1}b_i, \qquad \dot{\partial}_i\rho_1 = \rho_{-1}y_i + \rho_0 b_i, \tag{3.3}$$

where

$$\rho_{-2} = \frac{1}{2}\alpha^{-5}(3\beta^2-\alpha^2), \qquad \rho_{-1} = -\frac{\beta}{\alpha^3}, \qquad \rho_0 = \frac{1}{\alpha}.$$

Now, the Energy of a Lagrangian is given by

$$E_{\bar{L}} = y^i\frac{\partial \bar{L}}{\partial y^i} - \bar{L}.$$

Thus we have

Proposition 3.5. *The Energy of a Lagrangian $\bar{L}$ with Z. Shen square Randers metric is given by*

$$E_{\bar{L}} = \frac{(\alpha^2-\beta^2)(1-\alpha)}{\alpha^2}. \tag{3.4}$$

Now we can determine the fundamental tensor $\bar{g}_{ij}$ of the Lagrange space $\bar{L}$ with changed Z. Shen square Randers metric as follows:

Proposition 3.6. *The fundamental tensor $\bar{g}_{ij}$ of the Lagrange space $\bar{L}$ with Z. Shen square Randers metric is given as*

$$\bar{g}_{ij} = \frac{(\alpha^2-\beta^2)}{\alpha^3}g_{ij} + \alpha^{-1}b_ib_j - \frac{\beta}{\alpha^3}(b_iy_j + b_jy_i) + \frac{(3\beta^2-\alpha^2)}{2\alpha^5}y_iy_j. \tag{3.5}$$

The above equation can be rewritten as

$$\bar{g}_{ij} = \frac{(\alpha^2 - \beta^2)}{\alpha^3} g_{ij} + d_i d_j$$

where

$$d_i = \sqrt{\frac{(3\beta^2 - \alpha^2)}{2\alpha^5}}\ y_i + \frac{1}{\sqrt{\alpha}} b_i, \quad g_{ij} = \frac{1}{2}\frac{\partial^2 L}{\partial y^i \partial y^j}.$$

Proposition 3.7. *The reciprocal tensor $\bar{g}^{ij}$ of the fundamental tensor $\bar{g}_{ij}$ in $\bar{L}^n$ is given by*

$$\bar{g}^{ij} = \frac{\alpha^3}{(\alpha^2 - \beta^2)} g^{ij} - \frac{1}{(1 + d^2)} d^i d^j, \tag{3.6}$$

where,

$$d^i = \frac{\alpha^3}{(\alpha^2 - \beta^2)} g^{ij} d_j$$

and $d^i d_i = d^2$ and g^{ij} is reciprocal of the g_{ij}.

4. Euler - Lagrange equations in Lagrange spaces with changed Z. Shen square Randers metric

The Euler - Lagrange equations of the Lagrange spaces with changed Z. Shen square Randers metric are,

$$E_i(\bar{L}) = \frac{\partial \bar{L}}{\partial x^i} - \frac{d}{dt}\left(\frac{\partial \bar{L}}{\partial y^i}\right) = 0, \qquad y^i = \frac{dx^i}{dt}$$

Considering the relations

$$\frac{\partial \bar{L}}{\partial x^i} = \frac{1}{\alpha}\left[\left(\frac{\alpha^2 - \beta^2}{\alpha}\right)\frac{\partial \alpha}{\partial x^i} + (3\alpha + 2\beta)\frac{\partial \beta}{\partial x^i}\right],$$

$$\frac{\partial \bar{L}}{\partial y^i} = \frac{1}{\alpha}\left[\left(\frac{\alpha^2 - \beta^2}{\alpha}\right)\frac{\partial \alpha}{\partial y^i} + (3\alpha + 2\beta)\frac{\partial \beta}{\partial y^i}\right],$$

and

$$\frac{d}{dt}\frac{\partial \bar{L}}{\partial y^i} = \frac{d}{dt}\left\{\frac{\alpha^2 - \beta^2}{\alpha^2}\right\}\frac{\partial \alpha}{\partial y^i} + \frac{(\alpha^2 - \beta^2)}{\alpha^2}\frac{d}{dt}\left(\frac{\partial \alpha}{\partial y^i}\right) + \frac{d}{dt}\left\{\frac{(3\alpha + 2\beta)}{\alpha}\right\}\frac{\partial \beta}{\partial y^i} + \frac{(3\alpha + 2\beta)}{\alpha}\frac{d}{dt}\left(\frac{\partial \beta}{\partial y^i}\right).$$

