Journal of Finsler Geometry and its Applications
Vol. 4, No. 2 (2023), pp 104-112
https://doi.org/10.22098/jfga.2023.14025.1107

Geodesic vectors of infinite series (α, β)-metric on hypercomplex four dimensional Lie groups

Milad L. Zeinali ,

Department of Mathematics, University of Mohaghegh Ardabili, P.O.Box. 5619911367, Ardabil-Iran.
E-mail: miladzeinali@gmail.com

Abstract. In this paper, we consider invariant infinite series (α, β)-metrics. Then we describe all geodesic vectors of this spaces on the left invariant hypercomplex four dimensional simply connected Lie groups.

Keywords: complex structure, geodesic vector, hypercomplex structure, infinite series metric.

1. Introduction

Finsler geometry has its genesis in integrals of the form

$$\int_a^b F\Big(x^1, x^2, \ldots, x^n; \frac{dx^1}{dt}, \frac{dx^2}{dt}, \ldots, \frac{dx^n}{dt}\Big)dt.$$

The function $F(x^1, \ldots, x^n; y^1, \ldots, y^n)$ is positive unless all the y^i are zero. Finsler geometry also asserts itself in applications, most notably in theory of relativity, control theory and mathematical biology.

The special case of Finsler metrics we are going to discuss are expressed in terms of a Riemannian metric $\alpha = \sqrt{a_{ij}y^iy^j}$ and a 1-form $\beta = b_iy^i$. They are called (α, β)- metric. The notion of (α, β)- metrics are introduced by Matsumoto [11]. If $F = \alpha + \beta$, then we get the Randers metric. This metric is an (α, β)- metric that introduced by Ingarden. An (α, β)- metric is a Finsler

AMS 2020 Mathematics Subject Classification: 22E46, 53C60, 53C30

metric of the form $F = \alpha\varphi(s)$, $s = \beta/\alpha$ where $\alpha = \sqrt{\tilde{a}_{ij}(x)y^i y^j}$ is induced by a Riemannian metric $\tilde{a} = \tilde{a}_{ij}dx^i \otimes dx^j$ on a connected smooth n- dimensional manifold M and $\beta = b_i(x)y^i$ is a 1- form on M. We note that, the important kinds of (α, β)- metrics are Kropina metric $F = \alpha^2/\beta$, square metric $F = (\alpha+\beta)^2/\alpha$, exponential metric $F = \alpha\exp(\beta/\alpha)$, Matsumoto metric $F = \alpha^2/(\alpha-\beta)$ and infinite series metric $F = \beta^2/(\beta-\alpha)$. (For some details see [1, 4, 8, 9, 10, 14]).

The important concepts in Finsler geometry is geodesics. Geodesics in a manifold is the generalization of concept of a straight line in an Euclidean space. A geodesic in a homogeneous Finsler space $(G/H, F)$ is called homogeneous geodesic if it is an orbit of a one-parameter subgroup of G. Homogeneous geodesics on homogeneous Riemannian manifolds have been studied by many authors. Latifi has extended the concept of homogeneous geodesics in homogeneous Finsler spaces [7].

Suppose (M, F) be a connected homogeneous Finsler space, G is a connected transitive group of isometries of M and H is the isotropy subgroup at a point $o \in M$. Therefore, M is naturally identified with the coset space G/H with G- invariant Finsler metric F. Also, in this case the Lie algebra $\mathfrak{g}$ of G has a reductive decomposition

$$\mathfrak{g} = \mathfrak{m} + \mathfrak{h},$$

where $\mathfrak{m} \subset \mathfrak{g}$ is a subspace of $\mathfrak{g}$ isomorphic to the T_oM and $\mathfrak{h}$ is the Lie algebra of H.

In this paper, we study geodesics vectors of left invariant infinite series (α, β)-metrics on left invariant hypercomplex four dimensional simply connected Lie groups.

