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Research Paper

VAGUE SOFT HYPERMODULES; CONNECTIONS WITH SOFT HYPERMODULES AND HOMOMORPHISMS

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ABSTRACT

In this paper, the notion of vague soft hypermodules is introduced, which is an extension of the notions of vague soft hypergroups and vague soft hyperrings. Also, some basic properties of vague soft hypermodules and homomorphisms between vague soft hypermodules are presented and the image and pre-image of a vague soft hypermodule under a vague soft hypermodule homomorphism are studied.

1. Introduction

The notion of hyperstructures was introduced in 1934 by Marty [10]. Hyperstructures have many applications in several branches of both pure and applied sciences. A comprehensive review of the theory of hyperstructures can be found in [4, 5, 17]. The canonical hypergroups are a special type of hypergroups, which were first studied by Mittas [12]. Hypermodules which their additive structure is a canonical hypergroup, have been studied by several authors, for example see [6, 11].

The theory of fuzzy sets proposed by Zadeh [19] has achieved a numerous success in different fields. Rosenfeld [14] applied the notion of fuzzy sets to algebra and the literature of various fuzzy algebraic concepts has been growing very rapidly. There is a considerable

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amount of work on the connections between fuzzy sets and hyperstructures, such as [1, 9]. Molodtsov [13] introduced the concept of soft sets as a new mathematical tool for dealing with uncertainties that are free from the difficulties that have troubled the usual theoretical approaches. Molodtsov pointed out several directions for the applications of soft sets in many different fields.

The theory of vague sets was first proposed by Gau and Buehrer [7]. A vague set is defined by a truth-membership function t_v, and a false-membership function f_v, where t_v is a lower bound on the grade of membership of x derived from the evidence for x and $f_v(x)$ is a lower bound on the negation of x derived from the evidence against x. The values of $t_v(x)$ and $f_v(x)$ are both defined on the closed interval [0,1] with each point in a basic set X, where $t_v(x) + f_v(x) \leq 1$. Vague set theory is an extension of fuzzy set theory and vague sets are regarded as a special case of context-dependent fuzzy sets. The basic concepts of this theory and some of its interesting applications can be found in [3].

The study of soft hyperstructures began with the introduction of soft hypergroupoids and sub-hypergroupoids, which are an extension of the theories of soft sets and algebraic hyperstructures, and then, soft hypergroups, soft hyperrings and soft hypermodules were studied [2, 16]. Vague soft hypergroups and vague soft hyperrings were introduced in [15, 16].

In this paper, we develop the notion of soft hypermodules to the notion of vague soft hypermodules and give some basic properties of them. Also, we introduce vague soft hypermodule homomorphism and study homomorphic image and homomorphic pre-image of vague soft hypermodules.

2. Preliminaries

In the sequel, we recall some basic definitions and concepts regarding to hypergroups, hyperrings, hypermodules, soft sets, vague sets and vague soft sets.

Recall that a hyperstructure is a non-empty set H together with a mapping $\circ : H \times H \to P^*(H)$, where $P^*(H)$ is the set of all non-empty subsets of H. If $x \in H$ and $A, B \in P^*(H)$, then by $A \circ B$, $A \circ x$ and $x \circ B$, we mean $A \circ B = \bigcup\limits_{a\in A, b\in B} a \circ b$, $A \circ x = A \circ \{x\}$ and $x \circ B = \{x\} \circ B$, respectively.

A hyperstructure $(H, \circ)$ is said to be a canonical hypergroup [12] if the following conditions are fulfilled:

(i) for every $x, y, z \in H$, $x \circ (y \circ z) = (x \circ y) \circ z$,

(ii) for every $x, y \in H$, $x \circ y = y \circ x$,

(iii) there exists an element $0 \in H$ such that $0 \circ x = x$, for all $x \in H$;

(iv) for every $x \in H$, there exists a unique element $x' \in H$ such that $0 \in x \circ x'$ (the element x' is called the opposite of x and it is denoted by $-x$),

(v) $z \in x \circ y$ implies $y \in (-x) \circ z$ and $x \in z \circ (-y)$.

Following [8] a hyperring is an algebraic hyperstructure $(R, +, \cdot)$ which satisfies the following axioms:

(i) $(R, +)$ is a canonical hypergroup,

(ii) $(R, \cdot)$ is a semigroup having zero as a bilaterally absorbing element,

(iii) the multiplication is distributive with respect to the hyperoperation "$+$".

Let $(R, +, \cdot)$ be a hyperring and A a non-empty subset of R. Then A is called a sub-hyperring of R if $(A, +, \cdot)$ is itself a hyperring.

Following [4] a non-empty set M is called a left hypermodule over a hyperring R (R-hypermodule) if $(M, +)$ is a canonical hypergroup and there exists the map "$\cdot$" $: R \times M \to P^*(M)$ by $(r, m) \mapsto rm$ such that for all $r_1, r_2 \in R$ and $m_1, m_2 \in M$,

(i) $r_1(m_1 + m_2) = r_1m_1 + r_1m_2$,

(ii) $(r_1 + r_2)m_1 = r_1m_1 + r_2m_1$,

(iii) $(r_1r_2)m_1 = r_1(r_2m_1)$.