By direct calculations , we have

$$E_i(\bar{L}) = \left\{\frac{\alpha^2 - \beta^2}{\alpha^2}\right\} E_i(\alpha) + \frac{(3\alpha + 2\beta)}{\alpha} E_i(\beta) - \frac{\partial \alpha}{\partial y^i}\frac{d}{dt}\left\{\frac{(\alpha^2 - \beta^2)}{\alpha^2}\right\} - \frac{\partial \beta}{\partial y^i}\frac{d}{dt}\left\{\frac{(3\alpha + 2\beta)}{\alpha}\right\},$$

$$y^i = \frac{dx^i}{dt},$$

It's give

$$E_i(\bar{L}) = \{\frac{\alpha^2 - \beta^2}{\alpha^2}\}E_i(\alpha) + \frac{(3\alpha + 2\beta)}{\alpha}E_i(\beta) - \frac{\partial\alpha}{\partial y^i}(\frac{2\beta^2}{\alpha^3}\frac{d\alpha}{dt} - \frac{2\beta}{\alpha^2}\frac{d\beta}{dt}) + \frac{\partial\beta}{\partial y^i}(\frac{2\beta}{\alpha^2}\frac{d\alpha}{dt} - \frac{2}{\alpha}\frac{d\beta}{dt}).$$

As well have

$$E_i(\beta) = F_{ir}\frac{dx^r}{dt},$$

where,

$$F_{ir} = \frac{\partial A_r}{\partial x^i} - \frac{\partial A_i}{\partial x^r},$$

is the electromagnetic tensor field. Finally we have the following relation :

$$E_i(\bar{L}) = \{\frac{\alpha^2 - \beta^2}{\alpha^2}\}E_i(\alpha) + \frac{(3\alpha + 2\beta)}{\alpha}F_{ir}\frac{dx^r}{dt} - \frac{\partial\alpha}{\partial y^i}(\frac{2\beta^2}{\alpha^3}\frac{d\alpha}{dt} - \frac{2\beta}{\alpha^2}\frac{d\beta}{dt}) + \frac{\partial\beta}{\partial y^i}(\frac{2\beta}{\alpha^2}\frac{d\alpha}{dt} - \frac{2}{\alpha}\frac{d\beta}{dt}).$$

Proposition 4.1. *The Euler - Lagrange equation in the Lagrange space $\bar{L}^n$ with changed Z. Shen square Randers metric $\bar{L}$ are,*

$$E_i(\bar{L}) = E_i\left\{\frac{(\alpha + \beta)^2}{\alpha} + \beta\right\}, \qquad y^i = \frac{dx^i}{dt}. \tag{4.1}$$

For every smooth curve c on the base manifold M, the energy function of the Lagrangian $\bar{L}(x, y)$ can be written as

$$\frac{dE_{\bar{L}}}{dt} = -[\frac{\partial\bar{L}}{\partial x^i} - \frac{d}{dt}(\frac{\partial\bar{L}}{\partial y^i})]y^i = 0, \qquad where \quad y^i = \frac{dx^i}{dt},$$

$$or \qquad \frac{dE_{\bar{L}}}{dt} = -E_i(\bar{L})\frac{dx^i}{dt}.$$

Thus using proposition (4.1) we have

Theorem 4.2. *In a differentiable Lagrangian $\bar{L}(x, y)$, the energy function $E_{\bar{L}}$ is conserved along the solution curves c of the Euler - Lagrange equations for changed Z. Shen square Randers metric.*

If we have the natural parametrization of the curve $\in [0, 1] \to (x^i(t) \in M)$, then

$$L(x, \frac{dx}{dt}) = 1.$$

Thus we get:

Proposition 4.3. *In the canonical parametrization the Euler - Lagrange equations for changed Z. Shen square Randers metric in Lagrange space $\bar{L}^n$ are*

$$\begin{aligned} E_i(\bar{L}) = \{\frac{\alpha^2-\beta^2}{\alpha^2}\}E_i(\alpha) + \frac{(3\alpha+2\beta)}{\alpha}F_{ir}\frac{dx^r}{dt} \\ +\frac{\partial\beta}{\partial y^i}(\frac{2\beta}{\alpha^2}\frac{d\alpha}{dt} - \frac{2}{\alpha}\frac{d\beta}{dt}). \end{aligned} \tag{4.2}$$

Proposition 4.4. *If the 1- form β is constant on the integral curve c of the Euler - Lagrange equations for Z. shen square Randers metric , then (4.2) rewrite as the Lorentz equations of the $\bar{L}^n$*

$$E_i(\bar{L}) = \left\{\frac{\alpha^2-\beta^2}{\alpha^2}\right\}E_i(\alpha) + \frac{(3\alpha+2\beta)}{\alpha}F_{ir}\frac{dx^r}{dt}. \tag{4.3}$$

5. **Conclusion**

In this paper, we have continued the investigations on the new introduced changed Z. Shen square Randers metric which is defined as $\bar{L} = \frac{(\alpha+\beta)^2}{\alpha}+\beta$. The above generalization is very interesting because it enhance our understanding and geometric meaning of non - Riemannian quantities. Further, we obtained fundamental tensor fields for these spaces and the variational problem with fixed endpoints for the Lagrange spaces.

Acknowledgment: The authors are grateful for Professor S.K.Tiwari for his continuous help and encouragement.

References

1. B. Nicolaescu, *Lagrange spaces with* (α, β) *- metric*, A ppl. Sci.,3 :1 , 3(2001).
2. B. Nicolaescu, *The variational problem in Lagrange spaces endowed with* (α, β)*- metric*, in Balan, Vladimir (ed.), Proceedings of the 3rd international colloquium of Mathematics in engineering and numericalphysics (MENP -3) Bucharest, Romania, (2004), Mathematics sections, BSG proceedings, 12, Geometry Balken Press, Bucharest, (2005), 202 - 207.
3. C. Shibata, *On invariant tensors of* β*- changes of Finsler metrics*, J. Math. kyoto Univ., **24**(1984), 163-188.
4. J. Kern, *Lagrange geometry*, Arch Math., **25**(1974), 438-443.
5. I.M.Gelfand, S.V. Fomin, *Calculus of variations*, Dover publications, Mineola,(2000).
6. M. Matsumoto , *On some transformations of locally Minkowskian space*, Tensor. N. S. **22**(1971), 103-111.
7. M. Matsumoto , *Theory of Finsler spaces with* (α, β)*- metric*, Rep. Math. Phys., **31:1**(1992).

8. R. Miron, *A Lagrangian theory of relativity, I,II*, An. stiint . Univ. AI.I. cuza Iasi, N. S., sect. Ia,32 : 2,3, 37-62, 7 -16 (1986).
9. R. Miron, *Lagrange geometry*, Math. Comput. Modelling, **20**(1994), 4-5, 25-40.
10. R. Miron, M. Anastasiei, *The geometry of Lagrange spaces : theory and applications*, Kluwer Acad. Publ., Dordrecht, (1994).
11. S.S.Chern, Z. Shen, *Riemann - Finsler geometry*, Nankai Tracts in Mathematics, World Scientific, Hackensack, **06**(2005).
12. T. N. Pandey, V. K. Chaubey, *Lagrange spaces with $\beta-$ change*, Int. J. Contemp.Math. sci. **07**(2012), 45-48, 2363-2371.
13. T. N. Pandey, V. K. Chaubey, *The variational problem in Lagrange spaces endowed with (γ,β) metric*, Int. J. Pure.Appl. Math., **71:4**(2011), 633-638.
14. V.K.Chaubey, B.K. Tripathi, S.B.Chandak, *Lagrange spaces with change Z.shen square metric*, Siberian electronic Mathematical reports, **20:1**(2023), 17-24.
15. Z. Shen, G. C. Yildirim, *On a class of projectively flat metrics with constant flag curvature*, can. J. Math., **60:2**(2008), 443-456.

Received: 03.12.2024
Accepted: 23.12.2024