2. Preliminaries

Let M be an n-dimensional C^∞ manifold. Denote by T_xM the tangent space at $x \in M$ and by $TM := \cup_{x\in M} T_xM$ the tangent bundle of M. The dual space of T_xM is T_x^*M, called the cotangent space at x. The union $T^*M := \cup_{x\in M} T_x^*M$ is the cotangent bundle of M.

Definition 2.1. *A Finsler structure of M is a function*

$$F : TM \to [0, \infty),$$

with the following properties [2]:

(1) *F is smooth on the slit tangent bundle $TM^0 := TM\backslash\{0\}$.*
(2) *$F(x, \lambda y) = \lambda F(x, y)$ for any $x \in M$, $y \in T_xM$ and $\lambda > 0$.*
(3) *The $n \times n$ Hessian matrix*

$$(g_{ij}) := \left(\left[\frac{1}{2}F^2\right]_{y^i y^j}\right),$$

is positive-definite at every point of TM^0.

Let $\alpha = \sqrt{\widetilde{a}_{ij}\,(x)\,y^i y^j}$ be a norm induced by a Riemannian metric $\widetilde{a}$ and $\beta\,(x, y) = b_i(x)y^i$ be a 1-form on an n- dimensional manifold M. Let

$$b := \|\beta(x)\|_\alpha := \sqrt{\widetilde{a}^{ij}(x)b_i(x)b_j(x)}.$$

Now, let the function F is defined as follows

$$F := \alpha\varphi(s), \quad s = \frac{\beta}{\alpha}, \tag{2.1}$$

where $\varphi = \varphi(s)$ is a positive C^∞ function on $(-b_0, b_0)$ satisfying

$$\varphi\,(s) - s\varphi'\,(s) + \left(b^2 - s^2\right)\varphi''\,(s) > 0, \qquad |s| \leq \; b < b_0.$$

Then F is a Finsler metric if $\|\beta(x)\|_\alpha < b_0$ for any $x \in M$. A Finsler metric in the form (2.1) is called an (α, β)- metric [13].

A Finsler space having the Finsler function:

$$F\,(x, y) = \frac{\beta^2(x, y)}{\beta(x, y) - \alpha(x, y)},$$

is called a infinite series space. We note that the Riemannian metric $\tilde{a}$ induces an inner product on any cotangent space T^*_xM such that $\langle dx^i(x), dx^j(x)\rangle = \tilde{a}^{ij}(x)$. The induced inner product on T^*_xM induces a linear isomorphism between T^*_xM and T_xM. Then the 1-form β corresponds to a vector field $\tilde{X}$ on M such that

$$\tilde{a}(y, \tilde{X}(x)) = \beta(x, y). \tag{2.2}$$

Also we have $\|\beta(x)\|_\alpha = \|\tilde{X}(x)\|_\alpha$. Therefore we can write infinite series metric as follows:

$$F(x, y) = \frac{\tilde{a}(X, y)^2}{\tilde{a}(X, y) - \sqrt{\tilde{a}(y, y)}}. \tag{2.3}$$

Now, consider the Chern connection on π^*TM whose coefficients are denoted by Γ^i_{jk}. Let $\gamma(t)$ be a smooth regular curve in M with velocity field V. Suppose $W(t) := W^i(t)\frac{\partial}{\partial x^i}$ be a vector field along γ. Then the covariant derivative $D_V W$ with reference vector V have the form

$$\Big[\frac{dW^i}{dt} + W^j V^k (\Gamma^i_{jk})_{(\gamma,V)}\Big]\frac{\partial}{\partial x^i}|_{\gamma(t)}.$$

A curve $\gamma(t)$ with the velocity $V = \dot{\gamma}(t)$, is a Finslerian geodesic if

$$D_V\Big[\frac{V}{F(V)}\Big] = 0, \quad \text{with reference vector} \quad V.$$

Definition 2.2. *Suppose* $(G/H, F)$ *be a homogeneous Finsler manifold with a fixed origin* o. *Let* $\mathfrak{g}$ *and* $\mathfrak{h}$ *be the Lie algebra of* G *and* H *respectively and* $\mathfrak{g} = \mathfrak{m} + \mathfrak{h}$ *a reductive decomposition. Therefore, a homogeneous geodesic through the* $o \in G/H$ *is a geodesic* $\gamma(t)$ *of the form*

$$\gamma(t) = \exp(tZ)(o), \quad t \in \mathbb{R}, \tag{2.4}$$

where Z *is a nonzero vector of* $\mathfrak{g}$.