A non-empty subset M' of M is called a sub-hypermodule of the hypermodule $(M, +, \cdot)$ if $(M', +)$ is a sub-hypergroup of $(M, +)$ and $RM' \in \mathcal{P}^*(M')$.

Definition 2.1 ([20])**.** A fuzzy subset μ of a hypermodule M over a hyperring R is a fuzzy sub-hypermodule of M if

(i) $\inf_{z \in x+y} \mu(z) \geq \min\{\mu(x), \mu(y)\}$ for all $x, y \in M$,

(ii) $\mu(-x) \geq \mu(x)$, for all $x \in M$

(iii) $\mu(rx) \geq \mu(x)$ for all $r \in R$ and $x \in M$.

Let U be an initial universe set, E a set of parameters, $P(U)$ the power set of U, and $A \subseteq E$. Following [13] a soft set is defined as follows:

Definition 2.2. A pair (F, A) is called a soft set over U, where F is a mapping of A into $P(U)$. In other words, a soft set over U is a parameterized family of subsets of the universe set U. For $a \in A$, the set $F(a)$ is considered as the set of a-approximate elements of the soft set (F, A).

Definition 2.3 ([7])**.** Let $U = \{u_1, u_2, \ldots, u_n\}$ be an initial universe set. A vague set of U is characterized by a truth-membership function $t_v : U \to [0, 1]$ and a false-membership function $f_v : U \to [0, 1]$, where $t_v(u_i)$ is a lower bound on the grade of membership of u_i derived from the evidence for u_i, $f_v(u_i)$ is a lower bound on the negation of u_i derived from the evidence of $[0, 1]$ and $t_v(u_i) + f_v(u_i) \leq 1$.

Definition 2.4 ([18])**.** Let $x = [t_x, 1 - f_x]$ and $y = [t_y, 1 - f_y]$ be two vague values, where $t_x, t_y, f_x, f_y \in [0, 1]$, $0 \leq t_x \leq 1 - f_x \leq 1$, and $0 \leq t_y \leq 1 - f_y \leq 1$. Then

(i) if $t_x = 1$ and $f_x = 0$, then x is called a unit vague value.

(ii) if $t_x = 0$ and $f_x = 1$, then x is called a zero vague value.

(iii) if $t_x = t_y$ and $f_x = f_y$, then vague values x and y are called equal.

A vague set A of the initial universe set $U = \{u_1, u_2, \ldots, u_n\}$ can be shown by

$$A = \sum_{i=1}^{n} [t_A(u_i), 1 - f_A(u_i)]/u_i.$$

Definition 2.5 ([18])**.** Let A be a vague set of the initial universe set U. Then

(i) if for every $u \in U$, $t_A(u) = 1$ and $f_A(u) = 0$, then A is called a unit vague set.

(ii) if for every $u \in U$, $t_A(u) = 0$ and $f_A(u) = 1$, then A is called a zero vague set.

Let U be a initial universe set, E a set of parameters, $V(U)$ the power set of vague sets on U and $A \subseteq E$. Following [18] a vague soft set is defined as follows:

Definition 2.6 ([18])**.** A pair $(\hat{F}, A)$ is called a vague soft set over U, where $\hat{F}$ is a mapping given by $\hat{F} : A \to V(U)$.

In other words, a vague soft set over U is a parameterized family of vague sets of the universe U. For $\epsilon \in A$, $\mu_{\hat{F}(\epsilon)} : U \to [0,1]^2$ is considered as the set of ϵ-approximate elements of the vague soft set $(\hat{F}, A)$.

A null vague soft set is a vague soft set where both the truth and false membership functions are equal to zero [18].

The support of a vague soft set $(\hat{F}, A)$ is denoted by $\mathrm{Supp}(\hat{F}, A)$ and defined as following [16]:

$$\mathrm{Supp}(\hat{F}, A) = \{a \in A : t_{\hat{F}_a}(x) \neq 0,\ 1 - f_{\hat{F}_a}(x) \neq 0,\ \forall x \in U\}$$

Therefore, a vague soft set $(\hat{F}, A)$ is a non-null vague soft if $\mathrm{Supp}(\hat{F}, A) \neq \emptyset$.

