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Research Paper

SOME RESULTS ON IDEALS VIA HOMOMORPHISMS IN BL-ALGEBRAS

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ABSTRACT

In this paper, by considering the σ as a BL-homomorphism, we introduce the notion of σ-ideals in BL-algebras and obtain some new results about them. Also, while introducing the primary ideals in BL-algebras, we get some relationship between the primary ideals and obstinate ideals. Finally, by introducing σ-prime, σ-primary and σ-invariant ideals, we derived some new results related to them.

1. Introduction

Algebraic structures play a significant role in the study and analysis of many-valued logics, so that the nature of these logics becomes clearer with the help of algebraic tools. Various logical algebras have been presented and worked on as semantic systems for non-classical logical systems, among which two important structures called BL-algebras and MV-algebras can be mentioned.

In 1958, C. C. Chang [2] introduced MV-algebras in order to provide an algebraic proof for the completeness of Łukasiewicz many-valued logic. This algebraic structure quickly

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attracted the attention of many researchers, and interesting results were obtained in this field.

In 1998, Hájek introduced a generalization of many-valued logic called "basic logic" (BL, for short). This logic clarified the common connection between important many-valued logics such as Łukasiewicz and Gödel. He had two motivations for introducing basic logic from an algebraic point of view:
One is to work on logical propositions from an algebraic point of view, and the other is to obtain an algebraic concept for the study of continuous triangular norms on real unit interval $[0, 1]$. Since the double negation law ($x^{--} = x$) does not hold in BL-algebras, these algebraic structures are the generalization of MV-algebras.

The notion of ideals plays a fundamental and key role in many algebraic structures, such as commutative rings and MV-algebras, while in some other algebraic structures, such as BL-algebras and residuated lattices, due to the lack of a suitable additional operation, the focus is on the filters.

In 2013, C. Lele [7] et al. introduced the notion of ideals in BL-algebras and showed that filters and ideals behave quite differently. They proved that unlike MV-algebras, filters and ideals in BL-algebras are not dual of each other. They showed that quotient BL-algebras which are constructed via ideals are MV-algebras. Also, they demonstrated that $x \in I$ if and only if $x^{--} \in I$ for every ideal I, whereas if we replaced the ideals with filters, both of these corollaries does not hold in the case of filters in BL-algebras.

In 2019, N. Dolatabadi and J. Moghaderi [9] introduced σ-filters, where σ is a BL-homomorphism of A. They investigated some properties of σ-filters and obtained results in this context. For example, they mentioned the relation between special filters and σ-filters in BL-algebras.

In this paper, based on the above explanations and the fact that the behavior of ideals and filters is completely different in BL-algebras and that these two concepts are not dual to each other, we consider σ as a BL-homomorphism and define σ-ideals and primary ideals in BL-algebras. Then we introduce the σ-prime, σ-primary and σ-invariant ideals and obtain some results about them.

The structure of the paper is as follows:

In Section 2, we recall some definitions and results about BL-algebras that we use in the sequel. In Section 3, we introduce the notion of σ-ideals, primary ideals in BL-algebras and investigate some propositions about them. In Section 4, after introducing the σ-prime, σ-primary and σ-invariant, we obtain some relations between them.

2. Preliminaries

In this section, we review some definitions and properties about the BL-algebras and related topics that will be used in this paper. For more details, refer to the references.

Definition 2.1. [5]. An algebra $A = (A, \wedge, \vee, \odot, \longrightarrow, 0, 1)$ of type $(2, 2, 2, 2, 0, 0)$ is called a BL-algebra, if it satisfying the following conditions:
(BL_1) $(A, \wedge, \vee, 0, 1)$ is a bounded lattice,

(BL_2) $(A, \odot, 1)$ is a commutative monoid,

(BL_3) $(\odot, \longrightarrow)$ forms an add joint pair; that is $x \odot y \leq z$, if and only if $x \leq y \longrightarrow z$, for all $x, y, z \in A$,

(BL_4) $x \wedge y = x \odot (x \longrightarrow y)$,
(BL_5) $(x \longrightarrow y) \vee (y \longrightarrow x) = 1$.

For any BL-algebra A, $L(A) = (A, \wedge, \vee, 0, 1)$ is a bounded distributive lattice, which is called lattice reduct of A.

Theorem 2.2. [5, 12]. *Let A be a BL-algebra. Then the following hold for all $x, y, z \in A$:*
(1) $x \leq y$, *iff* $x \longrightarrow y = 1$,
(2) $x \longrightarrow (y \longrightarrow z) = (x \odot y) \longrightarrow z = y \longrightarrow (x \longrightarrow z)$,
(3) $x \odot y \leq x, y$, *hence* $x \odot y \leq x \wedge y$,
(4) $x \vee x^- = 1$ *implies* $x \wedge x^- = 0$,
(5) $1 \longrightarrow x = x$, $x \longrightarrow x = 1$, $x \leq y \longrightarrow x$ *and* $x^{\equiv} = x^-$,
(6) $0^- = 1$ *and* $1^- = 0$,
(7) $x \odot x^- = 0$, $x \odot y = 0$, *iff* $x \leq y^-$ *and* $x \leq x^=$,
(8) $(x \odot y)^- = x \longrightarrow y^-$,
(9) $(x \longrightarrow y)^= = x^= \longrightarrow y^=$, $(x \wedge y)^= = x^= \wedge y^=$, $(x \vee y)^= = x^= \vee y^=$ *and* $(x \odot y)^= = x^= \odot y^=$,
(10) *if* $x \leq y$, *then* $y \longrightarrow z \leq x \longrightarrow z$, $z \longrightarrow x \leq z \longrightarrow y$ *and* $x \odot z \leq y \odot z$,
(11) $x \odot (y \longrightarrow z) \leq (x \odot y) \longrightarrow z$,
(12) $(x \wedge y)^- = x^- \vee y^-$ *and* $(x \vee y)^- = x^- \wedge y^-$.

Definition 2.3. [5]. Let A be a BL-algebra and F be a non-empty subset of A. Then F is called a filter of A, if it satisfies:
(F_1) For every $x, y \in F$, $x \odot y \in F$,
(F_2) For every $x, y \in A$, if $x \leq y$ and $x \in F$, then $y \in F$.

Definition 2.4. [5, 12]. Let A be a BL-algebra. Then the order of an element $x \in A$ is the smallest integer n such that $x^n = 0$, where $x^0 = 1$, $x^n = x^{n-1} \odot x$, denoted by $\mathrm{ord}(x) = n$, and if no such n exists, then $\mathrm{ord}(x) = \infty$. An element $0 \neq a \in A$ is called a zero divisor element, if $a \odot b = 0$, for some $0 \neq b \in A$.
An element $a \in A$ is called a nilpotent element if $a^n = 0$, for some $n \in \mathbb{N}$.

Proposition 2.5. [13]. *A BL-algebra A is local if and only if for all $x \in A$, $\mathrm{ord}(x) < \infty$ or $\mathrm{ord}(x^-) < \infty$.*

Definition 2.6. [2]. An algebra $(\mathrm{M}, \oplus, -, 0)$ of type $(2, 1, 0)$ is an MV-algebra, if satisfying the following conditions:
(MV_1) $(\mathrm{M}, \oplus, 0)$ is an abelian monoid,
(MV_2) $(x^-)^- = x^= = x$,
(MV_3) $0^- \oplus x = 0^-$,
(MV_4) $(x^- \oplus y)^- \oplus y = (y^- \oplus x)^- \oplus x$.

A BL-algebra A is called Gödel algebra, if $x \odot x = x^2 = x$, for any $x \in A$. A BL-algebra A is an MV-algebra, if $x^= = x$, for any $x \in A$ [5].

A non-empty subset I of an MV-algebra A is called an ideal, if the following properties are satisfied:
(i) for any $a, b \in I$, $a \oplus b \in I$,
(ii) if $a \leq b$ and $b \in I$, then $a \in I$ [2].
An ideal P of an MV-algebra A is primary, if and only if for any $a, b \in A$, $a \odot b \in P$ implies $a^n \in P$ or $b^n \in P$, for some $n \in \mathbb{N}$ [1].