In Riemannian setting the authors in [6], proved that a $X \in \mathfrak{g} - \{0\}$ is a geodesic vector if and only if

$$\langle [X, Y]_{\mathfrak{m}}, X_{\mathfrak{m}} \rangle = 0, \quad \forall Y \in \mathfrak{m}. \tag{2.5}$$

After this, Latifi in Finsler setting shown that:

Lemma 2.3. [7] *Suppose* $(G/H, F)$ *be a homogeneous Finsler space with a reductive decomposition*

$$\mathfrak{g} = \mathfrak{h} + \mathfrak{m}.$$

Therefore, $Y \in \mathfrak{g} - \{0\}$ *is a geodesic vector if and only if*

$$g_{Y_{\mathfrak{m}}}(Y_{\mathfrak{m}}, [Y, Z]_{\mathfrak{m}}) = 0, \quad \forall Z \in \mathfrak{m}, \tag{2.6}$$

where the subscript $\mathfrak{m}$ *indicates the projection of a vector from* $\mathfrak{g}$ *to* $\mathfrak{m}$.

3. Geodesic Vectors of Infinite Series metric on Four Dimensional Lie Group

An almost complex structure on a real differentiable manifold M is a tensor field J which is an endomorphism of the tangent space T_xM such that $J^2 = -1$, where 1 denotes the identity transformation of T_xM, at every point x of M. Note that for any two vector fields X and Y, we define the Nijenhuis tensor N as

$$N(X, Y) = [X, Y] + J[JX, Y] + J[X, JY] - [JX, JY]. \tag{3.1}$$

A hypercomplex manifold is a manifold M with three globally-defined, integrable complex structures I, J, K satisfying the quaternion identities

$$I^2 = J^2 = K^2 = -1, \quad \text{and} \quad IJ = K = -JI. \tag{3.2}$$

Obata [12] proved that a hypercomplex manifold admits a unique torsion-free connection ∇ such that

$$\nabla I = \nabla J = \nabla K = 0.$$

Now let M be a 4-dimensional manifold. A hypercomplex structure on M is a family $\mathbb{H} = \{J_\alpha\}_{\alpha=1,2,3}$ of fiber-wise endomorphism of TM such that

$$-J_2J_1 = J_1J_2 = J_3, \quad J_\alpha^2 = -Id_{TM}, \quad \alpha = 1, 2, 3, \tag{3.3}$$

$$N_\alpha = 0, \quad \alpha = 1, 2, 3, \tag{3.4}$$

where N_α is the Nijenhuis tensor (torsion) corresponding to J_α.

We note that, an almost complex structure is a complex structure if and only if it has no torsion [5]. Then the complex structures $J_\alpha,\ \alpha = 1, 2, 3$, on a 4-dimensional manifold M form a hypercomplex if they satisfy in the relation 3.3.

Definition 3.1. *A Riemannian metric $\tilde{a}$ on a hypercomplex manifold $(M, \mathbb{H})$ is called hyper-Hermitian if for all vector fields X and Y on M and for all $\alpha = 1, 2, 3$ we have*

$$\tilde{a}(J_\alpha X, J_\alpha Y) = \tilde{a}(X, Y).$$

Definition 3.2. *A hypercomplex structure $\mathbb{H} = \{J_\alpha\}_{\alpha=1,2,3}$ on a Lie group G is said to be left invariant if for any $t \in G$ we have*

$$J_\alpha = Tl_t \circ J_\alpha \circ Tl_{t^{-1}},$$

where Tl_t is the differential function of the left translation l_t.