Definition 2.7 ([18])**.** The union of two vague soft sets $(\hat{F}, A)$ and $(\hat{G}, B)$ over an initial universe set U is a vague soft set $(\hat{H}, C)$, where $C = A \cup B$ and for all $e \in C$,

$$t_{\hat{H}(e)}(x) = \begin{cases} t_{\hat{F}(e)}(x), & \text{if } e \in A - B,\ x \in U, \\ t_{\hat{G}(e)}(x), & \text{if } e \in B - A,\ x \in U, \\ \max\left\{t_{\hat{F}(e)}(x), t_{\hat{G}(e)}(x)\right\}, & \text{if } e \in A \cap B,\ x \in U. \end{cases}$$

$$1 - f_{\hat{H}(e)}(x) = \begin{cases} 1 - f_{\hat{F}(e)}(x), & \text{if } e \in A - B,\ x \in U, \\ 1 - f_{\hat{G}(e)}(x), & \text{if } e \in B - A,\ x \in U, \\ \max\left\{1 - f_{\hat{F}(e)}(x), 1 - f_{\hat{G}(e)}(x)\right\}, & \text{if } e \in A \cap B,\ x \in U. \end{cases}$$

We denote it by $(\hat{F}, A)\tilde{\cup}(\hat{G}, B) = (\hat{H}, C)$.

Definition 2.8 ([18])**.** The intersection of two vague soft sets $(\hat{F}, A)$ and $(\hat{G}, B)$ over an initial universe set U is a vague soft set $(\hat{H}, C)$, where $C = A \cap B$ and for all $e \in C$,

$$t_{\hat{H}(e)}(x) = \begin{cases} t_{\hat{F}(e)}(x), & \text{if } e \in A - B,\ x \in U, \\ t_{\hat{G}(e)}(x), & \text{if } e \in B - A,\ x \in U, \\ \min\left\{t_{\hat{F}(e)}(x), t_{\hat{G}(e)}(x)\right\}, & \text{if } e \in A \cap B,\ x \in U. \end{cases}$$

$$1 - f_{\hat{H}(e)}(x) = \begin{cases} 1 - f_{\hat{F}(e)}(x), & \text{if } e \in A - B,\ x \in U, \\ 1 - f_{\hat{G}(e)}(x), & \text{if } e \in B - A,\ x \in U, \\ \min\left\{1 - f_{\hat{F}(e)}(x), 1 - f_{\hat{G}(e)}(x)\right\}, & \text{if } e \in A \cap B,\ x \in U. \end{cases}$$

We denote by $(\hat{F}, A)\tilde{\cap}(\hat{G}, B) = (\hat{H}, C)$.

Definition 2.9 ([18])**.** Let $(\hat{F}, A)$ and $(\hat{G}, B)$ be two vague soft sets over an initial universe set U. Then "$(\hat{F}, A)$ and $(\hat{G}, B)$" is denoted by $(\hat{F}, A) \wedge (\hat{G}, B)$ and defined by $(\hat{F}, A) \wedge (\hat{G}, B) = (\hat{H}, A \times B)$, where for all $(\alpha, \beta) \in A \times B$ and $x \in U$,

$$t_{\hat{H}(\alpha,\beta)}(x) = \min\{t_{\hat{F}(\alpha)}(x), t_{\hat{G}(\beta)}(x)\},$$
$$1 - f_{\hat{H}(\alpha,\beta)}(x) = \min\{1 - f_{\hat{F}(\alpha)}(x), 1 - f_{\hat{G}(\beta)}(x)\}.$$

Definition 2.10 ([18]). Let $(\hat{F}, A)$ and $(\hat{G}, B)$ be two vague soft sets over an initial universe set U. Then "$(\hat{F}, A)$ or $(\hat{G}, B)$" is denoted by $(\hat{F}, A) \vee (\hat{G}, B)$ and defined by $(\hat{F}, A) \vee (\hat{G}, B) = (\hat{O}, A \times B)$, where for all $(\alpha, \beta) \in A \times B$ and $x \in U$,

$$t_{\hat{O}(\alpha,\beta)}(x) = \max\{t_{\hat{F}(\alpha)}(x), t_{\hat{G}(\beta)}(x)\},$$
$$1 - f_{\hat{O}(\alpha,\beta)}(x) = \max\{1 - f_{\hat{F}(\alpha)}(x), 1 - f_{\hat{G}(\beta)}(x)\}.$$

In the following, the vague soft (α, β)-cut is introduced.

Definition 2.11 ([16]). Let $(\hat{F}, A)$ be a vague soft set over an initial universe set U. Then, for all $\alpha, \beta \in [0, 1]$, where $\alpha \leq \beta$, the (α, β)-cut or the vague soft (α, β)-cut of $(\hat{F}, A)$ is a subset of U which is defined as below:

$$(\hat{F}, A)_{(\alpha,\beta)} = \big\{x \in U : t_{\hat{F}_a}(x) \geq \alpha,\ 1 - f_{\hat{F}_a}(x) \geq \beta\big\},$$

for all $a \in A$.

If $\alpha = \beta$, then it is called the vague soft (α, α)-cut of $(\hat{F}, A)$ or the α-level set of $(\hat{F}, A)$, denoted by $(\hat{F}, A)_{(\alpha,\alpha)}$, is a subset of U which is as defined below:

$$(\hat{F}, A)_{(\alpha,\alpha)} = \big\{x \in U : t_{\hat{F}_a}(x) \geq \alpha,\ 1 - f_{\hat{F}_a}(x) \geq \alpha\big\},$$

for all $a \in A$.

