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Research Paper

A NOTE ON QUASI-HEMI SLANT SUBMANIFOLDS OF A NEARLY TRANS-SASAKIAN MANIFOLDS

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Abstract. Here our main objective is to introduce the notion of quasi hemi-slant submanifolds as a generalized case of slant submanifolds, semi-slant submanifolds and hemi-slant submanifolds of contact metric manifolds. We mainly focus on quasi hemi-slant submanifold of nearly trans-Sasakian manifold. During this manner, we tend to study and investigate integrability of distributions which are concerned in the definition of quasi hemi-slant submanifold of nearly trans-Sasakian manifold. Moreover, we tend to get necessary and sufficient conditions for quasi hemi-slant submanifold of nearly trans-Sasakian manifold to be totally geodesic for such manifolds.

Key Words: Quasi-hemi slant submanifolds, nearly trans-Sasakian manifolds, totally umbilical proper quasi-hemi slant submanifolds.
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1. Introduction

The concept of slant submanifolds of almost Hermitian manifolds has been studied by B.Y. Chen [6], and also studied on natural generalization of holomorphic immersions and totally real immersions and many more [6], [5]. A. Lotta [2] introduced and studied slant immersions of a Riemannian manifold into almost contact metric manifold. The Lorentzian para-Sasakian manifolds were defined K. Matsumoto [9]. I. Mihai and R. Rosca [7] are also studied.

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K. Matsumoto and I. Mihai [10] has defined and studied Lorentzian para-Sasakian manifolds. Later, many articles have appeared exploring the generalization of semi-slant submanifold, pseudo-slant submanifold, bi-slant submanifold and hemi-slant submanifold etc., in known differentiable manifolds [14], [15], [16].

2. Preliminaries

If $\bar{\mathcal{M}}$ is an $(2n+1)$- dimensional almost contact manifold, endowed with structure $(\phi, \xi, \upsilon, <,>)$, then we obtain

$$\phi^2 = \upsilon \otimes \xi - I, \quad 1 = \upsilon(\xi) \tag{2.1}$$

$$\phi\xi = 0, \quad 0 = \upsilon(\phi) \quad and \quad rank(\phi) = 2n$$

$$< \phi\mathcal{X}, \phi\mathcal{Y} > = < \mathcal{X}, \mathcal{Y} > - \upsilon(\mathcal{X})\upsilon(\mathcal{Y}), \tag{2.2}$$

$$\upsilon(\mathcal{X}) = < \mathcal{X}, \xi > \quad and \quad - < \mathcal{X}, \phi\mathcal{Y} > = < \phi\mathcal{X}, \mathcal{Y} > \tag{2.3}$$

where $\mathcal{X}$ and $\mathcal{Y}$ are vector fields on $\mathcal{M}$ and if the almost complex structure $\mathcal{J}$ on the product manifold $\bar{\mathcal{M}} \times \mathcal{R}$ satisfies

$$\mathcal{J}(\mathcal{X}, f d/dt) = (\phi\mathcal{X} - f\xi, \upsilon(\mathcal{X}) d/dt), \tag{2.4}$$

then the almost contact structure (ϕ, ξ, υ) has said to be normal. For trans-Sasakian manifold, the following conditions are equivalent

$$(\bar{\nabla}_{\mathcal{X}}\phi)\mathcal{Y} = -\mu\{\upsilon(\mathcal{Y})\mathcal{X} - < \mathcal{X}, \mathcal{Y} > \xi\} - \rho\{\upsilon(\mathcal{Y})\phi\mathcal{X} + < \mathcal{X}, \phi\mathcal{Y} >\}$$

$$\bar{\nabla}_{\mathcal{X}}\xi = -\mu\phi\mathcal{X} - \rho\phi^2\mathcal{X} \tag{2.5}$$

$$(\bar{\nabla}_{\mathcal{X}}\upsilon)\mathcal{Y} = -\mu < \phi\mathcal{X}, \mathcal{Y} > + \rho < \phi\mathcal{X}, \phi\mathcal{Y} > \quad and \quad \bar{\nabla}_{\xi}\phi = 0.$$

It is nearly trans-Sasakian manifolds if

$$\begin{aligned} (\bar{\nabla}_{\mathcal{Y}}\phi)\mathcal{X} + (\bar{\nabla}_{\mathcal{X}}\phi)\mathcal{Y} \ = \ & -\rho\{\upsilon(\mathcal{X})\phi\mathcal{Y} + \upsilon(\mathcal{Y})\phi\mathcal{X}\} \\ & -\mu\{\upsilon(\mathcal{Y})\mathcal{X} + \upsilon(\mathcal{X})\mathcal{Y} \\ & -2 < \mathcal{X}, \mathcal{Y} > \xi\}, \end{aligned} \tag{2.6}$$

for all vector fields $\mathcal{X}, \mathcal{Y}$ tangent to $\mathcal{M}$ where $\bar{\nabla}$ denotes Riemannian connection with respect to $<,>$.

Now, suppose $\mathcal{M}$ be a submanifold of a contact Lorentzian metric manifold $\bar{\mathcal{M}}$ with the induced metric $<,>$ and ξ be tangent to $\mathcal{M}$. Also suppose ∇ and $\nabla^{\perp}$ be the induced connections on the tangent bundle $T\mathcal{M}$ and the normal bundle $T^{\perp}\mathcal{M}$ of $\mathcal{M}$, respectively. Then the Gauss-Weingarten formulas are given by

$$\bar{\nabla}_{\mathcal{X}}\mathcal{Y} = \sigma(\mathcal{X}, \mathcal{Y}) + \nabla_{\mathcal{X}}\mathcal{Y} \tag{2.7}$$

$$\bar{\nabla}_{\mathcal{X}}\lambda = -\Lambda_{\lambda}\mathcal{X} + \nabla^{\perp}_{\mathcal{X}}\lambda, \tag{2.8}$$

for all vector fields $\mathcal{X}, \mathcal{Y}$ tangent to $\mathcal{M}$ and any vector filed λ normal to $\mathcal{M}$, where σ and Λ_{λ} are the second fundamental form and the shape operator for the immersion of $\mathcal{M}$ into $\bar{\mathcal{M}}$. The second fundamental form σ and shape operator Λ_{λ} are related by

$$< \sigma(\mathcal{X}, \mathcal{Y}), \lambda >=< \Lambda_{\lambda}\mathcal{X}, \mathcal{Y} >, \tag{2.9}$$

for all vector field $\mathcal{X}$ tangent to $\mathcal{M}$ and vector field λ normal to $\mathcal{M}$, we can write

$$\phi\mathcal{X} = T\mathcal{X} + N\mathcal{X} \tag{2.10}$$

$$\phi\lambda = t\lambda + s\lambda, \tag{2.11}$$

where $T\mathcal{X}$ and $t\lambda$ are the tangential components of $\phi\mathcal{X}$ and $\phi\lambda$, respectively, where as $N\mathcal{X}$ and $\mathcal{F}\lambda$ are the normal components of $\phi\mathcal{X}$ and $\phi\lambda$, respectively. Thus by using (2.10) and (2.11), we can obtain

$$(\nabla_{\mathcal{X}}T)\mathcal{Y} - T(\nabla_{\mathcal{X}}\mathcal{Y}) = (\bar{\nabla}_{\mathcal{X}}T)\mathcal{Y} \tag{2.12}$$