In this section, we consider left invariant hyper-Hermitian Riemannian metrics on left invariant hypercomplex 4-dimensional simply connected Lie groups. Barberis shown that in this spaces, $\mathfrak{g}$ is either Abelian or isomorphic to one of the following Lie algebras:

(1)
$$[e_2, e_3] = e_4, \quad [e_3, e_4] = e_2, \quad [e_4, e_2] = e_3, \quad e_1 : \text{central}, \tag{3.5}$$

(2)
$$[e_1, e_3] = e_1, \quad [e_2, e_3] = e_2, \quad [e_1, e_4] = e_2, \quad [e_2, e_4] = -e_1, \tag{3.6}$$

(3)
$$[e_1, e_2] = e_2, \quad [e_1, e_3] = e_3, \quad [e_1, e_4] = e_4, \tag{3.7}$$

(4)
$$[e_1, e_2] = e_2, \quad [e_1, e_3] = \frac{1}{2}e_3, \quad [e_1, e_4] = \frac{1}{2}e_4, \quad [e_3, e_4] = \frac{1}{2}e_2. \tag{3.8}$$

where $\{e_1, e_2, e_3, e_4\}$ is an orthonormal basis.

Now we want to describe all geodesics vectors of left invariant infinite series metrics F defined by relation

$$F(x, y) = \frac{\tilde{a}(X, y)^2}{\tilde{a}(X, y) - \sqrt{\tilde{a}(y, y)}}.$$

By using the formula

$$g_y(u, v) = \frac{1}{2}\frac{\partial^2}{\partial s \partial t}F^2(x, y + su + tv)|_{s=t=0},$$

and some computations we get:

$$\begin{aligned}g_y(u,v) =&\frac{\tilde{a}(X,y)^2}{\Big(\tilde{a}(X,y)-\sqrt{\tilde{a}(y,y)}\Big)^4}\Bigg[\tilde{a}(X,y)^2\tilde{a}(X,v)\tilde{a}(X,u)-4\tilde{a}(y,y)^{3/2}\tilde{a}(X,v)\tilde{a}(X,u)\\&+6\tilde{a}(y,y)\tilde{a}(X,v)\tilde{a}(X,u)+\frac{\tilde{a}(X,y)^2\tilde{a}(X,v)\tilde{a}(u,y)}{\tilde{a}(y,y)^{1/2}}-4\tilde{a}(X,y)\tilde{a}(X,v)\tilde{a}(u,y)\\&-\frac{\tilde{a}(X,y)^3\tilde{a}(u,y)\tilde{a}(v,y)}{\tilde{a}(y,y)^{3/2}}+\frac{\tilde{a}(X,y)^3\tilde{a}(u,v)}{\tilde{a}(y,y)^{1/2}}+\frac{4\tilde{a}(X,y)^2\tilde{a}(u,y)\tilde{a}(v,y)}{\tilde{a}(y,y)}\\&-\tilde{a}(X,y)^2\tilde{a}(u,v)+\frac{\tilde{a}(X,y)^2\tilde{a}(X,u)\tilde{a}(v,y)}{\tilde{a}(y,y)^{1/2}}-4\tilde{a}(X,y)\tilde{a}(X,u)\tilde{a}(v,y)\Bigg].\end{aligned} \tag{3.9}$$