Definition 2.12 ([16]). Let $\varphi : X \to Y$ and $\psi : A \to B$ be two functions, where A and B are parameter sets for the classical sets X and Y, respectively. Let $(\hat{F}, A)$ and $(\hat{G}, B)$ be two vague soft sets over X and Y, respectively. Then the ordered pair (φ, ψ) is called a vague soft function from $(\hat{F}, A)$ to $(\hat{G}, B)$, and it is denoted as $(\varphi, \psi) : (\hat{F}, A) \to (\hat{G}, B)$.

Definition 2.13 ([16]). Let $(\hat{F}, A)$ and $(\hat{G}, B)$ be two vague soft sets over X and Y, respectively. Let $(\varphi, \psi) : (\hat{F}, A) \to (\hat{G}, B)$ be a vague soft function. Then

(i) the image of $(\hat{F}, A)$ under the vague soft function (φ, ψ), denoted by $(\varphi, \psi)(\hat{F}, A)$, is a vague soft set over Y, which is defined as $(\varphi, \psi)(\hat{F}, A) = (\varphi(\hat{F}), \psi(A))$, where

$$\varphi(\hat{F}_a)\big(\varphi(x)\big) = \big(\varphi(\hat{F})\big)_{\psi(a)}(y) = [t_{\varphi(F)_k}(y), 1 - f_{\varphi(F)_k}(y)]$$

for all $a \in A$, $x \in X$, $y \in Y$, and $k \in B$ such that $\varphi(x) = y$, and $\psi(a) = k$, where

$$t_{\varphi(F)_k}(y) = \bigvee_{\varphi(x)=y} \bigvee_{\psi(a)=k} (t_{F_a})(x),$$

$$1 - f_{\varphi(F)_k}(y) = \bigvee_{\varphi(x)=y} \bigvee_{\psi(a)=k} (1 - f_{F_a})(x).$$

(ii) the pre-image of $(\hat{G}, B)$ under the vague soft function (φ, ψ), denoted by $(\varphi, \psi)^{-1}(\hat{G}, B)$, is a vague soft set over X, which is defined as $(\varphi, \psi)^{-1}(\hat{G}, B) = \big(\varphi^{-1}(\hat{G}), \psi^{-1}(B)\big)$, where

$$\varphi^{-1}(\hat{G}_b)\big(\varphi^{-1}(y)\big) = \big(\varphi^{-1}(\hat{G})\big)_{\psi^{-1}(b)}(x) = \hat{G}_{\psi(a)}\big(\varphi(x)\big)$$

for all $a \in A$, $b \in B$, $x \in X$, and $y \in Y$ such that $\psi(a) = b$, and $\varphi(x) = y$.

If φ and ψ are injective (surjective), then the vague soft function (φ, ψ) is said to be injective (surjective).

3. Vague soft hypermodules and soft hypermodules

In this section, we introduce the notions of vague sub-hypermodules and vague soft hypermodules and give some basic properties of vague soft hypermodules.

Let M be a hypermodule over a hyperring R and (F, A) a soft set over M. Then (F, A) is called a soft hypermodule over M if $F(a)$ is a sub-hypermodule of M, for each $a \in A$ [2].

Definition 3.1. Let M be a hypermodule over a hyperring R and $F = [t_F, 1 - f_F] : M \to [0,1]^2$ a vague set of M. Then F is called a vague sub-hypermodule of M if

(i) for all $x, y \in M$, $\min\{F(x), F(y)\} \leq \inf_{z \in x+y} F(z)$,

(ii) for all $x \in M$, $F(-x) \geq F(x)$,

(iii) for all $r \in R$ and $x \in M$, $F(rx) \geq F(x)$.

Let us consider the family of α-level subsets for F, given by $F_\alpha = \{x \in M : F(x) \geq \alpha\}$, where $\alpha \in [0,1]$. Then, for all $\alpha \in [0,1]$, F_α is a sub-hypermodule of M. Hence, $(F, [0,1])$ is a soft hypermodule over M.

Definition 3.2. Let M be a hypermodule over a hyperring R and $(\hat{F}, A)$ a non-null vague soft set over M. Then $(\hat{F}, A)$ is called a vague soft hypermodule over M if for every $a \in \mathrm{Supp}(\hat{F}, A)$, the following conditions are satisfied:

(i) for all $x, y \in M$,

$$\min\{t_{\hat{F}(a)}(x), t_{\hat{F}(a)}(y)\} \leq \inf\{t_{\hat{F}(a)}(z) : z \in x + y\},$$
$$\min\{1 - t_{\hat{F}(a)}(x), 1 - t_{\hat{F}(a)}(y)\} \leq \inf\{1 - t_{\hat{F}(a)}(z) : z \in x + y\}.$$

(ii) for all $x \in M$, $t_{\hat{F}(a)}(-x) \geq t_{\hat{F}(a)}(x)$ and $1 - t_{\hat{F}(a)}(-x) \geq 1 - t_{\hat{F}(a)}(x)$.