$$(\nabla^{\perp}_{\mathcal{X}}N)\mathcal{Y} - N(\nabla_{\mathcal{X}}\mathcal{Y}) = (\bar{\nabla}_{\mathcal{X}}N)\mathcal{Y}$$

$$(\nabla_{\mathcal{X}}t)\lambda - t(\nabla^{\perp}_{\mathcal{X}})\lambda = (\bar{\nabla}_{\mathcal{X}}t)\lambda, \tag{2.13}$$

$$(\nabla^{\perp}_{\mathcal{X}}s)\lambda - s(\nabla^{\perp}_{\mathcal{X}}\lambda) = (\bar{\nabla}_{\mathcal{X}}s)\lambda$$

for all vector fields $\mathcal{X}, \mathcal{Y}$ tangent to $\mathcal{M}$ and vector field λ normal to $\mathcal{M}$. The mean curvature vector σ of $\mathcal{M}$ is given by

$$\mathcal{H} = \frac{1}{m} trace(\sigma) = \frac{1}{m}\sum_{i=1}^{m}\sigma(\varepsilon_i, \varepsilon_i), \tag{2.14}$$

where m is the dimension of $\mathcal{M}$ and $\{\varepsilon_1, \varepsilon_2, ..., \varepsilon_m\}$ is a local orthonormal frame of $\mathcal{M}$. A submanifold $\mathcal{M}$ of an almost contact metric manifold $\bar{\mathcal{M}}$ is said to be totally umbilical if

$$< \mathcal{X}, \mathcal{Y} > \mathcal{H} = \sigma(\mathcal{X}, \mathcal{Y}), \tag{2.15}$$

where σ is the mean curvature vector. A submanifold $\mathcal{M}$ is said to be totally geodesic, if $\sigma(\mathcal{X}, \mathcal{Y}) = 0$. For all vector fields $\mathcal{X}, \mathcal{Y}$ tangent to $\mathcal{M}$ and $\mathcal{M}$ is said to be minimal if $\mathcal{H} = 0$.
Now, $\mathcal{P}_{\mathcal{X}}\mathcal{Y}$ and $\mathcal{F}_{\mathcal{X}}\mathcal{Y}$ are the tangential and normal parts of $(\bar{\nabla}_{\mathcal{X}}\phi)\mathcal{Y}$, then we decompose

$$\mathcal{F}_{\mathcal{X}}\mathcal{Y} + \mathcal{P}_{\mathcal{X}}\mathcal{Y} = (\bar{\nabla}_{\mathcal{X}}\phi)\mathcal{Y}, \tag{2.16}$$

for all vector fields $\mathcal{X}, \mathcal{Y}$ tangent to $\mathcal{M}$. Thus, we can obtain

$$\mathcal{P}_{\mathcal{X}}\mathcal{Y} = -t\sigma(\mathcal{X}, \mathcal{Y}) - \Lambda_{N\mathcal{Y}}\mathcal{X} + (\nabla_{\mathcal{X}}T)\mathcal{Y} \tag{2.17}$$

and

$$\mathcal{F}_{\mathcal{X}}\mathcal{Y} = \sigma(\mathcal{X}, T\mathcal{Y}) - \mathcal{F}\sigma(\mathcal{X}, \mathcal{Y}) + (\nabla_{\mathcal{X}}N)\mathcal{Y}. \tag{2.18}$$

Similarly, $\mathcal{P}_{\mathcal{X}}\lambda$ and $\mathcal{F}_{\mathcal{X}}\lambda$ are the tangential and normal parts of $(\bar{\nabla}_{\mathcal{X}}\phi)\lambda$, respectively, then we infer

$$\mathcal{P}_{\mathcal{X}}\lambda = (\nabla_{\mathcal{X}}t)\lambda - \Lambda_{\mathcal{F}\lambda}\mathcal{X} + T\Lambda_{\lambda}\mathcal{X} \tag{2.19}$$

and

$$\mathcal{F}_{\mathcal{X}}\lambda = (\nabla_{\mathcal{X}}\mathcal{F})\lambda + \sigma(t\lambda, \mathcal{X}) + N\Lambda_{\lambda}\mathcal{X}, \tag{2.20}$$

for the vector field λ normal to $\mathcal{M}$. By using (2.6), we deduce

$$\begin{aligned}(\bar{\nabla}_{\mathcal{Y}}\phi)\mathcal{X} &= -(\bar{\nabla}_{\mathcal{X}}\phi)\mathcal{Y} - \rho\{\upsilon(\mathcal{X})\phi\mathcal{Y} + \upsilon(\mathcal{Y})\phi\mathcal{X}\} \\ &\quad -\mu\{\upsilon(\mathcal{Y})\mathcal{X} + \upsilon(\mathcal{X})\mathcal{Y} - 2 < \mathcal{X}, \mathcal{Y} > \xi\}\end{aligned} \tag{2.21}$$

and

$$(\bar{\nabla}_{\mathcal{Y}}\phi)\mathcal{X} + (\bar{\nabla}_{\mathcal{X}}\phi)\mathcal{Y} = -\phi\bar{\nabla}_{\mathcal{Y}}\mathcal{X} + \bar{\nabla}_{\mathcal{Y}}\phi\mathcal{X} - \phi\bar{\nabla}_{\mathcal{X}}\mathcal{Y} + \bar{\nabla}_{\mathcal{X}}\phi\mathcal{Y}.$$

From (2.7), (2.8), (2.10) and (2.11), we get

$$\begin{aligned}(\bar{\nabla}_{\mathcal{Y}}\phi)\mathcal{X} &= -\phi(\sigma(\mathcal{X}, \mathcal{Y}) + \nabla_{\mathcal{X}}\mathcal{Y}) - \phi(\sigma(\mathcal{X}, \mathcal{Y}) + \nabla_{\mathcal{Y}}\mathcal{X}) \\ &\quad -(\bar{\nabla}_{\mathcal{X}}\phi)\mathcal{Y} + \bar{\nabla}_{\mathcal{X}}N\mathcal{Y} + \bar{\nabla}_{\mathcal{Y}}N\mathcal{X} + \bar{\nabla}_{\mathcal{X}}T\mathcal{Y} + \bar{\nabla}_{\mathcal{Y}}T\mathcal{X}\end{aligned}$$

$$\begin{aligned}\nabla_{\mathcal{X}}T\mathcal{Y} \;+\; & \sigma(\mathcal{X}, T\mathcal{Y}) - \Lambda_{N\mathcal{Y}}\mathcal{X} + \nabla^{\perp}_{\mathcal{X}}N\mathcal{Y} \\ & -T\nabla_{\mathcal{X}}\mathcal{Y} - N\nabla_{\mathcal{X}}\mathcal{Y} - 2t\sigma(\mathcal{X}, \mathcal{Y}) \\ & -2s\sigma(\mathcal{X}, \mathcal{Y}) + \nabla_{\mathcal{Y}}T\mathcal{X} + \sigma(\mathcal{Y}, T\mathcal{X}) - \Lambda_{N\mathcal{X}}\mathcal{Y} \\ & +\nabla^{\perp}_{\mathcal{Y}}N\mathcal{X} - T\nabla_{\mathcal{Y}}\mathcal{X} - N\nabla_{\mathcal{Y}}\mathcal{X} = 0.\end{aligned} \tag{2.22}$$

