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Generalized $\eta-$Ricci solitons on f-Kenmotsu manifolds admitting a quarter symmetric metric connection

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Abstract. In this paper, we study $\eta-$Ricci solitons on three-dimensional $f-$Kenmotsu manifolds with respect to a quarter symmetric metric connection. We obtain some results when the potential vector field is pointwise collinear with the Reeb vector field, conformal Killing vector field and a torqued vector field.

Keywords: Generalized $\eta-$Ricci soliton, $f-$Kenmotsu manifold, quarter symmetric metric connection.

1. Introduction

The concept of semi-symmetric metric connections on a differentiable manifold was introduced by Friedman and Schouten in 1924 [6]. As generalizations of these connections, the quarter symmetric metric connections were introduced by Golab in 1975 [7]. An affine connection $\tilde{\nabla}$ in a Riemannian manifold M is called a quarter symmetric metric connection if the torsion tensor T

$$T(U,V)=\tilde{\nabla}_U V-\tilde{\nabla}_V U-[U,V]$$

fulfills

$$T(U,V)=\eta(V)\phi U-\eta(U)\phi V,$$

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where U,V are vector fields, η is a 1-form and ϕ is a $(1,1)-$tensor field on M. When $\phi U = U$ the quarter symmetric connection becomes a semi-symmetric connection. If the connection $\tilde{\nabla}$ fulfills

$$(\tilde{\nabla}_U g)(V,W) = 0,$$

for all vector fields U,V,W on M, then the connection $\tilde{\nabla}$ is called quarter symmetric metric connection; contrarily, it is a non-metric connection.

Quarter symmetric metric connections have been studied extensively by many researchers, see [10],[11],[12],[19].

The notion of $f-$Kenmotsu manifolds was introduced by Jannsens and Vanhecke in 1981 [9] by considering the f is a real constant. Afterwards, in 1991, Olszak and Rosca defined the $f-$Kenmotsu manifolds by assuming the f as a function [14]. Here, they studied geometry of normal locally conformal almost cosymplectic manifolds.

On the other hand, let (M,g) be a Riemannian manifold of dimension $n, (n \geq 2)$ such that $\{g(t)\}$ is the 1-parameter family of metrics and $S(t)$ is its Ricci tensor. In this case, the equation of Ricci flow is defined by [8]

$$\frac{\partial g(t)}{\partial t} = -2S(t)g(t).$$

The special solutions of the Ricci flow are famous as Ricci solitons. A Ricci soliton is a triplet (g,X,ζ) on a Riemannian manifold satisfying

$$L_X g + 2S + 2\zeta g = 0,$$

where L_X is the Lie derivative in the direction of the potential vector field X, S is the Ricci tensor and ζ is a real constant [1]. The generalized Ricci soliton is defined by

$$L_X g + 2\nu X^b \otimes X^b - 2\alpha S - 2\zeta g = 0,$$

where X^b is the canonical 1-form associated to X [13]. The concept of $\eta-$Ricci soliton was defined by Cho and Kimura [5] as

$$L_X g + 2S + 2\zeta g + 2\sigma\eta \otimes \eta = 0.$$

The $\eta-$Ricci solitons are generalizations of Ricci solitons. Subsequently, M. D. Siddiqi defined the generalized $\eta-$Ricci soliton as [18]

$$L_X g + 2\nu X^b \otimes X^b + 2S + 2\zeta g + 2\sigma\eta \otimes \eta = 0.$$

In the present paper, we give some characterizations about generalized $\eta-$Ricci solitons on $f-$Kenmotsu manifolds admitting quarter symmetric metric connections. Throughout the paper, all geometric objects (curves, manifolds, vector fields, functions etc.) are assumed to be smooth.

2. **Preliminaries**

2.1. ***f*-Kenmotsu Manifolds.** Consider a 3-dimensional manifold M. If the (1,1)-tensor field φ, the vector field ξ, the 1-form η and the Riemannian metric g satisfy the following relations, we say that the quartet (φ, ξ, η, g) is a contact metric structure on M and the quintet $(M, \varphi, \xi, \eta, g)$ is a contact metric manifold:

$$\begin{aligned} &\eta \circ \varphi = 0, \\ &\varphi\xi = 0, \\ &\eta(\xi) = 1, \\ &g(U, \xi) = \eta(U), \\ &g(U, \varphi V) = -g(\varphi U, V), \\ &g(\varphi U, \varphi V) = g(U, V) - \eta(U)\eta(V), \\ &\varphi^2 U = -U + \eta(U)\xi, \end{aligned} \tag{2.1}$$