Therefore, for all $z\in\mathfrak{g}$ we have:

$$\begin{aligned}g_y(y,[y,z]) =&\frac{\tilde{a}(X,y)^3}{\Big(\tilde{a}(X,y)-\tilde{a}(y,y)^{1/2}\Big)^4}\Bigg[\tilde{a}(X,[y,z])\Big(\tilde{a}(X,y)^2-4\tilde{a}(y,y)^{3/2}\\&+\tilde{a}(X,y)\tilde{a}(y,y)^{1/2}+2\tilde{a}(y,y)\Big)+\tilde{a}(y,[y,z])\Big(\frac{\tilde{a}(X,y)^2}{\tilde{a}(y,y)^{1/2}}-\tilde{a}(X,y)\Big)\Bigg],\end{aligned}$$

which is equal to

$$\begin{aligned}g_y(y,[y,z]) &= \tilde{a}(MNX,[y,z])+\tilde{a}(MPy,[y,z])\\&= \tilde{a}(MNX+MPy,[y,z]),\end{aligned} \tag{3.10}$$

where

$$M=\frac{\tilde{a}(X,y)^3}{\Big(\tilde{a}(X,y)-\tilde{a}(y,y)^{1/2}\Big)^4},$$

$$N=\tilde{a}(X,y)^2-4\tilde{a}(y,y)^{3/2}+\tilde{a}(X,y)\tilde{a}(y,y)^{1/2}+2\tilde{a}(y,y),$$

$$P=\frac{\tilde{a}(X,y)^2}{\tilde{a}(y,y)^{1/2}}-\tilde{a}(X,y).$$

Now, by using Lemma 2.3 and equation (3) a vector $y=\sum_{i=1}^4 y_ie_i$ of $\mathfrak{g}$ is a geodesic vector if and only if for each $j=1,2,3,4$

$$\tilde{a}\Big(MN\sum_{i=1}^4 x_ie_i+MP\sum_{i=1}^4 y_ie_i,[\sum_{i=1}^4 y_ie_i,e_j]\Big)=0, \tag{3.11}$$

where

$$M=\frac{(\sum_{i=1}^4 x_iy_i)^3}{\Big(\sum_{i=1}^4 x_iy_i-(\sum_{i=1}^4 y_i^2)^{1/2}\Big)^4},$$

$$N=(\sum_{i=1}^4 x_iy_i)^2-4(\sum_{i=1}^4 y_i^2)^{3/2}+\sum_{i=1}^4 x_iy_i(\sum_{i=1}^4 y_i^2)^{1/2}+2\sum_{i=1}^4 y_i^2,$$

and

$$P = \frac{(\sum_{i=1}^{4} x_i y_i)^2}{(\sum_{i=1}^{4} y_i^2)^{1/2}} - \sum_{i=1}^{4} x_i y_i.$$

So we get the following cases:

3.1. **Case (1).**

$$\begin{cases} j = 2 \ \rightarrow \quad MN(x_3y_4 - x_4y_3) = 0, \\ j = 3 \ \rightarrow \quad MN(x_4y_2 - x_2y_4) = 0, \\ j = 4 \ \rightarrow \quad MN(x_2y_3 - x_3y_2) = 0. \end{cases}$$

As a special case, if $X = x_1e_1$, then a vector y of $\mathfrak{g}$ is a geodesic vector if and only if $y \in Span\{e_1\}$.

Corollary 3.3. *Let (M, F) be a Finsler space with infinite series metric F defined by an invariant metric $\tilde{a}$ and an invariant vector field $X = \sum_{i=1}^{4} x_ie_i$ on left invariant hypercomplex 4-dimensional simply connected Lie group and let (3.5) holds. Then geodesic vectors depending on x_2, x_3 and x_4.*

Theorem 3.4. *Let (M, F) be a Finsler space with infinite series metric F defined by an invariant metric $\tilde{a}$ and an invariant vector field $X = x_1e_1$ on left invariant hypercomplex 4-dimensional simply connected Lie group and let (3.5) holds. Then $y \in \mathfrak{g}$ is a geodesic vector of (M, F) if and only if y is a geodesic vector of $(M, \tilde{a})$.*

Proof. Let $y \in \sum_{i=1}^{4} y_ie_i \in \mathfrak{g}$. Let y is a geodesic vector of $(M, \tilde{a})$. By using (2.5) we have

$$\tilde{a}(y, [y, e_i]) = 0, \quad \text{for each } i = 1, 2, 3, 4.$$

Therefore by using (3.11), y is a geodesic of (M, F).