Then using (2.21) and (2.22), we deduce

$$\begin{aligned}(\nabla_{\mathcal{Y}}T)\mathcal{X} &= -(\nabla_{\mathcal{X}}T)\mathcal{Y} + \Lambda_{N\mathcal{Y}}\mathcal{X} + \Lambda_{N\mathcal{X}}\mathcal{Y} - 2t\sigma(\mathcal{X}, \mathcal{Y}) \\ &\quad -\mu\{\upsilon(\mathcal{X})\mathcal{Y} + \upsilon(\mathcal{Y})\mathcal{X} - 2 < \mathcal{X}, \mathcal{Y} > \xi\} \\ &\quad +\rho\{\upsilon(\mathcal{X})T\mathcal{Y} + \upsilon(\mathcal{Y})T\mathcal{X}\}\end{aligned} \tag{2.23}$$

$$\begin{aligned}(\nabla_{\mathcal{Y}}N)\mathcal{X} &= -(\nabla_{\mathcal{X}}N)\mathcal{Y} - \sigma(\mathcal{Y}, T\mathcal{X}) - \sigma(\mathcal{X}, T\mathcal{Y}) \\ &\quad +2f\sigma(\mathcal{X}, \mathcal{Y}) + \rho\{\upsilon(\mathcal{Y})N\mathcal{X} + \upsilon(\mathcal{X})N\mathcal{Y}\}.\end{aligned} \tag{2.24}$$

Take $\mathcal{Y} = \xi$ in (2.6) and by using (2.2), (2.7) and (2.8), we infer

$$T[\mathcal{X}, \xi] = \rho T\mathcal{X} - \mu\phi^2\mathcal{X} - 2t\sigma(\mathcal{X}, \xi) + (\nabla_{\xi}T)\mathcal{X} - T\nabla_{\xi}\mathcal{X} - \Lambda_{N\mathcal{X}}\xi \tag{2.25}$$

(2.26) $N[\mathcal{X},\xi] = (\nabla_\xi N)\mathcal{X} - N\nabla_\xi \mathcal{X} - 2f\sigma(\mathcal{X},\xi) + \sigma(T\mathcal{X},\xi) + \rho N\mathcal{X}$.

The submanifold $\mathcal{M}$ of an almost contact metric manifold $\bar{\mathcal{M}}$ is invariant for $\phi(T_\mathcal{X}\mathcal{M}) \subseteq T_\mathcal{X}\mathcal{M}$ for every point $\mathcal{X} \in \mathcal{M}$ and carrying a Riemannian manifold $\mathcal{M}$ isometrically absorbed in an almost contact metric manifold $\bar{\mathcal{M}}$.

The submanifold $\mathcal{M}$ of an almost contact metric manifold $\bar{\mathcal{M}}$ is anti-invariant for $\phi(T_\mathcal{X}\mathcal{M}) \subseteq T^\perp_\mathcal{X}\mathcal{M}$ for every point $\mathcal{X} \in \mathcal{M}$.

If ξ is tangential in $\mathcal{M}$ for a submanifold $\mathcal{M}$ of an almost contact metric manifold $\bar{\mathcal{M}}$ then, the submanifold $\mathcal{M}$ of an almost contact metric manifold $\bar{\mathcal{M}}$ is slant for each non zero vector $\mathcal{X}$ tangent to $\mathcal{M}$ at $\mathcal{X} \in M$ such that $\mathcal{X}$ is linearly independent to $\xi_\mathcal{X}$, the angle $\theta(\mathcal{X})$ between $\phi\mathcal{X}$ and $T_\mathcal{X}\mathcal{M}$ is constant i.e. it does not depend on the choice of the point $\mathcal{X} \in M$ and $\mathcal{X} \in T_\mathcal{X}\mathcal{M} - \{\xi\}$. In this case, the angle θ is called the slant angle of the submanifold. A slant submanifold $\mathcal{M}$ is proper slant submanifold for neither $\theta = 0$ nor $\theta = \pi/2$. Here $T\mathcal{M} = \mathcal{D}_\theta \oplus \{\xi\}$, where $\mathcal{D}_\theta$ is slant distribution with slant angle θ.

If $\theta = 0$, then slant submanifolds is said to be an invariant submanifolds and if $\theta = \pi/2$, then the slant submanifolds is said to be anti-invariant submanifolds.

The submanifold $\mathcal{M}$ of an almost contact metric manifold $\bar{\mathcal{M}}$ is semi-invariant if there exist two orthogonal complementary distributions $\mathcal{D}$ and $\mathcal{D}^\perp$ on $\mathcal{M}$ such that

$$T\mathcal{M} = \mathcal{D} \oplus \mathcal{D}^\perp \oplus \{\xi\},$$

where $\mathcal{D}$ is invariant i.e. $\phi\mathcal{D} \subseteq \mathcal{D}$ and $\mathcal{D}^\perp$ is anti -invariant i.e. $\phi\mathcal{D}^\perp \subset (T^\perp\mathcal{M})$.

The submanifold $\mathcal{M}$ of an almost contact metric manifold $\bar{\mathcal{M}}$ is semi-slant if there exist two orthogonal complementary distributions $\mathcal{D}$ and $\mathcal{D}_\theta$ on $\mathcal{M}$ such that

$$T\mathcal{M} = \mathcal{D} \oplus \mathcal{D}_\theta \oplus \{\xi\},$$

where $\mathcal{D}$ is invariant i.e. $\phi\mathcal{D} \subseteq \mathcal{D}$ and $\mathcal{D}_\theta$ is slant with slant angle θ here the angle θ is called semi-slant angle.

The submanifold $\mathcal{M}$ of an almost contact metric manifold $\bar{\mathcal{M}}$ is hemi-slant if there exist two orthogonal complementary distributions $\mathcal{D}_\theta$ and

$\mathcal{D}^{\perp}$ on $\mathcal{M}$ such that

$$T\mathcal{M} = \mathcal{D}_{\theta} \oplus \mathcal{D}^{\perp} \oplus \{\xi\},$$

where $\mathcal{D}^{\perp}$ is anti- invariant i.e. $\phi\mathcal{D}^{\perp} \subset (T^{\perp}\mathcal{M})$ and $\mathcal{D}_{\theta}$ is slant with slant angle θ here the angle θ is hemi-slant angle.

3. Quasi hemi-slant submanifolds of a nearly trans-Sasakian manifolds

Studying the existence of quasi hemi-slant submanifolds in a nearly trans-Sasakian manifolds is the goal of this section.