(iii) for all $r \in R$ and $x \in M$, $t_{\hat{F}(a)}(rx) \geq t_{\hat{F}(a)}(x)$ and $1 - t_{\hat{F}(a)}(rx) \geq 1 - t_{\hat{F}(a)}(x)$.

This means that, for every $a \in A$, $\hat{F}_a$ is a vague sub-hypermodule of M.

Example 3.3. Consider $M = R = \{0, 1, 2, 3\}$ with the following hyperoperations:

$+$	0	1	2	3
0	$\{0\}$	$\{1\}$	$\{2\}$	$\{3\}$
1	$\{1\}$	$\{0,1\}$	$\{3\}$	$\{2,3\}$
2	$\{2\}$	$\{3\}$	$\{0\}$	$\{1\}$
3	$\{3\}$	$\{2,3\}$	$\{1\}$	$\{0,1\}$

$\cdot$	0	1	2	3
0	$\{0\}$	$\{0\}$	$\{0\}$	$\{0\}$
1	$\{0\}$	$\{0\}$	$\{0\}$	$\{0\}$
2	$\{0\}$	$\{0\}$	$\{2\}$	$\{2\}$
3	$\{0\}$	$\{0\}$	$\{2\}$	$\{2\}$

Then $(M, +, \cdot)$ is a hypermodule over hyperring $(R, +, \cdot)$. Put $A = \{a, b, c\}$ and define the vague-valued function $\hat{F} : A \to V(M)$ by the following table, where $V(M)$ is the power set of vague sets on M:

	0	1	2	3
$\hat{F}_a$	$[0.3, 0.6]$	$[0.1, 0.4]$	$[0.2, 0.5]$	$[0.1, 0.4]$
$\hat{F}_b$	$[0.4, 0.7]$	$[0.2, 0.5]$	$[0.3, 0.6]$	$[0.2, 0.5]$
$\hat{F}_c$	$[0.5, 0.7]$	$[0.3, 0.4]$	$[0.4, 0.5]$	$[0.3, 0.4]$

Hence $(\hat{F}, A)$ is a vague soft hypermodule over M.

Example 3.4. A vague soft hypermodule $(\hat{F}, A)$, for which A is a singleton, is a vague sub-hypermodule. Therefore, vague sub-hypermodules and hypermodules are a particular type of vague soft hypermodules.

Example 3.5. Any soft hypermodule is a vague soft hypermodule, since any characteristic function of a sub-hypermodule is a vague sub-hypermodule.

Theorem 3.6. *Let $(\hat{F}, A)$ be a vague soft set over a hypermodule M. Then (F, A) is a vague soft hypermodule over M if and only if for all $t \in [0,1]$, $(\hat{F}, A)_{(t,t)}$ is a soft hypermodule over M.*

PROOF. Let $(\hat{F}, A)$ be a vague soft hypermodule over M. Then, for all $a \in \mathrm{Supp}(\hat{F}, A)$, $\hat{F}_a$ is a non-null vague sub-hypermodule of M. Let $x, y \in (\hat{F}_a)_{(t,t)}$. Hence,

$$\min\{t_{\hat{F}_a}(x), t_{\hat{F}_a}(y)\} \geq \min\{t, t\} = t,$$
$$\min\{1 - f_{\hat{F}_a}(x), 1 - f_{\hat{F}_a}(y)\} \geq \min\{t, t\} = t.$$

That is, $\min\{\hat{F}_a(x), \hat{F}_a(y)\} \geq t$.

Moreover, since $\hat{F}_a$ is a vague sub-hypermodule of M, we have

$$\inf\{t_{\hat{F}_a}(z) : z \in x + y\} \geq \min\{t_{\hat{F}_a}(x), t_{\hat{F}_a}(y)\} \geq t,$$
$$\inf\{1 - f_{\hat{F}_a}(z) : z \in x + y\} \geq \min\{1 - f_{\hat{F}_a}(x), 1 - f_{\hat{F}_a}(y)\} \geq t.$$

That is, $\inf\{\hat{F}_a(z) : z \in x + y\} \geq t$.

Hence, for all $z \in x + y$, we have $z \in (\hat{F}_a)_{(t,t)}$ and $x + y \subseteq (\hat{F}_a)_{(t,t)}$. Also, for all $x \in (\hat{F}_a)_{(t,t)}$, we have $t_{\hat{F}_a}(-x) \geq t_{\hat{F}_a}(x) \geq t$ and $1 - f_{\hat{F}_a}(-x) \geq 1 - f_{\hat{F}_a}(x) \geq t$. So, $-x \in (\hat{F}_a)_{(t,t)}$. For all $r \in R$ and $x \in (\hat{F}_a)_{(t,t)}$, we can write $t_{\hat{F}_a}(rx) \geq t_{\hat{F}_a}(x) \geq t$ and $1 - f_{\hat{F}_a}(rx) \geq 1 - f_{\hat{F}_a}(x) \geq t$. Thus $rx \in (\hat{F}_a)_{(t,t)}$. We obtain that $(\hat{F}_a)_{(t,t)}$ is a sub-hypermodule of M, for all $a \in \mathrm{Supp}(\hat{F}, A)$. Consequently, $(\hat{F}, A)_{(t,t)}$ is a soft hypermodule over M.

