Journal of Hyperstructures 5 *(2) (2016), 141-150.*
ISSN: 2322-1666 print/2251-8436 online

GENERATING THE GENERALIZED DISTRIBUTIONS BY USING FRACTIONAL CALCULUS

MASOUD GANJI AND FATEMEH GHARARI

Abstract. In this paper, we generalized some statistical distributions by using fractional calculus. The domain of parameter space for these distributions was expanded and their PDFs also presented as a product of fractional derivation of Dirac delta function of shape parameter order.

Key Words: Fractional calculus, Dirac delta function, Burr distribution, Beta type II distribution, Beta type III distribution.

2010 Mathematics Subject Classification: 60E05, 26A33, 46F25.

1. Introduction

Fractional calculus deals with the study of fractional order integral and derivatives and their diverse applications ([4], [5], [6], [7], [8]). Fractional calculus has widespread applications in different fields of science and engineering. The applications of fractional calculus in statistics will be ideal.

In this work, we follow our previous works, [1], [2] and [3], about applications of fractional calculus in statistics and present another application of fractional calculus in statistics. We know from a classic view point, continuous statistical distributions are not defined for improper values of parameter space, since, in such a case, nonintegrable functions will appear in the PDF formula. The Laplace and Fourier transforms of the PDF, which are the moment generating function and characteristic

Received: 4 September 2016, Accepted: 11 November 2016. Communicated by Behrouz Fathi-Vajargah ;

∗Address correspondence to M. Ganji; E-mail: mganji@uma.ac.ir.

function, respectively, are given by divergent integrals. However, the introduction of the generalized continuous random variable ($GCRV$) allow us to analytic continuation for the parameter. This approach is equivalent to generalized function approach. In this case, divergent integrals are called finite in partly integrals. Also, we must keep in mind that in applications the ordinary formulas of the PDF in its classical sense hold only for proper ordinary values of the parameter space and in generalized functions sense, for extended values of the parameter space. Therefore, formulation of an applicable problem and interpretation of its obtained results must be performed using the meaning of generalized functions.

We generalize some statistical distributions by having, generalized fractional integral and derivative operators which are fractional integral and derivatives of generalized functions, .
Generalized functions are defined as a linear functional on a space X of conveniently chosen test functions. For every locally integrable function $f \in \mathcal{L}^1_{loc}(\mathbb{R})$, there exists a distribution $F_f : X \to \mathbb{C}$ defined by

$$F_f(\varphi) = \langle f, \varphi \rangle = \int_{-\infty}^{\infty} f(x)\varphi(x)dx \tag{1.1}$$

where $\varphi \in X$ is test function from a suitable space X of test functions. A distribution that corresponds to functions via equation (1.1) are called regular distributions. Examples for regular distributions are the convolution kernels $K^\alpha_\pm \in \mathcal{L}^1_{loc}(\mathbb{R})$ defined as

$$K^\alpha_\pm = H(\pm x)\frac{\pm x^{\alpha-1}}{\Gamma(\alpha)} \tag{1.2}$$

for $\alpha > 0$ and $H(x)$ is a Heaviside unit step function. Distributions that are not regular are sometimes called singular. An example for a singular distribution is the Dirac delta function which is defined as $\delta : X \to \mathbb{C}$ by

$$\int \delta(x)\varphi(x)dx = \varphi(0) \tag{1.3}$$

for every test function $\varphi \in X$. The test function space X is usually chosen as a subspace of $C^\infty(\mathbb{R})$, the space of infinitely differentiable functions [4].
In the present work, like to our recent work, [3], we suppose that X is

a positive continuous random variable and α is the parameter of distribution. The new random variable is represented as the function $\Phi_\alpha(x)$ and defined by $\Phi_\alpha(x) = \frac{X_+^{\alpha-1}}{\Gamma(\alpha)}$. The function $\Phi_\alpha(x)$ can be extended to all complex values of α as a pseudo function and is a distribution whose support is $[0,\infty)$ except for the case $\alpha = 0, -1, \ldots$. The generalized functions in mathematical analysis are not really functions in the classical sense, for such reason we call our proposed random variable a generalized continuous random variable ($GCRV$). In the special case, when $\alpha = 2$, it becomes ordinary continuous random variable X.
The expectation of this generalized random variable coincides with Riemann-Liouville left fractional integral of the PDF, at the origin, for $\alpha > 0$ and Marchaud fractional derivative of the PDF, at the origin, for $-1 < \alpha < 0$, i.e.

$$E[\Phi_\alpha(X)] = \begin{cases} (I_-^\alpha f)(0), & \alpha > 0 \\ \\ (\mathbf{D}_-^\alpha f)(0), & -1 < \alpha < 0 \end{cases} \tag{1.4}$$

where

$$(I_-^\alpha f)(x) = \frac{1}{\Gamma(\alpha)} \int_0^\infty t^{\alpha-1} f(x+t)dt \tag{1.5}$$

is the Riemann-Liouville left fractional integral, while

$$(\mathbf{D}_-^\alpha f)(x) = \frac{1}{\Gamma(-\alpha)} \int_0^\infty t^{-\alpha-1} \{f(x+t) - f(x)\}dt \tag{1.6}$$

is the Marchaud fractional derivative [3].