Theorem 2.7. [3]. *For an MV-algebra A and a proper ideal $P \subseteq A$, the following conditions are equivalent:*

(i) *P is primary ideal,*

(ii) *A/P is a local MV-algebra,*

(iii) *$a \odot b \in P$ implies $a^n \in P$ or $b^n \in P$, for some $n \in \mathbb{N}$, where $a \odot b = (a^- \oplus b^-)^-$,*

(iv) *for any $a \in A$, there exists some $n \in \mathbb{N}$ such that $a^n \in P$ or $(a^-)^n \in P$.*

Proposition 2.8. [7]. *If A is a BL-algebra, then the operation "$\oslash$" is associative and compatible with the order relation, where $x \oslash y := x^- \longrightarrow y$, i.e., $x \leq y$ and $z \leq t$ imply $x \oslash z \leq y \oslash t$, for any $x, y, z, t \in A$.*

Through the paper, we mean A is a BL-algebra. C. Lele and et al. in [7], introduced the notion of ideals in BL-algebra as follows:
A non-empty subset I of A is an ideal, if it satisfying the following conditions:

(i) $x \oslash y \in I$, for every $x, y \in I$,

(ii) If $x \leq y$ and $y \in I$, then $x \in I$, for every $x, y \in A$.

The set of all ideals of A is denoted by $\mathrm{Id}(A)$.

Remark 2.9. [7]. For every ideal I of A, $x \in I$ if and only if $x^{=} \in I$, for every $x \in A$.

We recall that [5] if A and B are two BL-algebras, then a mapping $f : A \longrightarrow B$ is called a BL-homomorphism, if the following conditions hold, for all $x, y \in A$:

(i) $f(0_A) = 0_B$,

(ii) $f(x \longrightarrow y) = f(x) \longrightarrow f(y)$,

(iii) $f(x \odot y) = f(x) \odot f(y)$.

Also, $f(x \wedge y) = f(x) \wedge f(y)$, $f(x \vee y) = f(x) \vee f(y)$, $f(x^-) = (f(x))^-$, and if $x \leq y$, then $f(x) \leq f(y)$, for any $x, y \in A$.
The kernel of f is defined by $\mathrm{Ker} f = \{x \in A \ : \ f(x) = 0_B\}$.

Proposition 2.10. [7]. *Let I be an ideal of A. Then the quotient BL-algebra A/I is always an MV-algebra.*

Remark 2.11. [7]. If I is an ideal of A, then $A/I = \{a/I = [a] \ : \ a \in A\}$ and $[a] = [b]$ iff $a \odot b^- \in I$ and $a^- \odot b \in I$. Therefore, $a/I = 0/I$ iff $a \in I$ and $a/I = 1/I$ iff $a^- \in I$.

Theorem 2.12. [7]. *A set I containing 0 of A is an ideal, if and only if for every $x, y \in A$, $x^- \odot y \in I$ and $x \in I$ imply $y \in I$.*

From [7], for every subset X of A, the smallest ideal containing X is called the ideal generated by X and it is denoted by $\langle X \rangle$, i.e., $\langle X \rangle = \cap\{I \ : \ I \text{ is an ideal of } A, \ X \subseteq I\}$. Also, the authors in [7] proved that $\langle \emptyset \rangle = \{0\}$ and if $X \neq \emptyset$, then $\langle X \rangle = \{a \in A \ : \ a \leq (\ldots((x_1 \oslash x_2) \oslash x_3)\ldots) \oslash x_n$, for some $x_1, x_2, \ldots, x_n \in X\}$.

Theorem 2.13. [8]. *Let X be a non-empty subset of A. Then $\langle X \rangle = \{a \in A \ : \ (x_1^- \odot \ldots \odot x_n^-) \to a^- = 1$, for some $x_1, x_2, \ldots, x_n \in X\}$, where $\langle X \rangle$ is the ideal generated by X.*

Definition 2.14. Let I be a proper ideal of A. Then

(i) I is a prime ideal, if it satisfies for every $x, y \in A$, $(x \longrightarrow y)^- \in I$ or $(y \longrightarrow x)^- \in I$ [7],

(ii) I is a maximal ideal, if it does not properly contained in any other proper ideal of A, [7],

(iii) I is called an obstinate ideal, if $x, y \notin I$, then $x \odot y^- \in I$ and $y \odot x^- \in I$, for all $x, y \in A$, [10].

Corollary 2.15. [7]. *I is a prime ideal of A, if and only if for every $x, y \in A$, $x \wedge y \in I$ implies that $x \in I$ or $y \in I$.*

Definition 2.16. [11]. Let I be a proper ideal of A. The intersection of all maximal ideal of A that contain I is called the radical of I and it is denoted by $\mathrm{rad}(I)$. $\mathrm{rad}(I)$ is an ideal of A and $I \subseteq \mathrm{rad}(I)$.

Theorem 2.17. [11]. *Let I be a proper ideal of A. Then $\mathrm{rad}(I) = \{x \in A : (x \longrightarrow (x^-)^n)^- \in I, \ \forall\, n \in \mathbb{N}\}$.*

Lemma 2.18. [4]. *The following statements are equivalent, for all $x, y, z \in A$:*

(a) $((x \longrightarrow y) \longrightarrow y) \longrightarrow x = y \longrightarrow x$,
(b) $(x \longrightarrow y) \longrightarrow y = (y \longrightarrow x) \longrightarrow x$,
(c) *If $x \longrightarrow z \leq y \longrightarrow z$, $z \leq x$, then $y \leq x$,*
(d) *If $x \longrightarrow z \leq y \longrightarrow z$, $z \leq x, y$, then $y \leq x$,*
(e) *If $y \leq x$, then $(x \longrightarrow y) \longrightarrow y \leq x$,*
(f) *A is an MV-algebra.*

Theorem 2.19. [11]. *Let M be a proper ideal of A. Then the following conditions are equivalent:*

(1) *M is a maximal ideal of A,*
(2) *for all $x \notin M$, there exists $n \in \mathbb{N}$, $(x^-)^n \in M$,*
(3) *$\dfrac{A}{M}$ is a locally finite MV-algebra.*

Proposition 2.20. [7]. *An ideal I of A is a prime ideal, if and only if the quotient BL-algebra $\dfrac{A}{I}$ is an MV-chain.*

Definition 2.21. [13]. A proper filter F is called a primary filter, if for all $x, y \in A$, $(x \odot y)^- \in F$ implies $(x^n)^- \in F$ or $(y^n)^- \in F$, for some $n \in \mathbb{N} \cup \{0\}$.

Proposition 2.22. [14]. *Let $f : A \to B$ be a BL-homomorphism and $I \in \mathrm{Id}(B)$. Then $f^{\leftarrow}(I) \in \mathrm{Id}(A)$.*

Proposition 2.23. [14]. *Let $f : A \to B$ be a BL-epimorphism and I be an ideal of A. Then $f(I) \in \mathrm{Id}(B)$.*

3. σ-ideals and primary ideals in BL-algebra

In this section, by considering the mapping $\sigma : A \longrightarrow A$ as a BL-homomorphism, we introduce the notion of σ-ideals and primary ideals in BL-algebras and derive some results about them.

Lemma 3.1. *Let Y be an ideal of A. Then $M(Y)$, $K(A)$ and $E(Y)$ are ideals of A, where $K(A) = \{a \in A : a^{=} = 0\}$, $M(Y) = \{y \in A : y^{=} \in Y\}$ and $E(Y) = \{t \in A : s^- \leq t^-$, for some $s \in Y\}$.*