We say that $\mathcal{M}$ is quasi hemi-slant submanifold of a nearly trans-Sasakian manifold $\bar{\mathcal{M}}$, if there exist three orthogonal complementary distributions $\mathcal{D}$, $\mathcal{D}_{\theta}$ and $\mathcal{D}^{\perp}$ on $\mathcal{M}$ such that
(a) $T\mathcal{M}$ admits the orthogonal direct decomposition

$$T\mathcal{M} = \mathcal{D} \oplus \mathcal{D}_{\theta} \oplus \mathcal{D}^{\perp} \oplus \{\xi\}, \quad \xi \in \Gamma(\mathcal{D}_{\theta})$$

(b) $\phi\mathcal{D} = \mathcal{D}$
(c) $\phi\mathcal{D}^{\perp} \subseteq T^{\perp}\mathcal{M}$.
(d) The distribution $\mathcal{D}_{\theta}$ is a slant with slant constant angle θ, where $\theta =$ slant angle.
In this case, θ is said to be quasi hemi- slant angle of $\mathcal{M}$. If the dimension of distributions $\mathcal{D}, \mathcal{D}_{\theta}$ and $\mathcal{D}^{\perp}$ are m_1, m_2 and m_3 respectively, then
(a) $\mathcal{M}$ is a hemi-slant submanifold for $m_1 = 0$.
(b) $\mathcal{M}$ is a semi-invariant submanifold for $m_2 = 0$.
(c) $\mathcal{M}$ is a semi-slant submanifold for $m_3 = 0$.
The quasi hemi-slant submanifold $\mathcal{M}$ is proper if $\mathcal{D} \neq \{0\}$, $\mathcal{D}_{\theta} \neq \{0\}$, $\mathcal{D}^{\perp} = \{0\}$ and $\theta \neq 0, \pi/2$.

It represents that quasi hemi-slant submanifols is a generalization of invariant, anti-invariant, semi-invarint, slant, hemi-slant, semi-slant submanifolds.

It is clear from definition that if $\mathcal{D} \neq \{0\}$, $\mathcal{D}_{\theta} \neq \{0\}$ and $\mathcal{D}^{\perp} = \{0\}$, then $dim\mathcal{D} \geq 2, dim\mathcal{D}_{\theta} \geq 2$ and $\mathcal{D}^{\perp} \geq 1$. So for proper quasi hemi slant manifold $\mathcal{M}$, the $dim\mathcal{M} \geq 6$.

Suppose $\mathcal{M}$ be a quasi hemi-slant submanifold of Sasakian manifold $\bar{\mathcal{M}}$ and the projections on $\mathcal{D}$, $\mathcal{D}_{\theta}$ and $\mathcal{D}^{\perp}$ by $\mathcal{P}$, $\mathcal{Q}$ and $\mathcal{R}$ respectively, then for all vector field $\mathcal{X}$ tangent to $\mathcal{M}$, we infer

$$\mathcal{X} = \mathcal{R}\mathcal{X} + \mathcal{Q}\mathcal{X} + \mathcal{P}\mathcal{X} + \upsilon(\mathcal{X})\xi. \tag{3.1}$$

Now put

$$T\mathcal{X} + N\mathcal{X} = \phi\mathcal{X}, \tag{3.2}$$

where $T\mathcal{X}$ and $N\mathcal{X}$ are tangential and normal part of $\phi\mathcal{X}$ on M. From (3.1) and (3.2), we derive

$$\phi\mathcal{X} = N\mathcal{R}\mathcal{X} + T\mathcal{R}\mathcal{X} + N\mathcal{Q}\mathcal{X} + T\mathcal{Q}\mathcal{X} + N\mathcal{P}\mathcal{X} + T\mathcal{P}\mathcal{X}. \tag{3.3}$$

As $\phi\mathcal{D} = \mathcal{D}$ and $\phi\mathcal{D}^{\perp} \subseteq T^{\perp}\mathcal{M}$, we obtain $N\mathcal{P}\mathcal{X} = 0$, and $T\mathcal{R}\mathcal{X} = 0$ and

$$\phi\mathcal{X} = N\mathcal{R}\mathcal{X} + N\mathcal{Q}\mathcal{X} + T\mathcal{Q}\mathcal{X} + T\mathcal{P}\mathcal{X}. \tag{3.4}$$

For all vector field $\mathcal{X}$ tangent to $\mathcal{M}$, we infer

$$T\mathcal{X} = T\mathcal{P}\mathcal{X} + T\mathcal{Q}\mathcal{X}$$

and

$$N\mathcal{X} = N\mathcal{Q}\mathcal{X} + N\mathcal{R}\mathcal{X}.$$

Using (3.4) we deduce the following decomposition,

$$\phi(T\mathcal{M}) = \mathcal{D} \oplus T\mathcal{D}_{\theta} \oplus N\mathcal{D}_{\theta} \oplus N\mathcal{D}^{\perp}. \tag{3.5}$$

As $N\mathcal{D}_{\theta} \subseteq T^{\perp}\mathcal{M}$ and $N\mathcal{D}^{\perp} \subseteq T^{\perp}\mathcal{M}$, we obtain

$$T^{\perp}\mathcal{M} = N\mathcal{D}_{\theta} \oplus N\mathcal{D}^{\perp} \oplus \kappa, \tag{3.6}$$

where κ denotes the orthogonal component of $N\mathcal{D}_{\theta} \oplus N\mathcal{D}^{\perp}$ in $\Gamma(T^{\perp}\mathcal{M})$ and invariant with respect to ϕ.
For all non-zero vector field λ normal to $\mathcal{M}$, we infer

$$\phi\lambda = t\lambda + s\lambda, \tag{3.7}$$

where $t\lambda$ tangent to $\mathcal{M}$ and $s\lambda$ normal to $\mathcal{M}$.

Proposition 3.1. For a submanifold $\mathcal{M}$ of a nearly trans-Sasakian manifolds $\bar{\mathcal{M}}$, we infer

$$\begin{aligned}(\nabla_{\mathcal{Y}}T)\mathcal{X} &= -(\nabla_{\mathcal{X}}T)\mathcal{Y} + \Lambda_{N\mathcal{Y}}\mathcal{X} + \Lambda_{N\mathcal{X}}\mathcal{Y} + 2t\sigma(\mathcal{X},\mathcal{Y}) \\ &\quad +\mu\{\upsilon(\mathcal{X})\mathcal{Y} + \upsilon(\mathcal{Y})\mathcal{X} - 2 <\mathcal{X},\mathcal{Y}> \xi\} \\ &\quad -\rho\{\upsilon(\mathcal{Y})T\mathcal{X} + \upsilon(\mathcal{X})T\mathcal{Y}\}\end{aligned} \tag{3.8}$$

$$\begin{aligned}(\nabla_{\mathcal{Y}}N)\mathcal{X} &= -(\nabla_{\mathcal{X}}N)\mathcal{Y} + 2s\sigma(\mathcal{X},\mathcal{Y}) - \sigma(\mathcal{X},T\mathcal{Y}) \\ &\quad -\sigma(\mathcal{Y},T\mathcal{X}) - \rho\{\upsilon(\mathcal{Y})N\mathcal{X} + \upsilon(\mathcal{X})N\mathcal{Y}\},\end{aligned} \tag{3.9}$$

for all vector fields $\mathcal{X},\mathcal{Y}$ tangent to $\mathcal{M}$.

Proposition 3.2. For a quasi hemi-slant submanifold $\mathcal{M}$ of a nearly trans-Sasakian manifolds $\bar{\mathcal{M}}$, we infer

$$T\mathcal{D} = \mathcal{D}, \quad T\mathcal{D}_\theta = \mathcal{D}_\theta, \quad T\mathcal{D}^\perp = \{0\}, \tag{3.10}$$

$$tN\mathcal{D}_\theta = \mathcal{D}_\theta, \quad tN\mathcal{D}_\theta = \mathcal{D}^\perp.$$

From (3.2), (3.7) and $\phi^2 = -I + \upsilon \otimes \xi$, we get

Proposition 3.3. For the endomorphism T and N, t and s of a quasi hemi-slant submanifold $\mathcal{M}$ of a nearly trans-Sasakian manifolds $\bar{\mathcal{M}}$ in the tangent bundle of $\mathcal{M}$, we infer
(i) $T^2 + tN = -I + \upsilon \otimes \xi$ on tangent $\mathcal{M}$
(ii) $NT + sN = \{0\}$ on tangent $\mathcal{M}$
(iii) $Nt + s^2 = -I$ on normal $\mathcal{M}$
(iv) $Tt + ts = 0$ on on normal $\mathcal{M}$.