Conversely, assume that for $t \in [0,1]$, $(\hat{F}, A)_{(t,t)}$ is a soft hypermodule over M. Thus $\hat{F}_a(t,t)$ is a non-null sub-hypermodule of M. For every $x, y \in M$, we have $\min\{\hat{F}_a(x), \hat{F}_a(y)\} \leq \hat{F}_a(x)$ or $\min\{\hat{F}_a(x), \hat{F}_a(y)\} \leq \hat{F}_a(y)$. If we put $\min\{\hat{F}_a(x), \hat{F}(y)\} = t$, then $x, y \in (\hat{F}_a)_{(t,t)}$ and thus $x + y \subseteq (\hat{F}_a)_{(t,t)}$. Therefore, for every $z \in x + y$, $\hat{F}_a(z) \geq t$. It means

$$\min\{\hat{F}_a(x), \hat{F}_a(y)\} \leq \inf\{\hat{F}_a(z) : z \in x + y\}.$$

So, condition (i) of Definition 3.2 is proved.

Next, for $x \in M$, let $\hat{F}_a(x) = t_1$. Then $x \in (F_a)_{(t_1,t_1)}$ and since $(F_a)_{(t_1,t_1)}$ is a sub-hypermodule $-x \in (F_a)_{(t_1,t_1)}$. Therefore $F_a(-x) \geq t_1 = F_a(x)$. This proves condition (ii) of Definition 3.2. Finally, for condition (iii) of Definition 3.2, we let $\hat{F}_a(x) = t_2$, for every $x \in M$. Then $x \in (\hat{F}_a)_{(t_2,t_2)}$ and for every $r \in R$, we have $rx \in (\hat{F}_a)_{(t_2,t_2)}$, since $(\hat{F}_a)_{(t_2,t_2)}$ is a sub-hypermodule. It means that $(\hat{F}_a)(rx) \leq t_2 = \hat{F}_a(x)$. This completes the proof.

Corollary 3.7. *There is a one-to-one correspondence between vague soft hypermodules and soft hypermodules.*

Theorem 3.8. *Let $(\hat{F}, A)$ be a vague soft hypermodule over a hypermodule M. Then, for each $\alpha, \beta \in [0,1]$, $(\hat{F}, A)_{(\alpha,\beta)}$ is a sub-hypermodule of M.*

PROOF. Let $(\hat{F}, A)$ is a vague soft hypermodule over M. Then, for each $a \in \mathrm{Supp}(\hat{F}, A)$, $\hat{F}_a$ is a vague sub-hypermodule of M. Let $x, y \in (\hat{F}, A)_{(\alpha,\beta)}$. Then $t_{\hat{F}_a}(x) \geq \alpha$, $1 - f_{\hat{F}_a}(x) \geq \beta$

and $t_{\hat{F}_a}(y) \geq \alpha$, $1 - f_{\hat{F}_a}(y) \geq \beta$. Thus,

$$\min\{t_{\hat{F}_a}(x), t_{\hat{F}_a}(y)\} \geq \min\{\alpha, \alpha\} = \alpha,$$
$$\min\{1 - f_{\hat{F}_a}(x), 1 - f_{\hat{F}_a}(y)\} \geq \min\{\beta, \beta\} = \beta.$$

Since $\hat{F}_a$ is a vague sub-hypermodule of M, we have $\inf\{t_{\hat{F}_a}(z) : z \in x + y\} \geq \alpha$ and $\inf\{1 - f_{\hat{F}_a}(z) : z \in x + y\} \geq \beta$. Therefore, for every $z \in x + y$, we obtain $z \in (\hat{F}, A)_{(\alpha,\beta)}$. It means that $x + y \subseteq (\hat{F}, A)_{(\alpha,\beta)}$.

Now, let $r \in R$ and $x \in M$. If $x \in (\hat{F}, A)_{(\alpha,\beta)}$ then for all $a \in A$, we have

$$t_{\hat{F}_a}(-x) \geq t_{\hat{F}_a}(x) \geq \alpha,$$
$$1 - f_{\hat{F}_a}(-x) \geq 1 - f_{\hat{F}_a}(x) \geq \beta.$$

Thus $-x \in (\hat{F}, A)_{(\alpha,\beta)}$. Also, since for all $a \in A$,

$$t_{\hat{F}_a}(rx) \geq t_{\hat{F}_a}(x) \geq \alpha,$$
$$1 - f_{\hat{F}_a}(rx) \geq 1 - f_{\hat{F}_a}(x) \geq \beta,$$

we obtain $rx \in (\hat{F}, A)_{(\alpha,\beta)}$. Therefore $(\hat{F}, A)_{(\alpha,\beta)}$ is a sub-hypermodule of M.