Proof. (i) $K(A) \neq \varnothing$ (since $0 \in K(A)$). Suppose that $r, s \in K(A)$. Then $r^{=} = 0$ and $s^{=} = 0$. We have $(r \oslash s)^{=} = (r^{-} \longrightarrow s)^{=} = r^{\equiv} \longrightarrow s^{=} = (r^{-} \longrightarrow s^{=}) = (r^{-} \longrightarrow 0) = r^{=} = 0$, i.e., $r \oslash s \in K(A)$. Suppose that $r, s \in A$ with $r \leq s$ and $s \in K(A)$. Since $r^{=} \leq s^{=}$ and $s^{=} = 0$, $r^{=} = 0$. This means that $r \in K(A)$.
(ii) Let Y be an ideal of A. $0 \in M(Y)$ and $M(Y) \neq \varnothing$. If $r, s \in M(Y)$, then $r^{=}, s^{=} \in Y$. Therefore, by Remark 2.9, $r, s \in Y$. Consider $r^{=} \oslash s^{=} = r^{\equiv} \longrightarrow s^{=} = (r^{-} \longrightarrow s)^{=} = (r \oslash s)^{=}$. Since $r \oslash s \in Y$, $(r \oslash s)^{=} \in Y$. Therefore, $r \oslash s \in M(Y)$. If $r, s \in A$, $r \leq s$ and $s \in M(Y)$, then $r^{=} \leq s^{=}$. As $s^{=} \in Y$ and Y is an ideal of A, we get $r^{=} \in Y$ and $r \in M(Y)$.
(iii) $E(Y) \neq \varnothing$ (since $0 \in E(Y)$). If $s, t \in E(Y)$, then $a_1^{-} \leq s^{-}$, $a_2^{-} \leq t^{-}$, for some $a_1, a_2 \in Y$. We have $s^{=} \leq a_1^{=}$, $t^{=} \leq a_2^{=}$. By the compatibility of the operation "$\oslash$", we get that $(s \oslash t)^{=} = s^{=} \oslash t^{=} \leq a_1^{=} \oslash a_2^{=} = (a_1 \oslash a_2)^{=}$, i.e., $(a_1 \oslash a_2)^{\equiv} = (a_1 \oslash a_2)^{-} \leq (s \oslash t)^{\equiv} = (s \oslash t)^{-}$. Therefore, $(a_1 \oslash a_2)^{-} \leq (s \oslash t)^{-}$. Since $a_1, a_2 \in Y$ and Y is an ideal, $a_1 \oslash a_2 \in Y$. This means that $s \oslash t \in E(Y)$. Now, let $s, t \in A$, $s \leq t$ and $t \in E(Y)$. Then $a^{-} \leq t^{-}$, for some $a \in Y$. Since $t^{-} \leq s^{-}$, we conclude that $s \in E(Y)$. □

Theorem 3.2. *If I is an ideal of A and M is a non-empty subset of A, then the following statements hold:*

(i) $\sigma(E(Y)) \subseteq E(\sigma(Y))$ *and the equality holds, if σ is a BL-isomorphism,*
(ii) $\sigma(M(Y)) \subseteq M(\sigma(Y))$ *and the equality holds, if σ is a BL-isomorphism,*
(iii) $M(\sigma(I)) \subseteq M(\sigma(M(I)))$,
(iv) $\sigma(Y_l) \subseteq \sigma(Y)_l$, *where* $Y_l = \{a \in A \ : \ a \longrightarrow y = y, \text{ for all } y \in Y\}$,
(v) *If σ is surjective, and $\forall\, y \in A\ \exists\, n \in \mathbb{N}$; $\sigma(y) \leq y^n$, then $\sigma(I_l) = (\sigma(I))_l$.*

Proof. (i) Let $\alpha \in \sigma(E(Y))$. There exists $y \in E(Y)$ such that $\alpha = \sigma(y)$. We have $a^{-} \leq y^{-}$, for some $a \in Y$ and $a^{-} \longrightarrow y^{-} = 1$. So $\sigma(a^{-} \longrightarrow y^{-}) = \sigma(1) = 1$ and $\sigma(a^{-}) = (\sigma(a))^{-} \leq \sigma(y^{-}) = \sigma(y)^{-} = \alpha^{-}$. This means that $\alpha \in E(\sigma(Y))$. For the equality, if $t \in E(\sigma(Y))$, there exist $a_1 \in \sigma(Y)$, $y_1 \in Y$ such that $a_1^{-} \leq t^{-}$ and $a_1 = \sigma(y_1)$. It is enough to show that $t = \sigma(y_2)$, for some $y_2 \in E(Y)$. We take $a_2 = \sigma^{-1}(a_1)$ and $y_2 = \sigma^{-1}(t)$ (σ is isomorphism). Therefore,

$$\begin{aligned} a_2^{-} \leq y_2^{-} &\Longleftrightarrow (\sigma^{-1}(a_1))^{-} \leq (\sigma^{-1}(t))^{-} \\ &\Longleftrightarrow (\sigma^{-1}(a_1))^{-} \longrightarrow (\sigma^{-1}(t))^{-} = 1 \\ &\Longleftrightarrow \sigma(\sigma^{-1}(a_1))^{-} \longrightarrow \sigma(\sigma^{-1}(t))^{-} = 1 \\ &\Longleftrightarrow a_1^{-} \longrightarrow t^{-} = 1 \\ &\Longleftrightarrow a_1^{-} \leq t^{-}. \end{aligned}$$

The last relation is true by the fact that $t \in E(\sigma(Y))$. Therefore, $t \in \sigma(E(Y))$.
(ii) Let $\alpha \in \sigma(M(Y))$. Then $\alpha = \sigma(y)$, for some $y \in M(Y)$. Since $\alpha = \sigma(y)$, $\alpha^{-} = (\sigma(y))^{-} = \sigma(y^{-})$. Therefore, $\alpha^{=} = \sigma(y)^{=} = \sigma(y^{=})$ and $\alpha^{=} = \sigma(y^{=}) \in \sigma(Y)$. For the equality, let $t \in M(\sigma(Y))$. Then $t^{=} \in \sigma(Y)$. We show that $t = \sigma(y)$, for some $y \in M(Y)$. Take $y = \sigma^{-1}(t)$, so $y^{=} = (\sigma^{-1}(t))^{=} = \sigma^{-1}(t^{=}) \in \sigma^{-1}(\sigma(Y)) = Y$ (where σ isomorphism). This means that $y \in M(Y)$ and hence $t \in \sigma(M(Y))$.
(iii) Let $t \in M(\sigma(I))$. Then $t^{=} \in \sigma(y)$, for some $y \in I$. Based on Remark 2.9, $y^{=} \in I$ and so $y \in M(I)$. Therefore, $t \in M(\sigma(M(I))$.
(iv) Let $t \in \sigma(Y_l)$. Then there exists $z \in Y_l$ such that $t = \sigma(z)$. We have $z \longrightarrow y = y$, for any $y \in Y$. Therefore, $\sigma(z) \longrightarrow \sigma(y) = \sigma(y)$ and $\sigma(z) \in (\sigma(Y))_l$. Thus, $t \in (\sigma(Y))_l$.

(v) Since σ is onto, $I \subseteq \sigma(I)$ and $I = \sigma(I)$. Therefore, for the proof of $(\sigma(I))_l \subseteq \sigma(I_l)$, we show that $I_l \subseteq \sigma(I_l)$. Let $t \in I_l$, then $t \longrightarrow y = y$, for all $y \in I$. Since σ is onto, $\sigma(z) = t$, for some $z \in A$. Therefore, there exists $n \in \mathbb{N}$, $\sigma(z) \leq z^n$. Based on Theorem 2.2, $z^n \longrightarrow y \leq \sigma(z) \longrightarrow y = y$. But $z^n \leq z$, so $z \longrightarrow y \leq z^n \longrightarrow y$. Then $z \longrightarrow y = y$. This means that $z \in I_l$, i.e., $t = \sigma(z) \in \sigma(I_l)$. By (iv), the proof of the other side is clear. □

From now until the end of the paper, $\sigma : A \longrightarrow A$ is considered as a BL-homomorphism.

Definition 3.3. An ideal I of A is σ-ideal, if $\sigma(I) \subseteq I$.

Example 3.4. Let $A = \{0, a, b, 1\}$. Define " $\odot$ " and " $\longrightarrow$ " as follows:

$\odot$	0	a	b	1
0	0	0	0	0
a	0	a	0	a
b	0	0	b	b
1	0	a	b	1

$\longrightarrow$	0	a	b	1
0	1	1	1	1
a	b	1	b	1
b	0	a	1	1
1	0	a	b	1

Then it is easy to see that A is a BL-algebra and $I = \{0, a\}$ is an ideal of A. If we consider $\sigma : A \longrightarrow A$ by $\sigma(a) = 1$, $\sigma(b) = 0$, then $\sigma(I) = \{0, 1\} \nsubseteq I$. So I is not a σ-ideal. Also, we consider A in [7, Example $3 \cdot 5$], $\sigma = id_A$ and the ideal $J = \{0, d\}$ of A. Then J is a σ-ideal.