Lemma 3.4. For a quasi hemi- slant submanifold $\mathcal{M}$ of a nearly trans-Sasakian manifolds $\bar{\mathcal{M}}$, we infer
(1) $T^2\mathcal{X} = -(\cos^2\theta)\mathcal{X}$,
(2) $< T\mathcal{X}, T\mathcal{Y} >= (\cos^2\theta) < \mathcal{X}, \mathcal{Y} >$
(3) $< N\mathcal{X}, N\mathcal{Y} >= (\sin^2\theta) < \mathcal{X}, \mathcal{Y} >$
for all $\mathcal{X}, \mathcal{Y} \in D_\theta$.

Proof: The proof is the same as in [11].

Proposition 3.5. For a quasi hemi- slant submanifold $\mathcal{M}$ of a nearly trans-Sasakian manifolds $\bar{\mathcal{M}}$, we infer

$$\begin{aligned}(\bar{\nabla}_{\mathcal{Y}}T)\mathcal{X} &= -(\bar{\nabla}_{\mathcal{X}}T)\mathcal{Y} + 2t\sigma(\mathcal{X},\mathcal{Y}) + \Lambda_{N\mathcal{Y}}\mathcal{X} + \Lambda_{N\mathcal{X}}\mathcal{Y} \\ &\quad +\mu\{\upsilon(\mathcal{X})\mathcal{Y} + \upsilon(\mathcal{Y})\mathcal{X} - 2 < \mathcal{X},\mathcal{Y} > \xi\} \\ &\quad -\rho\{\upsilon(\mathcal{X})T\mathcal{Y} + \upsilon(\mathcal{Y})T\mathcal{X}\}\end{aligned}$$

$$\begin{aligned}(\bar{\nabla}_{\mathcal{Y}}N)\mathcal{X} &= -(\bar{\nabla}_{\mathcal{X}}N)\mathcal{Y} - \rho\{\upsilon(\mathcal{Y})N\mathcal{X} + \upsilon(\mathcal{X})N\mathcal{Y}\} \\ &\quad -\sigma(\mathcal{X},T\mathcal{Y}) - \sigma(\mathcal{Y},T\mathcal{X}) + 2s\sigma(\mathcal{X},\mathcal{Y})\end{aligned}$$

$$(\bar{\nabla}_{\mathcal{X}}t)\lambda = -(\bar{\nabla}_{\mathcal{Y}}t)\lambda + \Lambda_{s\lambda}\mathcal{X} + \Lambda_{s\lambda}\mathcal{Y} - T\Lambda_\lambda\mathcal{X} - T\Lambda_\lambda\mathcal{Y}$$

and

$$(\bar{\nabla}_{\mathcal{X}}s)\lambda = -(\bar{\nabla}_{\mathcal{Y}}s)\lambda - \sigma(\mathcal{X},t\lambda) + \sigma(\mathcal{Y},t\lambda) - N\Lambda_\lambda\mathcal{X} - N\Lambda_\lambda\mathcal{Y},$$

for all vector fields $\mathcal{X}, \mathcal{Y}$ tangent to $\mathcal{M}$ and vector fields λ normal to $\mathcal{M}$.

Proposition 3.6. For a quasi hemi-slant submanifold $\mathcal{M}$ of a nearly trans-Sasakian manifolds $\bar{\mathcal{M}}$, we infer

$$\nabla_{\mathcal{X}}\xi = -\mu T\mathcal{X} + \rho\mathcal{X}$$

and

$$\sigma(\mathcal{X}, \xi) = -\mu N\mathcal{X} - \rho\upsilon(\mathcal{X})\xi,$$

for all vector fields $\mathcal{X}$ tangent to $\mathcal{M}$.

Lemma 3.7. For a quasi hemi-slant submanifold $\mathcal{M}$ of a nearly trans-Sasakian manifolds $\bar{\mathcal{M}}$, we infer

$$\sigma_{\phi\mathcal{Z}}\mathcal{W} = \sigma_{\phi\mathcal{W}}\mathcal{Z},$$

for all $\mathcal{Z}, \mathcal{W} \in \mathcal{D}^{\perp}$.

Lemma 3.8. For a quasi hemi- slant submanifold $\mathcal{M}$ of a nearly trans-Sasakian manifolds $\bar{\mathcal{M}}$, we infer

$$< [\mathcal{Y}, \mathcal{X}], \xi > -2\mu < T\mathcal{Y}, \mathcal{X} > +2\rho < \mathcal{Y}, \mathcal{X} >= 0$$

$$< \bar{\nabla}_{\mathcal{Y}}\mathcal{X}, \xi > -\mu < T\mathcal{Y}, \mathcal{X} > +\rho < \mathcal{Y}, \mathcal{X} > -\rho\upsilon(\mathcal{Y})\upsilon(\mathcal{X}) = 0,$$

for all $\mathcal{Y}, \mathcal{X} \in \Gamma(\mathcal{D} \oplus \mathcal{D}_\theta \oplus \mathcal{D}^{\perp})$.

4. Integrability of Distributions and Decomposition Theorems

For invariant distributions $\mathcal{D}$, slant distributions $\mathcal{D}_\theta$ and anti-invariant distributions $\mathcal{D}^{\perp}$ we provide the integrability criteria.

Proposition 4.1. The invariant distribution $\mathcal{D}$ of a proper quasi hemi-slant submanifold $\mathcal{M}$ of nearly trans Sasakian manifold $\bar{\mathcal{M}}$ is not integrable.

Proof. If $\mathcal{X}, \mathcal{Y} \in \Gamma(\mathcal{D})$ and using (2.3), (2.5) and (2.7), we infer

$$< [\mathcal{X}, \mathcal{Y}], \xi >= 2\mu < \phi\mathcal{X}, \mathcal{Y} > -2\rho < \mathcal{X}, \mathcal{Y} >\neq 0. \tag{4.1}$$

Since $< \phi\mathcal{X}, \mathcal{Y} >\neq 0$, therefore $< [\mathcal{X}, \mathcal{Y}], \xi >\neq 0$. □

Theorem 4.2. The distribution $\mathcal{D} \oplus \{\xi\}$ of a proper quasi hemi-slant submanifold $\mathcal{M}$ of a nearly trans-Sasakian manifolds $\bar{\mathcal{M}}$ is integrable if and only if $\forall\ \mathcal{X}, \mathcal{Y} \in \Gamma(\mathcal{D} \oplus \{\xi\})$ and $\mathcal{Z} \in \Gamma(\mathcal{D}_\theta \oplus \mathcal{D}^\perp)$, we infer

$$(4.2)\quad < \nabla_\mathcal{X} T\mathcal{Y} - \nabla_\mathcal{Y} T\mathcal{X}, TQ\mathcal{Z} > \;=\; < \sigma(\mathcal{Y}, T\mathcal{X}) - \sigma(\mathcal{X}, T\mathcal{Y}), N\mathcal{Q}\mathcal{Z} + N\mathcal{R}\mathcal{Z} > .$$