Corollary 3.9. *Let $(\hat{F}, A)$ be a vague soft hypermodule over a hypermodule M. Then for each $\alpha \in [0, 1]$, $(\hat{F}, A)_{(\alpha,\alpha)}$ is a sub-hypermodule of M.*

Corollary 3.10. *Let $(\hat{F}, A)$ be a vague soft hypermodule over a hypermodule M and $0 \leq t_1 < t_2 \leq 1$. Then $(\hat{F}_a)_{(t_1,t_1)} = (\hat{F}_a)_{(t_2,t_2)}$ if and only if there does not exist $x \in M$ such that $t_1 \leq \hat{F}_a(x) \leq t_2$.*

Theorem 3.11. *Let $(\hat{F}, A)$ and $(\hat{G}, B)$ be two vague soft hypermodules over a hypermodule M. Then $(\hat{F}, A)\tilde{\cap}(\hat{G}, B)$ is a vague soft hypermodule over M if it is non-null.*

Proof. Since $(\hat{F}, A)$ and $(\hat{G}, B)$ are both non-null and $(\hat{H}, A \times B)$ represents the basic intersection between them, we conclude that $(\hat{H}, A \times B)$ is non-null. We prove that all the conditions of Definition 3.2 are satisfied.

(i) For every $x, y \in M$ and $(\alpha, \beta) \in \text{Supp}(\hat{H}, A \times B)$, we have

$$\begin{aligned}
\min\{t_{\hat{H}_{(\alpha,\beta)}}(x), t_{\hat{H}_{(\alpha,\beta)}}(y)\} &= \min\{t_{\hat{F}_\alpha}(x) \cap t_{\hat{G}_\beta}(x), t_{\hat{F}_\alpha}(y) \cap t_{\hat{G}_\beta}(y)\} \\
&\leq \min\{t_{\hat{F}_\alpha}(x) \cap t_{\hat{F}_\alpha}(y), t_{\hat{G}_\beta}(x) \cap t_{\hat{G}_\beta}(y)\} \\
&\leq \min\{t_{\hat{F}_\alpha}(x), t_{\hat{F}_\alpha}(y)\} \cap \min\{t_{\hat{G}_\beta}(x), t_{\hat{G}_\beta}(y)\} \\
&\leq \inf\{t_{\hat{F}_\alpha}(z) : z \in x + y\} \cap \inf\{t_{\hat{G}_\beta}(z) : z \in x + y\} \\
&\leq \inf\{t_{\hat{F}_\alpha}(z) \cap t_{\hat{G}_\beta}(z) : z \in x + y\} \\
&= \inf\{t_{\hat{H}_{(\alpha,\beta)}}(z) : z \in x + y\}.
\end{aligned}$$

(ii) For every $x \in M$, we have

$$\begin{aligned} t_{\hat{H}_{(\alpha,\beta)}}(x) &= \min\{t_{\hat{F}_\alpha}(x), t_{\hat{G}_\beta}(x)\} \\ &\le \min\{t_{\hat{F}_\alpha}(-x), t_{\hat{G}_\beta}(-x)\} \\ &= t_{\hat{H}_{(\alpha,\beta)}}(-x). \end{aligned}$$

(iii) For every $r \in R$ and $x \in M$, we have

$$\begin{aligned} t_{\hat{H}_{(\alpha,\beta)}}(x) &= \min\{t_{\hat{F}_\alpha}(x), t_{\hat{G}_\beta}(x)\} \\ &\le \min\{t_{\hat{F}_\alpha}(rx), t_{\hat{G}_\beta}(rx)\} \\ &= t_{\hat{H}_{(\alpha,\beta)}}(rx). \end{aligned}$$

The proof for the $1 - f_{\hat{H}_{(\alpha,\beta)}}$ can be derived in the same manner. Therefore, $(\hat{H}, A \times B) = (\hat{F}, A)\tilde{\cap}(\hat{G}, B)$ is a vague soft hypermodule over M.

Theorem 3.12. *Let $(\hat{F}, A)$ and $(\hat{G}, B)$ be two vague soft hypermodules over a hypermodule M. Then $(\hat{F}, A)\tilde{\cup}(\hat{G}, B), (\hat{F}, A)\wedge(\hat{G}, B)$ and $(\hat{F}, A)\vee(\hat{G}, B)$ are vague soft hypermodules over M if they are non-null.*

Proof. The proof is similar to the previous theorem.

4. Vague soft hypermodule homomorphisms

In this section, we introduce the notion of vague soft hypermodule homomorphisms and study homomorphic image and homomorphic pre-image of vague soft hypermodules.

Definition 4.1. Let $(M, +, \cdot)$ and $(M', +', \cdot')$ be two hypermodules over a hyperring R. Let $(\hat{F}, A)$ and $(\hat{G}, B)$ be two vague soft hypermodules over M and M', respectively, and let $(\varphi, \psi) : (\hat{F}, A) \to (\hat{G}, B)$ be a vague soft function. Then $(\varphi, \psi) : (\hat{F}, A) \to (\hat{G}, B)$ is called a vague soft hypermodule homomorphism if the following conditions are satisfied:

(i) φ is a hypermodule homomorphism from M_1 to M_2,

(ii) $\varphi\big(\hat{F}(x)\big) = \hat{G}\big(\psi(x)\big)$, for all $x \in \text{Supp}(\hat{F}, A)$.