Theorem 3.5. *Let I be an ideal of A and $\sigma : A \longrightarrow A$ be a BL-homomorphism. Then I is a σ-ideal of A, if and only if σ^- is a well define map, where $\sigma^- : \frac{A}{I} \longrightarrow \frac{A}{I}$, by $\sigma^-([y]) = [\sigma(y)]$.*

Proof. We suppose that I is a σ-ideal of A. Let $[y], [z] \in \frac{A}{I}$ such that $[y] = [z]$. Then $y^- \odot z \in I$ and $z^- \odot y \in I$. We have $\sigma(y^- \odot z) \in \sigma(I) \subseteq I$, i.e., $\sigma(y^-) \odot \sigma(z) \in I$. Also, $\sigma(z^-) \odot \sigma(y) \in I$. Therefore, $\sigma(y) \sim \sigma(z)$ and hence $[\sigma(y)] = [\sigma(z)]$. Conversely, let $t \in \sigma(I)$. Then $t = \sigma(x)$, for some $x \in I$. Since $x \in I$, we have $[x] = [0]$. Therefore, $\sigma^-[x] = \sigma^-[0]$. This means that $[\sigma(x)] = [\sigma(0)]$, and hence $t = \sigma(x) \in I$. □

Remark 3.6. We note that, if the σ-ideal property of Theorem 3.5 is drop, then in general, σ^- is not well define. If we consider A in Example 3.4, $I = \{0, a\}$ as an ideal which is not σ-ideal $(\sigma(I) = \{0, 1\} \nsubseteq I)$ and $\sigma : A \longrightarrow A$ by $\sigma(a) = 1$, $\sigma(b) = 0$ as a BL-homomorphism, we have $a \in I$ and $[a] = [0]$. We know that $1 \notin I$, i.e., $[1] \neq [0]$, in other words, $[\sigma(a)] \neq [\sigma(0)]$. This means that $\sigma^-[a] \neq \sigma^-[0]$ and hence σ^- is not well define.

Lemma 3.7. *If I and J are σ-ideals of A, then the following conditions are satisfied:*

(i) *If $\sigma^n(I)$ is an ideal, then for every $n \in \mathbb{N}$, $\sigma^n(I)$ is a σ-ideal,*
(ii) *$I \cap J$ is a σ-ideal,*
(iii) *$\langle I \cup J \rangle$ is a σ-ideal,*
(iv) *$M(I)$ and $\mathrm{rad}(I)$ are σ-ideals of A,*
(v) *$E(I)$ is a σ-ideal.*

Proof. (i) By induction, let $n = 2$. Then $\sigma^2(I) = \sigma(\sigma(I)) \subseteq \sigma(I) \subseteq I$. Now, we suppose that for $n = k$. $\sigma^k(I) \subseteq I$, then $\sigma^{k+1}(I) = \sigma(\sigma^k(I)) \subseteq \sigma(I) \subseteq I$.
(ii) It is clear by the fact that $I \cap J \subseteq I, J$.
(iii) Let $t \in \sigma(\langle I \cup J \rangle)$. Then there exists $l \in \langle I \cup J \rangle$ such that $t = \sigma(l)$. Since $l \in \langle I \cup J \rangle$, $(a_1^- \odot \ldots \odot a_n^-) \longrightarrow l^- = 1$, for some $a_1, a_2, \ldots, a_n \in I \cup J$, $n \in \mathbb{N}$. Therefore, $\sigma(a_1^-) \odot \ldots \odot \sigma(a_n^-) \longrightarrow \sigma(l^-) = 1$. In other words, $(\sigma(a_1^-) \odot \ldots \odot \sigma(a_n^-)) \longrightarrow t^- = 1$. Now,

the proof is complete, since each $\sigma(a_i) \in I \cup J$, $1 \leq i \leq n$.
(iv) Let I be a σ-ideal of A. We show that $\sigma(M(I)) \subseteq M(I)$. Let $\alpha \in \sigma(M(I))$. Then $\alpha = \sigma(a)$, for some $a \in M(I)$. Since $a \in M(I)$, $a^{=} \in I$ and $\sigma(a^{=}) \in \sigma(I) \subseteq I$. Therefore, $(\sigma(a))^{=} = \sigma(a^{=}) \in I$, i.e., $\alpha^{=} \in I$, and hence $\alpha \in M(I)$. Now, let $t \in \sigma(\mathrm{rad}(I))$. Then $t = \sigma(b)$, for some $b \in \mathrm{rad}(I)$. By Theorem 2.17, $[b \longrightarrow (b^-)^n]^- \in I$, for every $n \in \mathbb{N}$. We get $\sigma[b \longrightarrow (b^-)^n] \in \sigma(I) \subseteq I$, for every $n \in \mathbb{N}$. This means that $[\sigma(b) \longrightarrow (\sigma(b))^-)^n]^- \in I$. Therefore, $[t \longrightarrow (t^-)^n]^- \in I$ and hence $t \in \mathrm{rad}(I)$.
(v) Let $t \in \sigma(E(I))$. Then $t = \sigma(y)$, for some $y \in E(I)$. We have $a^- \leq y^-$, for some $a \in I$. Also, $\sigma(a^-) \leq \sigma(y^-)$, i.e., $(\sigma(a))^- \leq (\sigma(y))^-$. In other words, $\sigma((a))^- \leq t^-$. Since $a \in I$, $\sigma(a) \in \sigma(I) \subseteq I$, so $t \in E(I)$. □

Corollary 3.8. *The structure $\langle I^{\sigma}(A), \wedge, \vee, \{0\}, A\rangle$ is a bounded lattice, where*

$$I^{\sigma}(A) = \{I \ : \ I \textit{ is a } \ \sigma - \textit{ideal of } A\}$$

denotes the set of all σ-ideals of A.

Proof. Let K and J be two σ-ideals of A. Then $K \vee J = \langle K \cup J\rangle$, $K \wedge J = K \cap J$. By Lemma 3.7, both $K \vee J$, $K \wedge J$ are σ-ideals of A. As for every $I \in I^{\sigma}(A)$, $\{0\} \subseteq I \subseteq A$, we conclude the lattice is bounded. □

Lemma 3.9. *Let A be an MV-algebra, and $M(I)$ a σ-ideal of A. Then I is a σ-ideal of A.*

Proof. Let I be an ideal of A and $t \in \sigma(I)$. Then $t = \sigma(a)$, for some $a \in I$. We know that $I \subseteq M(I)$. From Remark 2.9, for every $a \in I$, $a^{=} \in I$, i.e., $a \in M(I)$. So $\sigma(I) \subseteq \sigma(M(I)) \subseteq M(I)$. By hypothesis, $t \in M(I)$. Therefore, we get $t^{=} \in I$ and hence $t \in I$. □

Proposition 3.10. *Let A_1, A_2 be two BL-algebras and I_1, I_2 ideals of A_1 and A_2 respectively. If $\alpha : A_1 \longrightarrow A_1$, $\beta : A_2 \longrightarrow A_2$ and $f : A_1 \times A_2 \longrightarrow A_1 \times A_2$ by $f(x, y) = (\alpha(x), \beta(y))$ are BL-homomorphisms, for all $(x, y) \in A_1 \times A_2$, then $I_1 \times I_2$ is an f-ideal of $A_1 \times A_2$, if and only if I_1 is an α-ideal of A_1 and I_2 is a β-ideal of A_2.*

Proof. Consider that $(t, s) \in f(I_1 \times I_2)$. Then $(t, s) = f(x, y)$, for some $(x, y) \in I_1 \times I_2$. This means that $(t, s) = (\alpha(x), \beta(y))$. Since I_1 and I_2 are α and β-ideal, respectively, we have $\alpha(x) \in \alpha(I_1) \subseteq I_1$ and $\beta(y) \in \beta(I_2) \subseteq I_2$, i.e., $(t, s) = (\alpha(x), \beta(y)) \in I_1 \times I_2$.