where U, V are vector fields on M. The contact metric manifold M is called $f-$Kenmotsu if it fulfills the following relation

$$(\nabla_U \varphi)(V) = f[g(\varphi U, V)\xi - \eta(V)\varphi(U)], \tag{2.2}$$

where f is a function. This gives us

$$\nabla_U \xi = f[U - \eta(U)\xi], \tag{2.3}$$

and

$$(\nabla_U \eta)(V) = f[g(U, V) - \eta(U)\eta(V)]. \tag{2.4}$$

Using (2.3) and (2.4), we obtain

$$\begin{aligned} R(U, V)\xi &= -(f^2 + \xi(f))[\eta(V)U - \eta(U)V], \\ R(U, \xi)V &= (f^2 + \xi(f))[g(U, V)\xi - \eta(V)U], \\ R(\xi, U)\xi &= -(f^2 + \xi(f))[\eta(U)\xi - U], \end{aligned}$$

for every vector fields U, V on M. Here, R denotes the Riemannian curvature tensor of M. The Ricci tensor of the $f-$Kenmotsu manifold M is expressed as

$$S(U, V) = \Big(\xi(f) + \frac{r}{2} + f^2\Big)g(U, V) - (3\xi(f) + \frac{r}{2} + 3f^2)\eta(U)\eta(V), \tag{2.5}$$

for every vector fields U, V on M. Here, r denotes the scalar curvature of M. From (2.5), we get

$$S(U, \xi) = -2\Big(f^2 + \xi(f)\Big)\eta(U), \tag{2.6}$$

for every vector fields U on M.

2.2. **A quarter symmetric metric connection on a $f-$Kenmotsu manifold.** Let $\tilde{\nabla}$ be an affine connection and ∇ be the Levi-Civita connection of $f-$Kenmotsu manifold M. The connection $\tilde{\nabla}$ is said to be a quarter symmetric metric connection on M if

$$\tilde{\nabla}_U V = \nabla_U V - \eta(U)\varphi V, \tag{2.7}$$

for every vector fields U, V on M. From (2.1), (2.2) and (2.7), we get

$$(\tilde{\nabla}_U \varphi)V = f\Big[g(\varphi U, V)\xi - \eta(V)\varphi U\Big]. \tag{2.8}$$

From (2.3) and (2.7), we have

$$\tilde{\nabla}_U \xi = f[U - \eta(U)\xi]. \tag{2.9}$$

From (2.4) and (2.7), we occur

$$(\tilde{\nabla}_U \eta)V = fg(\varphi U, \varphi V). \tag{2.10}$$

The curvature tensor $\tilde{R}$, the Ricci tensor $\tilde{S}$, the scalar curvature $\tilde{r}$ and the Ricci operator $\widetilde{Q}$ of the connection $\tilde{\nabla}$ in (2.7) are given by respectively:

$$\begin{aligned} \tilde{R}(U,V)W &= R(U,V)W + f(\eta(V)\varphi(U) - \eta(U)\varphi(V))\eta(W) \\ &\quad + f(\eta(U)g(\phi V, W) - \eta(V)g(\phi U, W))\xi, \end{aligned}$$

$$\begin{aligned} \tilde{S}(U,V) &= S(U,V) + fg(\varphi U, V) \\ &= (\xi(f) + \frac{r}{2} + f^2)g(U,V) \\ &\quad -(3\xi(f) + \frac{r}{2} + 3f^2)\eta(U)\eta(V) \\ &\quad + fg(\varphi U, V), \end{aligned} \tag{2.11}$$

$$\widetilde{Q}U = (\xi(f) + \frac{r}{2} + f^2)U - (3\xi(f) + \frac{r}{2} + 3f^2)\eta(U)\xi + fg\varphi U,$$

$$\tilde{r} = r \tag{2.12}$$

see [2],[15]. We also have

$$\tilde{R}(U,V)\xi = -(f^2 + \xi(f))(\eta(V)U - \eta(U)V) + f(\eta(V)\varphi U - \eta(U)\varphi V),$$

$$\tilde{R}(\xi,V)\xi = -(f^2 + \xi(f))(\eta(V)\xi - V) - f\varphi V,$$

$$\widetilde{S}(V,\xi) = -2(f^2 + \xi(f))\eta(V).$$

For more details, see [17].