Conversely, let

$$y = \sum_{i=1}^{5} y_ie_i \in \mathfrak{g}$$

be a geodesic vector of (M, F), because $\tilde{a}(X, [y, e_i]) = 0$ for each $i = 1, 2, 3, 4$, by using (3.11) we have

$$\tilde{a}(y, [y, e_i]) = 0.$$

This completes the proof. □

3.2. **Case (2).**

$$\begin{cases} j = 1 \ \rightarrow \quad MNx_1y_3 + MPy_1y_3 + MNx_2y_4 + MPy_2y_4 = 0, \\ j = 2 \ \rightarrow \quad MNx_1y_4 + MPy_1y_4 - (MNx_2y_3 + MPy_2y_3) = 0, \\ j = 3 \ \rightarrow \quad MNx_1y_1 + MPy_1^2 + MNx_2y_2 + MPy_2^2 = 0, \\ j = 4 \ \rightarrow \quad MN(x_2y_1 - x_1y_2) = 0. \end{cases}$$

As a special case, if $X = x_3e_3 + x_4e_4$, then a vector y of $\mathfrak{g}$ is a geodesic vector if and only if $y \in Span\{e_3, e_4\}$.

Corollary 3.5. *Let (M, F) be a Finsler space with infinite series metric F defined by an invariant metric $\tilde{a}$ and an invariant vector field $X = \sum_{i=1}^{4} x_ie_i$ on left invariant hypercomplex 4-dimensional simply connected Lie group and let (3.6) holds. Then geodesic vectors depending on x_1 and x_2.*

Theorem 3.6. *Let (M, F) be a Finsler space with infinite series metric F defined by an invariant metric $\tilde{a}$ and an invariant vector field $X = x_3e_3 + x_4e_4$ on left invariant hypercomplex 4-dimensional simply connected Lie group and let (3.6) holds. Then $y \in \mathfrak{g}$ is a geodesic vector of (M, F) if and only if y is a geodesic vector of $(M, \tilde{a})$.*

Proof. The proof is the same as before. □

3.3. Case (3).

$$\begin{cases} j = 1 \;\rightarrow & MN(x_2y_2 + x_3y_3 + x_4y_4) + MP(y_2^2 + y_3^2 + y_4^2) = 0, \\ j = 2 \;\rightarrow & MNx_2y_1 + MPy_2y_1 = 0, \\ j = 3 \;\rightarrow & MNx_3y_1 + MPy_3y_1 = 0, \\ j = 4 \;\rightarrow & MNx_4y_1 + MPy_4y_1 = 0. \end{cases}$$

As a special case, if $X = x_1e_1$, then a vector y of $\mathfrak{g}$ is a geodesic vector if and only if $y \in Span\{e_1\}$.

Corollary 3.7. *Let (M, F) be a Finsler space with infinite series metric F defined by an invariant metric $\tilde{a}$ and an invariant vector field $X = \sum_{i=1}^{4} x_ie_i$ on left invariant hypercomplex 4-dimensional simply connected Lie group and let (3.7) holds. Then geodesic vectors depending on x_2, x_3 and x_4.*

Theorem 3.8. *Let (M, F) be a Finsler space with infinite series metric F defined by an invariant metric $\tilde{a}$ and an invariant vector field $X = x_1e_1$ on left invariant hypercomplex 4-dimensional simply connected Lie group and let (3.7) holds. Then $y \in \mathfrak{g}$ is a geodesic vector of (M, F) if and only if y is a geodesic vector of $(M, \tilde{a})$.*