Conversely, we show that I_1 and I_2 are α and β-ideals, respectively. Let $x \in \alpha(I_1)$ and $y \in \beta(I_2)$, then $x = \alpha(a_1)$ and $y = \beta(a_2)$, for some $a_1 \in I_1$ and $a_2 \in I_2$. We know that $(x, y) = (\alpha(a_1), \beta(a_2)) = f(a_1, a_2) \in f(I_1 \times I_2) \subseteq I_1 \times I_2$. Therefore, $(x, y) \in I_1 \times I_2$ and hence $x \in I_1$, $y \in I_2$. □

Theorem 3.11. *If I is an ideal of A, then the following statements are equivalent:*

(i) *I is a σ-ideal of A,*

(ii) *For any $x \in I$, $x \oslash \sigma(x) \in I$,*

(iii) $\sigma^- : \dfrac{A}{I} \longrightarrow \dfrac{A}{I}$, *by $\sigma^-[a] = [\sigma(a)]$ is a BL-homomorphism.*

Proof. (i)$\Longrightarrow$ (ii) Let $x \in I$. Then $\sigma(x) \in \sigma(I) \subseteq I$ and $\sigma(x) \in I$. Since I is an ideal of A, we get $x \oslash \sigma(x) \in I$, for any $x \in I$.
(ii)$\Longrightarrow$ (i) Let $t \in \sigma(I)$. Then $t = \sigma(a)$, for some $a \in I$. Based on (ii), we conclude $a \oslash \sigma(a) \in I$. We know that $\sigma(a) \odot a^- \leq \sigma(a)$. Therefore, $\sigma(a) \leq a^- \longrightarrow \sigma(a) = a \oslash \sigma(a) \in I$, i.e., $t = \sigma(a) \in I$.
(iii)$\Longleftrightarrow$ (i) The proof is clear from Theorem 3.5. □

Proposition 3.12. *If I, J are ideals of A and $J \subseteq I$, then $\frac{I}{J}$ is σ^--ideal of $\frac{A}{J}$ if and only if I is a σ-ideal of A.*

Proof. Let $t \in \sigma(I)$. Then $t = \sigma(a)$, for some $a \in I$ and $[a] \in \frac{I}{J}$. Therefore, $\sigma^-[a] \in \sigma^-\left[\frac{I}{J}\right] \subseteq \frac{I}{J}$. Since $[\sigma(a)] \in \frac{I}{J}$, $\sigma(a) \in I$, i.e., $t \in I$. Hence $\sigma(I) \subseteq I$.

Conversely, let $[x] \in \sigma^-\left[\frac{I}{J}\right]$. Then there exists $[y] \in \frac{I}{J}$, $[x] = \sigma^-([y]) = [\sigma(y)]$. Since $[y] \in \frac{I}{J}$, we have $[y] = \frac{y}{J} \in \frac{I}{J}$ and $y \in I$. Therefore, $\sigma(y) \in \sigma(I) \subseteq I$ and $\sigma(y) \in I$. This means that $[x] = [\sigma(y)] = \frac{\sigma(y)}{J} \in \frac{I}{J}$. Therefore, $[x] \in \frac{I}{J}$ and hence $\sigma(y) \in I$. □

Corollary 3.13. *If A is a Gödel algebra, then any ideal of A is a σ-ideal, if and only if for all $x \in A$, $x^- \leq \sigma^-(x)$.*

Proof. Let I be an ideal of A and for every $x \in A$, $x^- \leq \sigma^-(x)$. We show that $\sigma(I) \subseteq I$. If $t \in \sigma(I)$, we have $t = \sigma(s)$, for some $s \in I$. Since $s \in I$, $s \in A$ and $s^- \leq (\sigma(s))^- = t^-$. Therefore, $t \leq t^= \leq s^=$. From Remark 2.9 and the fact that $s \in I$, we get $t \in I$.

Conversely, let $x \in A$. We know that $x \in \langle x \rangle$ and by hypothesis, $\sigma(x) \in \sigma(\langle x \rangle) \subseteq \langle x \rangle$, i.e., $\sigma(x) \in \langle x \rangle$. Therefore, $(x^- \odot \ldots \odot x^-) \longrightarrow \sigma(x^-) = 1$, for some $n \in \mathbb{N}$. This means that $(x^-)^n = x^- \longrightarrow \sigma^-(x) = 1$, i.e., $x^- \leq (\sigma(x))^-$. □

Theorem 3.14. *Let $\sigma : A \longrightarrow A$ be a BL-homomorphism such that $(x^- \odot \sigma^-(x)) \longrightarrow (x^-)^n = (x^-)^n$, for every $x \in A$ and $n \in \mathbb{N}$. If $x^= = x$, for every $x \in A$, then $x^- \leq \sigma^-(x)$.*

Proof. We know that $x \in \langle x \rangle$, for every $x \in A$. Then $\sigma(x) \in \sigma(\langle x \rangle) \subseteq \langle x \rangle$ and by Theorem 2.13, we get $(x^-)^n \longrightarrow (\sigma(x))^- = 1$, for some $n \in \mathbb{N}$. This means that $(x^-)^n \leq (\sigma(x))^-$. By Theorem 2.2 (11), we obtain $x^- \odot ((\sigma(x))^- \longrightarrow (x^-)^n) \leq (x^- \odot (\sigma(x))^-) \longrightarrow (x^-)^n = (x^-)^n$. Therefore, $(\sigma(x))^- \longrightarrow (x^-)^n \leq x^- \longrightarrow (x^-)^n$. As $(x^-)^n \leq (\sigma(x))^-$ and $(\sigma(x))^- \longrightarrow (x^-)^n \leq x^- \longrightarrow (x^-)^n$, we apply Lemma 2.18 $(c) \Longleftrightarrow (f)$, so $x^- \leq (\sigma(x))^-$. □

Proposition 3.15. *If $g : A_1 \longrightarrow A_2$, $\sigma : A_1 \longrightarrow A_1$ and $\alpha : A_2 \longrightarrow A_2$ are BL-homomorphism and $\alpha g = g\sigma$, then the following statements hold:*

(i) *If g is an epimorphism and I is a σ-ideal of A_1, then $g(I)$ is a σ-ideal of A_2,*

(ii) *$g^{-1}(J)$ is a σ-ideal of A_1, for every α-ideal J of A_2.*

Proof. (i) Let $t \in \alpha(g(I))$. Then there exists $s \in g(I)$ such that $t = \alpha(s)$. Since $\alpha(g(I)) = g(\sigma(I))$ and $\sigma(I) \subseteq I$, we get $\alpha(g(I)) \subseteq g(I)$ and $t \in g(I)$.

(ii) Let $y \in \sigma(g^{-1}(J))$. Then $y = \sigma(x)$, for some $x \in g^{-1}(J)$. Since $\alpha g = g\sigma$, we have $g(\sigma(x)) = \alpha(g(x)) \in \alpha(J) \subseteq J$. Thus, $g(\sigma(x)) \in J$ and $\sigma(x) \in g^{-1}(J)$. This means that $g^{-1}(J)$ is a σ-ideal of A. □

Definition 3.16. An ideal P of A is called primary if for every $x, y \in A$, $x^= \wedge y^= \in P$ implies $x^n \in P$ or $y^n \in P$, for some $n \in \mathbb{N}$.

Example 3.17. (i) If we consider the BL-algebra A in [7, Example $3 \cdot 5$], then the ideal $I = \{0, d\}$ is primary, but $J = \{0\}$ is not primary, since $a^= \wedge d^= = c \wedge d = c \odot (c \to d) = c \odot d = 0 \in J$, but for any $n \in \mathbb{N}$, $a^n = a \notin J$, $d^n = d \notin J$.

(ii) In Example 3.4, $I = \{0, a\}$ is a primary ideal of A.

Theorem 3.18. *An ideal P of A is primary if and only if the quotient* BL*-algebra A/P is local.*

Proof. Let P be a primary ideal of A and $x \in A$. Then $(x^{\equiv} \wedge x^{=}) = (x^{-} \wedge x^{=}) = 0 \in P$. Since P is primary, we get $(x^{-})^n \in P$ or $x^n \in P$, for some $n \in \mathbb{N}$. Therefore, $(x^{-})^n/P = (x^{-}/P)^n = 0/P$ or $x^n/P = (x/P)^n = 0/P$. This means that $\mathrm{ord}(x^{-}/P) < \infty$ or $\mathrm{ord}(x/P) < \infty$ and hence A/P is a local BL-algebra.
Conversely, let A/P be a local BL-algebra and $x, y \in A$, $x^{=} \wedge y^{=} \in P$. Then $x \odot y \leq x^{=} \odot y^{=} \leq x^{=} \wedge y^{=}$. We get $x \odot y \in P$. From Proposition 2.10, since P is an ideal of A, A/P is a local MV-algebra and by Theorem 2.7, there exists $n \in \mathbb{N}$ such that $x^n \in P$ or $y^n \in P$. Therefore, P is a primary ideal of MV-algebra A. Now, we conclude P is a primary ideal in BL-algebra A by the fact that P is a primary ideal of MV-algebra A and $x \odot y \leq x^{=} \odot y^{=} \leq x^{=} \wedge y^{=}$. □

It is easy to see that any prime ideal is primary. Indeed, let P be a prime ideal of A and $x, y \in A$ such that $x^{=} \wedge y^{=} \in P$. By the fact that $x \wedge y \leq x^{=} \wedge y^{=}$, we conclude $x \wedge y \in P$. Since P is a prime ideal, we get $x \in P$ or $y \in P$ and hence P is primary.