Proof. Using (2.2), (2.5) and (3.2) we obtain

$$\begin{aligned} < [\mathcal{X}, \mathcal{Y}], \mathcal{Z} > &= - < \bar{\nabla}_\mathcal{Y} \mathcal{X}, \mathcal{Z} > + < \bar{\nabla}_\mathcal{X} \mathcal{Y}, \mathcal{Z} > \\ &= - < \phi\bar{\nabla}_\mathcal{Y} \mathcal{X}, \phi\mathcal{Z} > + < \phi\bar{\nabla}_\mathcal{X} \mathcal{Y}, \phi\mathcal{Z} > . \end{aligned}$$

After some computation, we get

$$(4.3)\quad < \nabla_\mathcal{X} T\mathcal{Y} - \nabla_\mathcal{Y} T\mathcal{X}, TQ\mathcal{Z} > \;=\; < \sigma(\mathcal{Y}, T\mathcal{X}) - \sigma(\mathcal{X}, T\mathcal{Y}), N\mathcal{Q}\mathcal{Z} + N\mathcal{R}\mathcal{Z} > .$$

□

Proposition 4.3. A slant distribution $\mathcal{D}_\theta$ of proper quasi hemi- slant submanifold $\mathcal{M}$ of a nearly trans-Sasakian manifolds $\bar{\mathcal{M}}$ is not integrable.

Proof. : If $\mathcal{W}, \mathcal{X} \in \Gamma(\mathcal{D}_\theta)$ and using (2.3), (2.5) and (2.7), we infer

$$< [\mathcal{W}, \mathcal{X}], \xi >= 2\mu < \phi\mathcal{W}, \mathcal{X} > -2\rho < \mathcal{W}, \mathcal{X} > \neq 0.$$

Since $< \phi\mathcal{W}, \mathcal{X} > \neq 0$, therefore $< [\mathcal{W}, \mathcal{X}], \xi > \neq 0$. □

Theorem 4.4. The distribution $\mathcal{D}_\theta \oplus \{\xi\}$ of a proper quasi hemi-slant submanifolds $\mathcal{M}$ of a nearly trans-Sasakian manifold $\bar{\mathcal{M}}$ is integrable if and only if $\forall\ \mathcal{Y}, \mathcal{Z} \in \Gamma(\mathcal{D}_\theta \oplus \{\xi\})$ and $\mathcal{W} \in \Gamma(\mathcal{D} \oplus \mathcal{D}^\perp)$, we infer

$$\begin{aligned} (4.4) < \Lambda_{NT\mathcal{Z}} \mathcal{Y} - \Lambda_{NT\mathcal{Y}} \mathcal{Z}, \quad \mathcal{W} \quad &>=< \Lambda_{N\mathcal{Z}} \mathcal{Y} - \Lambda_{N\mathcal{Y}} \mathcal{Z}, T\mathcal{P}\mathcal{W} > \\ &+ < \nabla^\perp_\mathcal{Z} N\mathcal{Y} - \nabla^\perp_\mathcal{Y} N\mathcal{Z}, N\mathcal{R}\mathcal{W} > . \end{aligned}$$

Proof. If $\mathcal{Y}, \mathcal{Z} \in \Gamma(\mathcal{D}_\theta \oplus \{\xi\})$ and $\mathcal{W} = \mathcal{P}\mathcal{W} + \mathcal{R}\mathcal{W} \in \Gamma(\mathcal{D} \oplus \mathcal{D}^\perp)$ and using (2.2), (2.5) and (3.2), we infer

$$< [\mathcal{X}, \mathcal{Y}], \mathcal{Z} >=< \phi\bar{\nabla}_\mathcal{Y} \mathcal{Z}, \phi\mathcal{W} > - < \phi\bar{\nabla}_\mathcal{Z} \mathcal{Y}, \phi\mathcal{W} > .$$

By using (2.8), (3.2) and lemma 3.4 we infer,

$$\begin{aligned} (\sin^2\theta) < [\mathcal{Y}, \mathcal{Z}], \mathcal{W} > &= < \Lambda_{NT\mathcal{Z}} \mathcal{Y} - \Lambda_{NT\mathcal{Y}} \mathcal{Z}, \mathcal{W} > \\ &+ < \nabla^\perp_\mathcal{Y} N\mathcal{Z} - \nabla^\perp_\mathcal{Z} N\mathcal{Y}, N\mathcal{R}\mathcal{W} > \\ &- < \Lambda_{N\mathcal{Z}} \mathcal{Y} - \Lambda_{N\mathcal{Y}} \mathcal{Z}, T\mathcal{P}\mathcal{W} > . \end{aligned}$$

□

This leads to the following conclusion:

Theorem 4.5. The distribution $\mathcal{D}_\theta \oplus \{\xi\}$ of a proper quasi hemi-slant submanifolds $\mathcal{M}$ of a nearly trans-Sasakian manifold $\bar{\mathcal{M}}$ is integrable if

$$\nabla^\perp_{\mathcal{U}} N\mathcal{V} - \nabla^\perp_{\mathcal{V}} N\mathcal{U} \in N\mathcal{D}_\theta \oplus \kappa,$$

$$\Lambda_{NT\mathcal{V}}\mathcal{U} - \Lambda_{NT\mathcal{U}}\mathcal{V} \in \mathcal{D}_\theta$$

and

$$\Lambda_{N\mathcal{V}}\mathcal{U} - \Lambda_{N\mathcal{U}}\mathcal{V} \in \mathcal{D}^\perp \oplus \mathcal{D}_\theta,$$

for all $\mathcal{V}, \mathcal{U} \in \Gamma(\mathcal{D}_\theta \oplus \{\xi\})$.

.

Theorem 4.6. The anti-invariant distribution $\mathcal{D}^\perp$ of a quasi hemi-slant submanifold $\mathcal{M}$ of a nearly trans-Sasakian manifolds $\bar{\mathcal{M}}$ is integrable if and only if $\forall\ \mathcal{Z}, \mathcal{W} \in \Gamma(\mathcal{D}^\perp)$, we infer

$$\nabla^\perp_{\mathcal{Z}} N\mathcal{W} - \nabla^\perp_{\mathcal{W}} N\mathcal{Z} \in N\mathcal{D}^\perp \oplus \kappa.$$

Proof. If $\mathcal{Z}, \mathcal{W} \in \Gamma(\mathcal{D}^\perp), \mathcal{Y} = \mathcal{P}\mathcal{Y} + \mathcal{Q}\mathcal{Y} \in \Gamma(\mathcal{D} \oplus \mathcal{D}_\theta)$ and using (2.2), (2.5), (2.8), (3.2) and lemma 3.7, we infer

$$\begin{aligned} < [\mathcal{Z}, \mathcal{W}], \mathcal{Y} > \; &= \; < \bar{\nabla}_{\mathcal{Z}} \phi\mathcal{W}, \phi\mathcal{Y} > - < \bar{\nabla}_{\mathcal{W}} \phi\mathcal{Z}, \phi\mathcal{Y} > \\ &= < \Lambda_{\phi\mathcal{Z}}\mathcal{W}, T\mathcal{P}\mathcal{Y} > - < \Lambda_{\phi\mathcal{W}}\mathcal{Z}, T\mathcal{P}\mathcal{Y} > \\ &\quad - < \nabla^\perp_{\mathcal{W}} \phi\mathcal{Z}, N\mathcal{Q}\mathcal{Y} > + < \nabla^\perp_{\mathcal{Z}} \phi\mathcal{W}, N\mathcal{Q}\mathcal{Y} > \\ &= < \nabla^\perp_{\mathcal{Z}} N\mathcal{W}, N\mathcal{Q}\mathcal{Y} > - < \nabla^\perp_{\mathcal{W}} N\mathcal{Z}, N\mathcal{Q}\mathcal{Y} > . \end{aligned}$$