3. **Main Results**

The generalized $\eta-$Ricci soliton with respect to the quarter symmetric metric connection is defined by

$$\alpha\widetilde{S}+\frac{\beta}{2}\tilde{L}_X g+\nu X^b\otimes X^b+\sigma\eta\otimes\eta+\zeta g=0, \tag{3.1}$$

where $\widetilde{S}$ is the Ricci tensor of the connection $\tilde{\nabla}$, X^b is the canonical 1-form associated to X, i.e., $X^b(U)=g(U,X)$ for every vector fields U, ζ is a function and α,β,ν,σ are real constants satisfying $(\alpha,\beta,\nu)\neq(0,0,0)$. The particular cases of the generalized $\eta-$Ricci soliton are listed below:

(a) If $\alpha=1,\ \nu=\sigma=0$, we obtain the Ricci soliton.
(b) If $\alpha=1,\ \nu=0$, we obtain the $\eta-$Ricci soliton.
(c) If $\sigma=0$, we obtain the generalized Ricci soliton.

On the other hand, an $f-$Kenmotsu manifold is called $\eta-$Einstein if

$$S=f_1 g+f_2\eta\otimes\eta,$$

where f_1,f_2 are functions on M. Now, assume that M is an $f-$Kenmotsu manifold satisfying the generalized $\eta-$Ricci soliton with respect to the quarter symmetric metric connection (3.1). Consider the potential vector field $X=\theta\xi$, in other words, let X be a pointwise collinear with the Reeb vector field ξ. Using (2.9), we get

$$(\tilde{L}_{\theta\xi}g)(U,V)=(U\theta)\eta(V)+(V\theta)\eta(U)+2f\theta\Big\{g(U,V)-\eta(U)\eta(V)\Big\}, \tag{3.2}$$

for every vector fields U,V on M. It is clear that

$$\xi^b\otimes\xi^b(U,V)=\eta(U)\eta(V). \tag{3.3}$$

Putting $X=\theta\xi$ and the relations (2.11), (3.2), (3.3) in (3.1), we deduce

$$\begin{aligned}\alpha\Big[S(U,V)+fg(U,\varphi V)\Big]+\tfrac{\beta}{2}\Big\{(U\theta)\eta(V)+(V\theta)\eta(U)\Big\}\\ +\beta f\theta\Big\{g(U,V)-\eta(U)\eta(V)\Big\}+\big(\nu\theta^2+\sigma\big)\eta(U)\eta(V)+\zeta g(U,V)=0.\end{aligned} \tag{3.4}$$

Taking $V=\xi$ in (3.4) and using (2.6) we obtain

$$\alpha\Big[-2(f^2+\xi(f))\eta(U)\Big]+\frac{\beta}{2}U(\theta)+\frac{\beta}{2}\xi(\theta)\eta(U)+(\nu\theta^2+\sigma+\zeta)\eta(U)=0. \tag{3.5}$$

Taking $U=\xi$ in (3.5) we get

$$\beta\xi(\theta)=2\alpha(f^2+\xi(f))-(\nu\theta^2+\sigma+\zeta). \tag{3.6}$$

Substituting (3.6) in (3.5) we have

$$\beta U(\theta)=[2\alpha(f^2+\xi(f))-(\nu\theta^2+\sigma+\zeta)]\eta(U),$$

which leads to

$$\beta d\theta=\Big[2\alpha(f^2+\xi(f))-(\nu\theta^2+\sigma+\zeta)\Big]\eta. \tag{3.7}$$

Putting (3.7) in (3.4) we get

$$\alpha\widetilde{S}(U,V)=\Big(\zeta+\beta f\theta\Big)\Big[-g(U,V)+\eta(U)\eta(V)\Big]. \tag{3.8}$$

Equation (3.8) gives us

$$\alpha\widetilde{r}=-2\zeta-2\beta f\theta.$$

Now, we can express the following theorem and corollary.

Theorem 3.1. *Let (M,g,φ,ξ,η) be an $f-$Kenmotsu manifold admitting the quarter symmetric metric connection defined by (2.7). If M is a generalized $\eta-$Ricci soliton with the septet $(g,X,\alpha,\beta,\nu,\sigma,\zeta)$ such that $\alpha\neq 0$ and $X=\theta\xi$ for a function θ on M, then M is an $\eta-$Einstein soliton and an $\eta-$Einstein manifold with respect to the quarter symmetric metric connection.*

Corollary 3.2. *Let (M,g,φ,ξ,η) be an $f-$Kenmotsu manifold admitting the quarter symmetric metric connection defined by (2.7). If M is a generalized $\eta-$Ricci soliton with the septet $(g,X,\alpha,\beta,\nu,\sigma,\zeta)$ such that $\alpha\neq 0$ and $X=\theta\xi$ for a function θ on M, then $\alpha\widetilde{r}=-2\zeta-2\beta f\theta$.*

Now, we recall the definition of the conformal Killing and torse-forming vector fields and give some results about them.