Let $f : A \longrightarrow B$ be a BL-homomorphism. One can see that the $f^{\leftarrow}(J)$ is a primary ideal, for every primary ideal $J \in \mathrm{Id(B)}$ and $f(I)$ is a primary ideal of B, if $I \in \mathrm{Id(A)}$ is a primary ideal and f is surjective.

Recall [7] for every filter F of A, $N(F)$ is an ideal of A, where $N(F) = \{x \in A \,:\, x^{-} \in F\}$. Now, let F be a primary filter of A and $x, y \in A$, such that $x^{=} \wedge y^{=} \in N(F)$. Then $(x^{=} \wedge y^{=})^{-} \in F$. We get $x^{\equiv} \vee y^{\equiv} = x^{-} \vee y^{-} \in F$. Also, $x^{-} \vee y^{-} = (x \wedge y)^{-} \leq (x \odot y)^{-}$. Since F is a filter, $(x \odot y)^{-} \in F$. This means that $(x^n)^{-} \in F$ or $(y^n)^{-} \in F$, for some $n \in \mathbb{N} \cup \{0\}$ and hence $x^n \in N(F)$ or $y^n \in N(F)$, for some $n \in \mathbb{N} \cup \{0\}$. Therefore, we have the following corollary:

Corollary 3.19. *If F is a primary filter of A, then $N(F)$ is a primary ideal of A.*

Proposition 3.20. *Let P be a proper ideal of A such that A/P does not contain any non-nilpotent zero divisor elements. Then P is a primary ideal.*

Proof. Let $x, y \in A$; $x^{=} \wedge y^{=} \in P$. Then $x^{=} \odot y^{=} \in P$, as $x^{=} \odot y^{=} \leq x^{=} \wedge y^{=}$. By Remark 2.11, $(x^{=} \odot y^{=})/P = 0/P$. This means that $(x^{=}/P) \odot (y^{=}/P) = 0/P$. Now we are considering the two following cases:
(i) A/P does not contain any zero divisor elements. Therefore, $x^{=}/P = 0/P$ or $y^{=}/P = 0/P$, i.e., $x^{=} \in P$ or $y^{=} \in P$. By Remark 2.9, $x \in P$ or $y \in P$. Thus P is Primary.
(ii) A/P contains some zero divisor element. Then $x^{=}/P \neq 0/P$ and $y^{=}/P \neq 0/P$. This means that $x^{=}/P$ and $y^{=}/P$ are zero divisor. Therefore, by hypothesis, we conclude $(x^{=}/P)^n = 0/P$ or $(y^{=}/P)^m = 0/P$, for some $m, n \in \mathbb{N}$. We get $(x^{=})^n \in P$ or $(y^{=})^m \in P$. By Theorem 2.2 (7), $x^n \in P$ or $y^m \in P$ and hence P is primary. □

Theorem 3.21. *Let A be a Gödel algebra and I be a primary ideal of A. Then* $\mathrm{rad}(I)$ *is primary.*

Proof. Let $x, y \in A$; $x^{=} \wedge y^{=} \in \mathrm{rad}(I)$. Then by Theorem 2.17, $[(x^{=} \wedge y^{=}) \to ((x^{=} \wedge y^{=})^{-})^n]^{-} \in I$, for any $n \in \mathbb{N}$. Take $n = 1$, we get $[(x^{=} \wedge y^{=}) \to (x^{=} \wedge y^{=})^{-}]^{-} \in I$. From

Theorem 2.2(8), $[((x^= \wedge y^=) \odot (x^= \wedge y^=))^-]^- = ((x^= \wedge y^=)^2)^= \in I$. Since A is a Gödel algebra, we conclude $(x^= \wedge y^=)^= = (x^= \wedge y^=) \in I$. Therefore, $x^n \in I$ or $y^n \in I$, for some $n \in \mathbb{N}$, as I is a primary ideal. Now, the proof is completed by the fact that $I \subseteq \text{rad}(I)$. □

Theorem 3.22. *Every obstinate ideal of A is primary.*

Proof. Let I be an obstinate ideal of A and $x^= \wedge y^= \in I$ such that $x^n \notin I$, for every $n \in \mathbb{N}$. We show that $y^m \in I$, for some $m \in \mathbb{N}$. We obtain $x \notin I$ by the fact that $x^n \leq x$, $x^n \notin I$ and I is an ideal. Since I is an obstinate ideal and $1 \notin I$ and $x \notin I$, we get $x \odot 1^- \in I$ and $x^- \odot 1 \in I$. Therefore, $x^- \in I$. Now, if we apply the ideal property, so $x^- \oslash (x^= \wedge y^=) \in I$, i.e., $x^= \to (x^= \wedge y^=) \in I$. From BL_3 and Theorem 2.2 (3), we conclude $y^= \leq x^= \to x^= \wedge y^=$. As $x^= \to x^= \wedge y^= \in I$, we get $y^= \in I$ and by Remark 2.9, $y \in I$. This means that, there exists $m = 1$, $y = y^1 \in I$ and hence I is primary. □

Now, we give an example to conclude that the converse of the above theorem does not hold.

Example 3.23. We consider $A = \{0, a, b, c, d, 1\}$, with the following operations on A, where $0 < a < b < 1$, $0 < a < d < 1$, $0 < c < d < 1$.

$\odot$	0	a	b	c	d	1
0	0	0	0	0	0	0
a	0	0	a	0	0	a
b	0	a	b	0	a	b
c	0	0	0	c	c	c
d	0	0	a	c	c	d
1	0	a	b	c	d	1

$\to$	0	a	b	c	d	1
0	1	1	1	1	1	1
a	d	1	1	d	1	1
b	c	d	1	c	d	1
c	b	b	b	1	1	1
d	a	b	b	d	1	1
1	0	a	b	c	d	1

Then $(A, \wedge, \vee, \odot, \to, 0, 1)$ is a BL-algebra [6].

It is easy to see that $I = \{0, c\}$ is a primary ideal but, is not obstinate, since $a, b \notin I$, $a \odot b^- = a \odot c = 0 \in I$ but $a^- \odot b = d \odot b = a \notin I$.

Definition 3.24. Let Y be a subset of A. We say that Y holds in maximal property, if for any $y \notin Y$, there exists $n \in \mathbb{N}$ such that $(y^n)^- \in Y$.

Lemma 3.25. *If I is a σ-ideal of A and $\sigma^n(I)$ hold in maximal property, for some $n \in \mathbb{N}$, then I is a maximal ideal.*

Proof. Since I is a σ-ideal, $\sigma^n(I) \subseteq I$, for some $n \in \mathbb{N}$. If $x \notin I$, then $x \notin \sigma^n(I)$. Now from hypothesis, we get $(x^m)^- \in \sigma^n(I) \subseteq I$, for some $m \in \mathbb{N}$. This means that $(x^m)^- = (x^-)^m \in I$. Therefore, the proof is complete by Theorem 2.19. □

Proposition 3.26. *Let I be a σ-ideal of A and there exists $n \in \mathbb{N}$, with $(x \longrightarrow y)^- \in \sigma^n(I)$ or $(y \longrightarrow x)^- \in \sigma^n(I)$, for every $x, y \in I$. Then $\dfrac{A}{I}$ is an MV-chain.*

Proof. At first, we show that I is a prime ideal of A. Let $(x \longrightarrow y)^- \notin I$, since I is a σ-ideal, so $(x \longrightarrow y)^- \notin \sigma^n(I)$. By hypothesis, we get $(y \longrightarrow x)^- \in \sigma^n(I) \subseteq I$. Therefore, $(y \longrightarrow x)^- \in I$ and I is a prime ideal. The proof is complete by Proposition 2.20. □

We note that if σ is onto and I is a prime σ-ideal of A, then the converse of Proposition 3.26 is hold, for every $x, y \in A$ and $n \in \mathbb{N}$. Since let $x, y \in I$, $(x \longrightarrow y)^- \notin \sigma^n(I) = I$ (σ is onto). Then, by Definition 2.14(i), $(y \longrightarrow x)^- \in I = \sigma^n(I)$.