□

Theorem 4.7. If $\mathcal{M}$ is a proper quasi hemi-slant submanifold of a nearly trans-Sasakian manifolds $\bar{\mathcal{M}}$, then $\mathcal{M}$ is totally geodesic if and only if

$$\begin{aligned} (4.5) < \sigma(\mathcal{W}, \mathcal{P}\mathcal{X}), \quad \mathcal{Y} \quad &> = < \nabla^\perp_{\mathcal{W}} NT\mathcal{Q}\mathcal{X}, \mathcal{Y} > + < \Lambda_{N\mathcal{Q}\mathcal{X}}\mathcal{W}, t\mathcal{Y} > \\ &+ < \Lambda_{N\mathcal{R}\mathcal{X}}\mathcal{W}, t\mathcal{Y} > - < \nabla^\perp_{\mathcal{W}} N\mathcal{X}, s\mathcal{Y} > \\ &- \cos^2\theta < \sigma(\mathcal{W}, \mathcal{Q}\mathcal{X}), \mathcal{Y} > . \end{aligned}$$

Proof. If $\mathcal{W}, \mathcal{X} \in \Gamma(T\mathcal{M}), \mathcal{Y} \in \Gamma(T^\perp\mathcal{M})$ and using (2.2), (2.5), we infer

$$\begin{aligned} < \bar{\nabla}_{\mathcal{W}}\mathcal{X}, \mathcal{Y} > \; &= \; < \bar{\nabla}_{\mathcal{W}}\mathcal{P}\mathcal{X}, \mathcal{Y} > + < \bar{\nabla}_{\mathcal{W}}\mathcal{Q}\mathcal{X}, \mathcal{Y} > + < \bar{\nabla}_{\mathcal{W}}\mathcal{R}\mathcal{X}, \mathcal{Y}) \\ &= < \bar{\nabla}_{\mathcal{W}}\phi\mathcal{P}\mathcal{X}, \phi\mathcal{Y} > + < \bar{\nabla}_{\mathcal{W}} T\mathcal{Q}\mathcal{X}, \phi\mathcal{Y} > \\ &\quad + < \bar{\nabla}_{\mathcal{W}} N\mathcal{Q}\mathcal{X}, \phi\mathcal{Y} > + < \bar{\nabla}_{\mathcal{W}}\phi\mathcal{R}\mathcal{X}, \phi\mathcal{Y} > . \end{aligned}$$

Using (2.3), (2.7), (2.8), (3.2) and lemma 3.4, we get

$$\begin{aligned} < \bar{\nabla}_{\mathcal{W}}\mathcal{X}, \mathcal{Y} > &= < \bar{\nabla}_{\mathcal{W}}\mathcal{P}\mathcal{X}, \mathcal{Y} > - < \bar{\nabla}_{\mathcal{W}}T^2\mathcal{Q}\mathcal{X}, \mathcal{Y} > \\ &- < \bar{\nabla}_{\mathcal{W}}NT\mathcal{Q}\mathcal{X}, \mathcal{Y} > + < \bar{\nabla}_{\mathcal{W}}N\mathcal{Q}\mathcal{X}, \phi\mathcal{Y} > \\ &+ < \bar{\nabla}_{\mathcal{W}}N\mathcal{R}\mathcal{X}, \phi\mathcal{Y} > \\ &=< \sigma(\mathcal{W}, \mathcal{P}\mathcal{X}), \mathcal{Y} > + \cos^2\theta < \nabla_{\mathcal{W}}\mathcal{Q}\mathcal{X}, \mathcal{Y} > \\ &+ \cos^2\theta < \sigma(\mathcal{W}, \mathcal{Q}\mathcal{X}), \mathcal{Y} > - < \nabla^{\perp}_{\mathcal{W}}NT\mathcal{Q}\mathcal{X}, \mathcal{Y} > \\ &+ < -\Lambda_{N\mathcal{Q}\mathcal{X}}\mathcal{W} + \nabla^{\perp}_{\mathcal{W}}N\mathcal{Q}\mathcal{X}, \phi\mathcal{Y} > \\ &+ < -\Lambda_{N\mathcal{R}\mathcal{X}}\mathcal{W} + \nabla^{\perp}_{\mathcal{W}}N\mathcal{R}\mathcal{X}, \phi\mathcal{Y} > . \end{aligned}$$

$$\begin{aligned} (4.6) < \bar{\nabla}_{\mathcal{W}}\mathcal{X}, \mathcal{Y} > &= < \sigma(\mathcal{W}, \mathcal{P}\mathcal{X}), \mathcal{Y} > \\ &- < \nabla^{\perp}_{\mathcal{W}}N\mathcal{T}\mathcal{Q}\mathcal{X}, \mathcal{Y} > + < \nabla^{\perp}_{\mathcal{W}}N\mathcal{X}, f\mathcal{Y} > \\ &- < \Lambda_{N\mathcal{Q}\mathcal{X}}\mathcal{W} + \Lambda_{N\mathcal{R}\mathcal{X}}\mathcal{W}, t\mathcal{Y} > \\ &+ \cos^2\theta < \sigma(\mathcal{W}, \mathcal{Q}\mathcal{X}), \mathcal{Y} > . \end{aligned}$$

□

Examine the geometry of the leaves of the slant, anti-slant, and invariant distributions now.

Proposition 4.8. An invariant distribution $\mathcal{D}$ of proper quasi hemi-slant submanifold $\mathcal{M}$ of a nearly trans -sasakian manifold $\bar{\mathcal{M}}$ is not define a totally geodesic foliation on $\mathcal{M}$.