Definition 3.3. *A vector field X is called a conformal Killing vector field if*

$$(L_X g)(U,V)=2hg(U,V),$$

for every vector fields U,V, where h is a function. The particular cases of a conformal Killing vector field are listed below:
(i) If $h=0$, we obtain Killing vector fields.
(ii) If h is a constant, we obtain homothetic vector fields.
(iii) If h is not a constant, we obtain proper vector fields.

Suppose that X is called a conformal Killing vector field with respect to the quarter symmetric metric connection $\tilde{\nabla}$, i.e.,

$$(\widetilde{L}_X g)(U,V)=2hg(U,V).$$

By (3.1), we have

$$\alpha\widetilde{S}(U,V)+\beta hg(U,V)+\nu X^b(U)X^b(V)+\sigma\eta(U)\eta(V)+\zeta g(U,V)=0. \tag{3.9}$$

Taking $V=\xi$ in (3.9), we get

$$g\Big(-2(f^2+\xi(f)\Big)\xi+\beta h\xi+\nu\eta(X)X+\sigma\xi+\zeta\xi,U)=0.$$

So, we have

Theorem 3.4. *Let $(M, g, \varphi, \xi, \eta)$ be an $f-$Kenmotsu manifold admitting the quarter symmetric metric connection defined by (2.7). If M is a generalized $\eta-$Ricci soliton with the septet $(g, X, \alpha, \beta, \nu, \sigma, \zeta)$ such that X is a conformal Killing vector field, then*

$$\Big[-2(f^2 + \xi(f)) + \beta h + \sigma + \zeta\Big]\xi + \nu\eta(X)X = 0.$$

Definition 3.5. *A non-zero vector field X is called a torse-forming vector field on a Riemannian manifold (M, g) [20] if*

$$\nabla_U X = fU + \omega(U)X, \tag{3.10}$$

for every vector field U, where ∇ is the Levi-Civita connection of g, f is a function and ω is a 1-form. The particular cases of a torse-forming vector field are listed below:

(i) If $\omega(U) = 0$ in (3.10), we obtain torqued vector fields [3].
(ii) If $f = \omega = 0$, we obtain parallel vector fields.
(iii) If $\omega = 0$ and $f = 1$, we obtain concurrent vector fields [16].
(iv) If $\omega = 0$, we obtain concircular vector fields [4].

Assume that $(g, X, \alpha, \beta, \nu, \sigma, \zeta)$ is a generalized $\eta-$Ricci soliton on an $f-$Kenmotsu manifold M such that X is a a torse-forming vector field. Then we have

$$\alpha\widetilde{S}(U,V)+\frac{\beta}{2}(\tilde{L}_X g)(U,V)+\nu X^b(U)X^b(V)+\sigma\eta(U)\eta(V)+\zeta g(U,V) = 0. \tag{3.11}$$

Since

$$(\tilde{L}_X g)(U,V) = 2fg(U,V) + \omega(U)g(X,V) + \omega(V)g(X,U),$$

we rewrite (3.11) as

$$\alpha\widetilde{S}(U,V) + [\beta f + \zeta]g(U,V) + \sigma\eta(U)\eta(V) + \frac{\beta}{2}\Big[\omega(U)g(X,V) + \omega(V)g(X,U)\Big] + \nu g(X,U)g(X,V) = 0.$$

Taking contraction in the above equation we get

$$\alpha\widetilde{r} + 3[\beta f + \zeta] + \sigma + \beta\omega(X) + \nu\,|X|^2 = 0.$$

Using (2.12) we can express the final theorem of the paper.

Theorem 3.6. *Let $(M, g, \varphi, \xi, \eta)$ be an $f-$Kenmotsu manifold admitting the quarter symmetric metric connection defined by (2.7). If M is a generalized $\eta-$Ricci soliton with the septet $(g, X, \alpha, \beta, \nu, \sigma, \zeta)$ such that X is a torse-forming vector field, then*

$$\zeta = -\frac{1}{3}\Big[\alpha r + \sigma + \beta\omega(X) + \nu\,|X|^2\Big] - \beta f.$$

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