Corollary 3.27. *If I is a σ-ideal of A, then I is a prime ideal, if the following property holds:*

There exists $m \in \mathbb{N}$ such that $x \wedge y \in \sigma^m(I)$ implies $x \in \sigma^m(I)$ or $y \in \sigma^m(I)$, for every $x, y \in A$.

Proof. It is clear from Corollary 2.15 and the fact that $\sigma^m(I) \subseteq I$. □

4. σ-invariant, σ-prime and σ-primary ideals in BL-algebras

In this section, we define the notion of σ-invariant, σ-prime and σ-primary ideals in BL-algebras and we derive some results about them.

Definition 4.1. If $\sigma : A \longrightarrow A$ is a BL-homomorphism, then the ideal I is called an σ-invariant ideal, if $\sigma^{-1}(I) = I$.

It is easy to see that the ideal $\{0\}$ of A is a σ-invariant ideal.

Lemma 4.2. *If I is a σ-invariant ideal, then the following conditions hold:*
(i) *I is a σ-ideal,*
(ii) $\mathrm{Ker}(\sigma) \subseteq I$.

Proof. (i) Let I be a σ-invariant ideal of A. Then $\sigma^{-1}(I) = I$ and $\sigma(I) = \sigma(\sigma^{-1}(I)) \subseteq I$,
(ii) Let $t \in \mathrm{Ker}(\sigma)$. Then $\sigma(t) = 0 \in I$ and hence $t \in \sigma^{-1}(I) = I$. □

One should note that the converse of the above Lemma holds, if we add the condition $\sigma^2 = \sigma$. In fact, it is enough to show that $\sigma^{-1}(I) = I$. Let $a \in \sigma^{-1}(I)$. Then $\sigma(a) \in I$. Consider $\sigma((\sigma(a))^- \odot a) = (\sigma(\sigma(a)))^- \odot \sigma(a) = (\sigma^2(a))^- \odot \sigma(a) = \sigma^-(a) \odot \sigma(a) = 0$. This means that $(\sigma(a))^- \odot a \in \mathrm{Ker}(\sigma) \subseteq I$, i.e. $\sigma^-(a) \odot a \in I$. Now, by applying Theorem 2.12 and the fact that $\sigma(a) \in I$, we obtain $a \in I$. Hence, $\sigma^{-1}(I) \subseteq I$. Also, since I is a σ-ideal, $\sigma(I) \subseteq I$. Therefore, $I \subseteq \sigma^{-1}(I)$.

Proposition 4.3. *If I and J are σ-invariant ideals, then the following conditions hold:*
(i) *$I \cap J$ is a σ-invariant ideal,*
(ii) *$\langle I \cup J \rangle$ is a σ-invariant ideal, when σ is an epimorphism,*
(iii) $\mathrm{rad}(I)$ *is a σ-invariant ideal.*

Proof. (i) It is clear that by the fact that $I \cap J \subseteq I, J$. So $\sigma^{-1}(I \cap J) \subseteq \sigma^{-1}(I) = I$ and $\sigma^{-1}(I \cap J) \subseteq \sigma^{-1}(J) = J$. Thus, $\sigma^{-1}(I \cap J) \subseteq I \cap J$. Since $I \subseteq \sigma^{-1}(I)$, $J \subseteq \sigma^{-1}(J)$. Therefore, $I \cap J \subseteq \sigma^{-1}(I \cap J)$, and hence $I \cap J$ is a σ-invariant ideal.
(ii) By Lemma 3.7 (iii), $\sigma(\langle I \cup J \rangle) \subseteq \langle I \cup J \rangle$. So $\langle I \cup J \rangle \subseteq \sigma^{-1}(\langle I \cup J \rangle)$. We show that $\sigma^{-1}(\langle I \cup J \rangle) \subseteq \langle I \cup J \rangle$. Let $t \in \sigma^{-1}(\langle I \cup J \rangle)$. Then there exists $l \in \langle I \cup J \rangle$ such that $t = \sigma^{-1}(l)$ for some $a_1, \ldots, a_n \in I \cup J$. We have $(a_1^- \odot \ldots \odot a_n^-) \longrightarrow l^- = 1$. So $\sigma^{-1}(a_1^-) \odot \ldots \odot \sigma^{-1}(a_n^-) \longrightarrow \sigma^{-1}(l^-) = 1$. Consider that there exist $b_1, \ldots, b_n \in I \cup J$, such that $(b_1^- \odot \ldots \odot b_n^-) \longrightarrow t^- = 1$. We take $b_1 = \sigma^{-1}(a_1), \ldots, b_n = \sigma^{-1}(a_n)$, then $b_1^- \odot \ldots \odot b_n^- \longrightarrow t^- = 1$, i.e., $t \in \langle I \cup J \rangle$.
(iii) By Lemma 3.7 (iv), $\sigma(\mathrm{rad}(I)) \subseteq \mathrm{rad}(I)$. So $\mathrm{rad}(I) \subseteq \sigma^{-1}(\mathrm{rad}(I))$. We show that $\sigma^{-1}(\mathrm{rad}(I)) \subseteq \mathrm{rad}(I)$. Let $t \in \sigma^{-1}(\mathrm{rad}(I))$. Then there exists $a \in \mathrm{rad}(I)$ such that $t = \sigma^{-1}(a)$. As $a \in \mathrm{rad}(I)$, by Theorem 2.17, for any $n \in \mathbb{N}$, $[a \longrightarrow (a^-)^n]^- \in I$, then $\sigma^{-1}[a \longrightarrow (a^-)^n]^- \in \sigma^{-1}(I) = I$. Therefore, $[\sigma^{-1}(a) \longrightarrow \sigma^{-1}(a^-)^n]^- \in I$. Since $t = \sigma^{-1}(a)$, then $[t \longrightarrow (t^-)^n]^- \in I$. Hence $t \in \mathrm{rad}(I)$ and $\sigma^{-1}(\mathrm{rad}(I)) = \mathrm{rad}(I)$. □

Proposition 4.4. *If σ is onto and $\sigma^2 = \sigma$, then for any σ-ideal I of A, $\mathrm{Ker}(\sigma) \subseteq I$ implies that $\sigma(\mathrm{rad}(I)) = \mathrm{rad}(I)$.*

Proof. Let $t \in \mathrm{rad}(I)$. Then by Theorem 2.17 $[t \longrightarrow (t^-)^n]^- \in I$ for all $n \in \mathbb{N}$. Since $t \in \mathrm{rad}(I) \subseteq A$ and $\sigma : A \longrightarrow A$ is onto, there exists $b \in A$ such that $\sigma(b) = t$. Therefore, $[\sigma(b) \longrightarrow ((\sigma(b))^-)^n]^- \in I$. So $\sigma[b \longrightarrow (b^-)^n]^- \in I$. In other words, $[b \longrightarrow (b^-)^n]^- \in \sigma^{-1}(I) = I$, i.e., $b \in \mathrm{rad}(I)$. Thus, $\sigma(b) \in \sigma(\mathrm{rad}(I))$. This means that $t \in \sigma(\mathrm{rad}(I))$.

Conversely, we show that $\sigma(\mathrm{rad}(I)) \subseteq \mathrm{rad}(I)$. Since $\sigma^2 = \sigma$ and I is a σ-invariant ideal. Also, by Proposition 4.3 (iii), $\mathrm{rad}(I)$ is a σ-invariant ideal. Then $\mathrm{rad}(I)$ is a σ-ideal, and hence $\sigma(\mathrm{rad}(I)) \subseteq \mathrm{rad}(I)$. □

Definition 4.5. If I is a prime ideal of A, then I is called a σ-prime ideal, when for each $a \in I$, there exists $b \notin I$ such that $a \wedge b \in \mathrm{Ker}(\sigma)$.

Example 4.6. Let A be a BL-algebra in [7, Example $3 \cdot 5$], $\sigma = \mathrm{id}_A$ and $I = \{0, d\}$. Then I is a σ-prime ideal of A, as by [7, Example $3 \cdot 5$], I is a prime ideal of A and if $x = 0 \in I$, then for every $y \notin \{0, d\}$, $0 = x \wedge y \in \mathrm{Ker}(\sigma)$. Also, if $x = d$, then there exists $a \notin I$, $d \wedge a = d \odot (d \longrightarrow a) = d \odot c = 0 \in \mathrm{Ker}(\sigma)$.