Theorem 4.2. *Let M and M' be two hypermodules over a hyperring R. Also, let $(\varphi, \psi) : (\hat{F}, A) \to (\hat{G}, B)$ be a vague soft hypermodule homomorphism and ψ be an injective mapping. Then $\big(\varphi(F), \psi(A)\big)$ is a vague soft hypermodule over M_2.*

Proof. Let $k \in \psi(A)$ and $y_1, y_2 \in M'$. If $\varphi^{-1}(y_1) = \emptyset$ or $\varphi^{-1}(y_2) = \emptyset$, the proof is routine. Let there exist $x_1, x_2 \in M$ such that $\varphi(x_1) = y_1$ and $\varphi(x_2) = y_2$. For all $x \in M$, we have $\varphi(x) = z \in y_1 + y_2$ and

$$\begin{aligned}
\inf_{z\in y_1+y_2} t_{\varphi(F)_k}(z) &= \inf_{z\in y_1+y_2}\big(\bigvee_{\varphi(x)=z}\bigvee_{\psi(a)=k}(t_{F_a})(x)\big)\\
&\geq \bigvee_{\varphi(x)=z}\bigvee_{\psi(a)=k}\inf_{x\in x_1+x_2}(t_{F_a})(x)\\
&\geq \bigvee_{\psi(a)=k}\inf_{x\in x_1+x_2}(t_{F_a})(x)\\
&\geq \bigvee_{\psi(a)=k}\min\{t_{F_a}(x_1), t_{F_a}(x_2)\}\\
&= \min\{\bigvee_{\psi(a)=k} t_{F_a}(x_1), \bigvee_{\psi(a)=k} t_{F_a}(x_2)\}.
\end{aligned}$$

This inequality holds for each $x_1, x_2 \in M$, which $\varphi(x_1) = y_1$ and $\varphi(x_2) = y_2$. Set $\alpha = \bigvee_{\psi(a)=k} t_{F_a}(x_1)$ and $\beta = \bigvee_{\psi(a)=k} t_{F_a}(x_2)$. Since " min " is a continuous T-norm, we have

$$\begin{aligned}
\inf_{z\in y_1+y_2} t_{\varphi(F)_k}(z) &\geq \bigvee_{\varphi(x_2)=y_2}\big(\bigvee_{\varphi(x_1)=y_1}\min\{\alpha,\beta\}\big)\\
&= \min\big\{\bigvee_{\varphi(x_1)=y_1}\alpha, \bigvee_{\varphi(x_2)=y_2}\beta\big\}\\
&= \min\{t_{\varphi(F)_k}(y_1), t_{\varphi(F)_k}(y_2)\}.
\end{aligned}$$

The proof for $1 - f_{\varphi(F)_k}$ can be derived in the same way. So, condition (i) of Definition 3.2 has been verified. Moreover, for all $y \in M'$, where $\varphi(x) = y$ and $x \in M$, we have

$$\begin{aligned}
t_{\varphi(F)_k}(-y) &= \bigvee_{\varphi(-x)=-y}\bigvee_{\psi(a)=k} t_{F_a}(-x)\\
&\geq \bigvee_{\varphi(x)=y}\bigvee_{\psi(a)=k} t_{F_a}(x)\\
&= t_{\varphi(F)_k}(y).
\end{aligned}$$

Similarly, $1 - f_{\varphi(F)_k}(-y) \geq 1 - f_{\varphi(F)_k}(y)$. Also,

$$t_{\varphi(F)_k}(ry) \geq t_{\varphi(F)_k}(y),$$

$$1 - f_{\varphi(F)_k}(ry) \geq 1 - f_{\varphi(F)_k}(y),$$

for all $r \in R$ and $y \in M'$. Hence, conditions (ii) and (iii) have been verified and this completes the proof.

Theorem 4.3. *Let M and M' be two hypermodules over a hyperring R. Let $(\hat{F}, A)$ and $(\hat{G}, B)$ be two vague soft hypermodules over M and M', respectively and $(\varphi, \psi) : (\hat{F}, A) \to (\hat{G}, B)$ be a vague soft hypermodule homomorphism. Then $\big(\varphi^{-1}(\hat{G}), \psi^{-1}(B)\big)$ is a vague soft hypermodule over M.*

Proof. The proof is similar to the previous theorem.

5. Conclusions

In this paper, we introduced the notion of vague soft hypermodules as an extension of vague soft hypergroups and vague soft hyperrings. Also, some basic properties of vague soft hypermodules and homomorphisms between vague soft hypermodules are presented and discussed. This study can be extended by considering vague soft quotient hypermodules or roughness of vague soft hypermodules.

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