Proof. If $\mathcal{Y}, \mathcal{Z} \in \Gamma(\mathcal{D})$ and using (2.3), (2.5), (2.7), we infer

$$\begin{aligned} (4.7) \quad < \bar{\nabla}_{\mathcal{Y}}\mathcal{Z}, \xi > &= < \nabla_{\mathcal{Y}}\mathcal{Z}, \xi > \\ &= \mu < \phi\mathcal{Y}, \mathcal{Z} > -\rho < \mathcal{Y}, \mathcal{Z} > +\rho\upsilon(\mathcal{Y})\upsilon(\mathcal{Z}) \\ (4.8) \quad &\neq 0. \end{aligned}$$

Since $< \phi\mathcal{Y}, \mathcal{Z} > \neq 0$, therefore $< \bar{\nabla}_{\mathcal{Y}}\mathcal{Z}, \xi > \neq 0$. □

Theorem 4.9. The distribution $\mathcal{D} \oplus \{\xi\}$ of a proper quasi hemi-slant submanifold $\mathcal{M}$ of nearly trans-Sasakian manifold $\bar{\mathcal{M}}$ is define totally geodesic foliation on $\mathcal{M}$ if and only if for all $\mathcal{X}, \mathcal{Y} \in \Gamma(\mathcal{D})$, $\mathcal{Z} \in \Gamma(\mathcal{D}_\theta \oplus \mathcal{D}^{\perp})$ and $\lambda \in (T^{\perp}\mathcal{M})$, we infer

$$< \nabla_{\mathcal{X}}T\mathcal{Y}, T\mathcal{Q}\mathcal{Z} >= - < \sigma(\mathcal{X}, T\mathcal{Y}), N\mathcal{Q}\mathcal{Z} + N\mathcal{R}\mathcal{Z} >$$

and

$$< \nabla_{\mathcal{X}}T\mathcal{Y}, t\lambda >= - < \sigma(\mathcal{X}, T\mathcal{Y}), s\lambda > .$$

Proof. If $\mathcal{X}, \mathcal{Y} \in \Gamma(\mathcal{D}), \mathcal{Z} = \mathcal{Q}\mathcal{Z} + \mathcal{R}\mathcal{Z} \in \Gamma(\mathcal{D}_\theta \oplus \mathcal{D}^\perp)$ and using (2.2), (2.5), (3.2) and $N\mathcal{Y} = 0$, we infer

$$\begin{aligned} <\bar{\nabla}_\mathcal{X}\mathcal{Y}, \mathcal{Z}> &= <\bar{\nabla}_\mathcal{X} T\mathcal{Y}, \phi\mathcal{Z}> \\ &=<\nabla_\mathcal{X} T\mathcal{Y}, T\mathcal{Q}\mathcal{Z}> + <\sigma(\mathcal{X}, T\mathcal{Y}), N\mathcal{Q}\mathcal{Z} + N\mathcal{R}\mathcal{Z}>, \end{aligned}$$

for all $\lambda \in (T^\perp\mathcal{M})$ and $\mathcal{X}, \mathcal{Y} \in \Gamma(\mathcal{D})$, we infer

$$<\bar{\nabla}_\mathcal{X}\mathcal{Y}, \lambda>=<\nabla_\mathcal{X} T\mathcal{Y}, t\lambda> + <\sigma(\mathcal{X}, T\mathcal{Y}), s\lambda>.$$

□

Proposition 4.10. The slant distribution $\mathcal{D}_\theta$ of a proper quasi hemi-slant submanifold $\mathcal{M}$ of a nearly trans-Sasakian manifold $\bar{\mathcal{M}}$ is not define a totally geodesic foliation on $\mathcal{M}$.

Proof. If $\mathcal{Y}, \mathcal{Z} \in \Gamma(\mathcal{D}_\theta)$ and using (2.3), (2.5) and (2.7), we infer

$$\begin{aligned} (4.9)\quad <\bar{\nabla}_\mathcal{Y}\mathcal{Z}, \xi> &= <\nabla_\mathcal{Y}\mathcal{Z}, \xi> \\ &= \mu <\phi\mathcal{Y}, \mathcal{Z}> -\rho <\mathcal{Y}, \mathcal{Z}> +\rho\upsilon(Y)\upsilon(\mathcal{Z}) \\ (4.10)\qquad\qquad &\neq 0, \qquad for some \quad \mathcal{Y}, \mathcal{Z} \in \Gamma(\mathcal{D}_\theta). \end{aligned}$$

Since $<\phi\mathcal{Y}, \mathcal{Z}>\neq 0$, therefore $<\bar{\nabla}_\mathcal{Y}\mathcal{Z}, \xi>\neq 0$. □

Theorem 4.11. The distribution $\mathcal{D}_\theta \oplus \{\xi\}$ of a proper quasi hemi-slant submanifold $\mathcal{M}$ of a nearly trans-Sasakian manifold $\bar{\mathcal{M}}$ is to define a totally geodesic foliation on $\mathcal{M}$ iff $\forall \quad \mathcal{U}, \mathcal{V} \in \Gamma(\mathcal{D}_\theta \oplus \{\xi\}), \mathcal{W} \in \Gamma(\mathcal{D} \oplus \mathcal{D}^\perp)$ and $\lambda \in (T^\perp\mathcal{M})$, we infer

$$<\nabla^\perp_\mathcal{U} N\mathcal{V}, N\mathcal{R}\mathcal{W}>=<\Lambda_{N\mathcal{V}}\mathcal{U}, T\mathcal{P}\mathcal{W}> - <\Lambda_{NT\mathcal{V}}\mathcal{U}, \mathcal{W}>$$

and

$$<\Lambda_{N\mathcal{V}}\mathcal{U}, t\lambda>=<\nabla^\perp_\mathcal{U} N\mathcal{V}, s\lambda> - <\nabla^\perp_\mathcal{U} NT\mathcal{V}, \lambda>.$$

Proof. If $\mathcal{U}, \mathcal{V} \in \Gamma(\mathcal{D}_\theta \oplus \{\xi\}), \mathcal{W} = \mathcal{P}\mathcal{W} + \mathcal{R}\mathcal{W} \in \Gamma(\mathcal{D} \oplus \mathcal{D}^\perp)$, and using (2.2), (2.5) and (3.2), we infer

$$<\bar{\nabla}_\mathcal{U}\mathcal{V}, \mathcal{W}>=<\bar{\nabla}_\mathcal{U}\phi\mathcal{V}, \phi\mathcal{W}>=<\bar{\nabla}_\mathcal{U} T\mathcal{V}, \phi\mathcal{W}> + <\bar{\nabla}_\mathcal{U} N\mathcal{V}, \phi\mathcal{W}>.$$

Then using (2.8), (3.2) and Lemma 3.4 and the fact that $N\mathcal{P}\mathcal{W} = 0$, we infer

$$\begin{aligned} <\bar{\nabla}_\mathcal{U}\mathcal{V}, \mathcal{W}> &= \cos^2\theta <\bar{\nabla}_\mathcal{U}\mathcal{V}, \mathcal{W}> - <\bar{\nabla}_\mathcal{U} NT\mathcal{V}, \mathcal{W}> \\ &\quad + <\bar{\nabla}_\mathcal{U} N\mathcal{V}, \phi\mathcal{W}> \end{aligned}$$

$$
\begin{aligned}
(4.11)\qquad \sin^2\theta < \bar{\nabla}_{\mathcal{U}}\mathcal{V}, \mathcal{W} > &= < \Lambda_{NT\mathcal{V}}\mathcal{U}, \mathcal{W} > \\
&+ < \nabla^{\perp}_{\mathcal{U}} N\mathcal{V}, N\mathcal{R}\mathcal{W} > \\
&- < \Lambda_{N\mathcal{V}}\mathcal{U}, T\mathcal{P}\mathcal{W} > .
\end{aligned}
$$

Similarly, we get

$$
\begin{aligned}
(4.12)\qquad \sin^2\theta < \bar{\nabla}_{\mathcal{U}}\mathcal{V}, V > &= - < \nabla^{\perp}_{\mathcal{U}} NTW, \lambda > \\
&- < \Lambda_{N\mathcal{V}}\mathcal{U}, t\lambda > \\
&+ < \nabla^{\perp}_{\mathcal{U}} N\mathcal{V}, s\lambda > .
\end{aligned}
$$

□

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