Proposition 4.7. *Let I be a σ-prime σ-invariant ideal of A. Then I is a minimal prime σ-invariant ideal.*

Proof. Suppose that I is a σ-prime σ-invariant ideal of A and J is a prime σ-invariant ideal, such that $J \subseteq I$, for $a \in I$ and $a \notin J$. By hypothesis, since I is σ-prime, there exists $b \notin I$, such that $a \wedge b \in \mathrm{Ker}(\sigma)$. On the other hand, J is σ-invariant. By Lemma 4.2, $\mathrm{Ker}(\sigma) \subseteq J$ and $a \wedge b \in J$. As $b \notin I$, we take $b \notin J$. Thus, $a \wedge b \notin J$; this is a contradiction. Therefore, I is a minimal prime σ-invariant ideal. □

Definition 4.8. Let A be BL-algebra and I an ideal. Then I is called σ-primary, if for every $x \in A$, $x^= \wedge \sigma^=(x) \in I$, implies $x \in I$.

Example 4.9. (i) Let $A = \{0, a, b, c, 1\}$, $0 < a < c < 1$, $0 < b < c < 1$. Define $\odot$ and $\to$ as follow:

$\odot$	0	a	b	c	1
0	0	0	0	0	0
a	0	a	0	a	a
b	0	0	b	b	b
c	0	a	b	c	c
1	0	a	b	c	1

$\to$	0	a	b	c	1
0	1	1	1	1	1
a	b	1	b	1	1
b	a	a	1	1	1
c	0	a	b	1	1
1	0	a	b	c	1

Then it is easy to see that $I = \{0, a\}$ is an ideal of A. If we consider the mapping $\sigma : A \to A$ by $\sigma(0) = 0$, $\sigma(a) = b$, $\sigma(b) = a$, $\sigma(c) = c$, $\sigma(1) = 1$, so I is not a σ-primary ideal of A.
(ii) Let $A = \{0, a, b, 1\}$, $0 < a, b < 1$. Define $\odot$ and $\to$ as follow:

$\odot$	0	a	b	1
0	0	0	0	0
a	0	a	0	a
b	0	0	b	b
1	0	a	b	1

$\to$	0	a	b	1
0	1	1	1	1
a	b	1	b	1
b	a	a	1	1
1	0	a	b	1

Then it is easy to see that $I = \{0, a\}$ with $\sigma = \mathrm{id}_A$ is a σ-primary ideal of A.

Proposition 4.10. *If I, J are ideals of BL-algebras A, B respectively, then $I \times J$ is a σ-primary ideal of $A \times B$, if and only if I is a σ_1-primary ideal of A and J be a σ_2-primary ideal of B, where $\sigma : A \times B \longrightarrow A \times B$ by $\sigma(a, b) = (\sigma_1(a), \sigma_2(b))$ is a BL-homomorphism.*

Proof. Let $(a, b) \in A \times B$; $((a, b)^= \wedge \sigma^=(a, b)) \in I \times J$. Then $((a^=, b^=) \wedge (\sigma_1^=(a), \sigma_2^=(b)) = ((a^= \wedge \sigma_1^=(a)), (b^= \wedge \sigma_2^=(b)) \in I \times J$. This means that $((a^= \wedge \sigma_1^=(a)) \in I$ and $(b^= \wedge \sigma_2^=(b)) \in J$. Since, I is a σ_1-primary ideal of A and J is a σ_2-primary ideal of B, we get $a \in I$ and $b \in J$. Therefore, $(a, b) \in I \times J$.

Conversely, let $a \in A$ and $b \in B$; $(a^= \wedge \sigma_1^=(a)) \in I$ and $(b^= \wedge \sigma_2^=(b)) \in J$. Then $((a^= \wedge \sigma_1^=(a)), (b^= \wedge \sigma_2^=(b))) \in I \times J$. We get $((a^=, b^=) \wedge (\sigma_1^=(a), \sigma_2^=(b))) = ((a^=, b^=) \wedge \sigma^=(a, b)) \in I \times J$. Since $I \times J$ is a σ-primary ideal of $A \times B$, we conclude $(a, b) \in I \times J$,i.e., $a \in I$, $b \in J$. Therefore, I is a σ_1-primary of A and J is a σ_2-primary of B. □

Now, we have the following proposition, by considering Propositions 2.22 and 2.23.

Proposition 4.11. *Let $g : A_1 \longrightarrow A_2$ be a BL-epimorphism, $\sigma : A_1 \longrightarrow A_1$, $\gamma : A_2 \longrightarrow A_2$ be BL-homomorphism and $\gamma g = g\sigma$. Then the following statements hold:*

(i) *If I is a σ-primary ideal of A_1 and* $\mathrm{Ker}(g) \subseteq I$, *then $g(I)$ is a γ-primary ideal of A_2,*

(ii) *If J is a γ-primary ideal of A_2, then $g^{-1}(J)$ is a σ-primary ideal of A_1.*

Proof. (i) Let $b \in A_2$; $(b^= \wedge \gamma^=(b)) \in g(I)$. Since g is onto, there exists $a \in A_1$, such that $g(a) = b$. Therefore, $(g^=(a) \wedge \gamma g^=(a)) \in g(I)$. By hypothesis, we get $(g^=(a) \wedge g(\sigma^=(a)) = g(a^= \wedge \sigma^=(a)) \in g(I)$. Therefore, $g(a^= \wedge \sigma^=(a)) = g(c)$, for some $c \in I$ and $g(a^= \wedge \sigma^=(a)) \wedge g^-(c) = g(c) \wedge g^-(c) = 0$. This means that $g(a^= \wedge \sigma^=(a) \wedge c^-) = 0$ and $a^= \wedge \sigma^=(a) \wedge c^- \in \mathrm{Ker} g \subseteq I$. Since $c^- \odot (a^= \wedge \sigma^=(a)) \leq c^- \wedge (a^= \wedge \sigma^=(a))$, we have $c^- \odot (a^= \wedge \sigma^=(a)) \in I$. By applying Theorem 2.12 and the fact that $c \in I$, we get $(a^= \wedge \sigma^=(a)) \in I$. Since I is a σ-primary, $a \in I$ and hence $b = g(a) \in g(I)$.

(ii) Let $a \in A_1$; $(a^= \wedge \sigma(a^=)) \in g^{-1}(J)$. Then $g(a^= \wedge \sigma(a^=)) \in J$. We get $g(a^=) \wedge g\sigma(a^=) = (g(a))^= \wedge \sigma(g(a))^=) \in J$ and by hypothesis, $g(a^=) \wedge \gamma g(a^=) \in J$. Since J is a γ-primary, we get $g(a) \in J$ and hence $a \in g^{-1}(J)$. □

Proposition 4.12. *If I is a σ-primary ideal of A, then $M(I)$ is σ-primary.*

Proof. Let $(x^= \wedge \sigma^=(x)) \in M(I)$, then we show that $x \in M(I)$. As $(x^= \wedge \sigma^=(x)) \in M(I)$, $((x^= \wedge \sigma^=(x))^= \in I$. This means that $(x^\equiv \vee \sigma^\equiv(x))^- \in I$. By Theorem 2.2 (5), $(x^- \vee \sigma^-(x))^- \in I$, so $(x^= \wedge \sigma^=(x)) \in I$. Based on hypothesis, $x \in I$ and by Remark 2.9, $x^= \in I$ and hence, $x \in M(I)$.

□

Remark 4.13. $K(A)$ is a σ-primary ideal of A, if $\{0\}$ is a σ-primary. Indeed, let $z^= \wedge \sigma^=(z) \in K(A)$, then $(z^= \wedge \sigma^=(z))^= = 0$. This means that $z^= \wedge \sigma^=(z) = 0 \in \{0\}$. Since $\{0\}$ is σ-primary ideal, so $z \in \{0\}$ and $z = 0 = 0^= = z^=$, therefore, $z \in K(A)$.

5. Conclusion

The main results of this paper are related to the ideals that depend on a BL-homomorphism. We note that the notions of ideals and filters in BL-algebras are not dual to each other. At first, by considering σ as a BL-homomorphism, we introduced the notion of σ-ideals in BL-algebras and obtained some new results about them. Also, while introducing the primary ideals in BL-algebras, we get some relationship between the primary ideals and obstinate ideals. Finally, by introducing σ-prime, σ-primary and σ-invariant ideals, we derived some new results related to them. As a future research, these concepts can be extended to other algebraic structures such as residuated lattices and pseudo BL-algebras that they are extensions of BL-algebras.

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