Proof. The proof is the same as before. □

3.4. Case (4).

$$\begin{cases} j = 1 \;\rightarrow & MN(2x_2y_2 + x_3y_3 + x_4y_4) + MP(2y_2^2 + y_3^2 + y_4^2) = 0, \\ j = 2 \;\rightarrow & MNx_2y_1 + MPy_2y_1 = 0, \\ j = 3 \;\rightarrow & MN(x_3y_1 - x_2y_4) + MP(y_3y_1 - y_2y_4) = 0, \\ j = 4 \;\rightarrow & MN(x_2y_3 + x_4y_1) + MP(y_4y_1 + y_2y_3) = 0. \end{cases}$$

As a special case, if $X = x_1e_1$, then a vector y of $\mathfrak{g}$ is a geodesic vector if and only if $y \in Span\{e_1\}$.

Corollary 3.9. *Let (M, F) be a Finsler space with infinite series metric F defined by an invariant metric $\tilde{a}$ and an invariant vector field $X = \sum_{i=1}^{4} x_i e_i$ on left invariant hypercomplex 4-dimensional simply connected Lie group and let (3.8) holds. Then geodesic vectors depending on x_2, x_3 and x_4.*

Theorem 3.10. *Let (M, F) be a Finsler space with infinite series metric F defined by an invariant metric $\tilde{a}$ and an invariant vector field $X = x_1 e_1$ on left invariant hypercomplex 4-dimensional simply connected Lie group and let (3.8) holds. Then $y \in \mathfrak{g}$ is a geodesic vector of (M, F) if and only if y is a geodesic vector of $(M, \tilde{a})$.*

Proof. The proof is the same as before. □

References

1. P. Bahmandoust, D. Latifi, *Naturally reductive homogeneous (α, β)-spaces*, Int. J. Geom. Methods Mod. Phys., **17**(8) (2020), 2050117.
2. D. Bao, S. S. Chern, Z. Shen, *An introduction to Riemann-Finsler geometry*, Springer-Verlage, 2000.
3. M. L. Barberis, *Hypercomplex structures on four-dimensional Lie groups*, Proc. Am. Math. Soc., **125**(1997), 1043-1054.
4. M. Ebrahimi, D. Latifi, *On Flag Curvature and Homogeneous Geodesics of Left Invariant Randers Metrics on the Semi-Direct Product $a \oplus_p r$*, Journal of Lie Theory, **29**(2019), 619-627.
5. S. Kobayashi and K. Nomizu, *Foundations of differential Geometry*, Interscience Publishers, (1969).
6. O. Kowalski, J. Szenthe, *On the Existence of Homogeneous Geodesics in Homogeneous Riemannian manifolds*, Geom. Dedicata, **81**(2000), 209-214.
7. D. Latifi,*Homogeneous geodesics in homogeneous Finsler spaces*, J. Geom. Phys., **57**(2007), 1421–1433.
8. D. Latifi, M. Toomanian, *Invariant naturally reductive Randers metrics on homogeneous spaces*, Math Sci, **6**(63) (2012), 1-5.
9. D. Latifi, M. Toomanian, *On the existence of bi-invariant Finsler metrics on Lie groups*, Math Sci, **7**(37) (2013), 1-5.
10. D. Latifi, *Bi-invariant (α, β)-metrics on Lie groups*, Acta Universitatis Apulensis, **65**, (2021), 121-131.
11. M. Matsumoto, *Theory of Finsler spaces with (α, β)-metric*, Rep. Math. Phys., **31**(1992), 33–65.
12. M. Obata, *Affine connections on manifolds with almost complex, quaternion or Hermitian structure*, Jap. J. Math, **26**(1956), 43-79.
13. M. Parhizkar, D. Latifi, *Geodesic vectors of Randers metrics on nilpotent Lie groups of dimension five*, Global. J. Adv. Res. Class. Moder. Geom, **7**(2018), 92-101.
14. M. L. Zeinali, *On generalized symmetric Finsler spaces with some special (α, β) -metrics*, Journal of Finsler Geometry and its Applications, **1**(1), (2020), 45-53.

Received: 12.11.2023
Accepted: 24.12.2023