Some distributions like Weibull, Gamma and Beta type I distributions have been generalized in [3]. Here, we generalize some distributions like Burr, Beta type II and Beta type III distributions. Like our previous work, by this generalized random variable, we more expand the domain of shape parameter space. For example, the domain of shape parameter which has been already expanded for Burr distribution from $(0,\infty)\times(0,\infty)$ to $(-1,\infty)\times(0,\infty)$ and for Beta type II and type III distributions from $(0,\infty)\times(0,\infty)$ to $(-1,\infty)\times(-1,\infty)$. In case of negative values of the shape parameter space, the relationship between fractional derivatives and statistics theory can be obtained, this means we can write the PDF of distributions like Gamma, Weibull, Beta as a product of fractional derivatives of Dirac delta function. Another property of the

$GCRV$ is

$$\Phi_{\alpha}(x-a) \,_* \, \Phi_{\beta}(x) = \Phi_{\alpha+\beta}(x-a), \tag{1.7}$$

where $\alpha > -1$ and $\beta > -1$ such that $\alpha + \beta > -1$ and we used the star notation for the convolution operation.
The proof is easy for $\alpha > 0$ and $\beta > 0$, as instance, see [8]. Other values of α and β can be proved by using analytic continuation. Also, we have

$$(I_0^{\alpha}\Phi_{\beta}(t))(x) = \Phi_{\beta+\alpha}(x), \quad \alpha > 0, \beta > -1 \tag{1.8}$$

and

$$(D_0^{\alpha}\Phi_{\beta}(t))(x) = \Phi_{\beta-\alpha}(x), \quad \alpha \geq 0, \beta > -1. \tag{1.9}$$

that is, integrals and derivatives of order $\alpha > 0$ of the $GCRV$ is the $GCRV$.

2. Preliminaries

In this section, we introduce notations, definitions and preliminary facts which are used throughout this paper. We need some basic definitions and properties of the fractional calculus theory and the generalized functions theory which are used further in this paper. As mentioned in references [5] and [6], the definitions 1 and 2 of fractional calculus are as following:

Definition 1. For a function f defined on an interval $[a, b]$, the Riemann-Liouville (R-L) integrals $I_{a+}^{\alpha} f$ and $I_{b-}^{\alpha} f$ of order $\alpha \in \mathbb{C}$, $(\mathcal{R}(\alpha) > 0)$ are defined by

$$(I_{a+}^{\alpha} f)(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t-\xi)^{\alpha-1} f(\xi) d\xi \tag{2.1}$$

and

$$(I_{b-}^{\alpha} f)(t) = \frac{1}{\Gamma(\alpha)} \int_t^b (\xi-t)^{\alpha-1} f(\xi) d\xi \tag{2.2}$$

respectively. Also, the left and right R-L fractional derivations $D_{a+}^{\alpha} f$ and $D_{b-}^{\alpha} f$ of order $\alpha \in \mathbb{C}$, $(\mathcal{R}(\alpha) > 0)$ are defined by

$$(D_{a+}^{\alpha} f)(t) = (\frac{d}{dt})^n (I_{a+}^{n-\alpha} f)(t), \tag{2.3}$$

and

$$(D_{b-}^{\alpha} f)(t) = (\frac{-d}{dt})^n (I_{a+}^{n-\alpha} f)(t). \tag{2.4}$$

respectively.

Definition 2. Let f be a generalized function $f \in C_0^\infty(\mathbb{R})'$, (namely the dual space of the vector space $C_0^\infty(\mathbb{R})$) with $supp f \subset \mathbb{R}^+$. Then its fractional integral is the distribution $I_{0+}^\alpha f$ defined as:

$$\langle I_{0+}^\alpha f, \varphi\rangle = \langle I^\alpha f, \varphi\rangle = \langle K_+^\alpha * f, \varphi\rangle \tag{2.5}$$

for $\mathcal{R}(\alpha) > 0$ [4]. Also, the fractional derivative of order α with lower limit 0 is the distribution $D^\alpha\{f(z)\}$defined as:

$$\langle D_{0+}^\alpha f, \varphi\rangle = \langle D^\alpha f, \varphi\rangle = \langle K_+^{-\alpha} * f, \varphi\rangle \tag{2.6}$$

where $\alpha \in \mathbb{C}$ and

$$K_+^\alpha(x) = \begin{cases} H(x)\frac{x^{\alpha-1}}{\Gamma(\alpha)}, & \mathcal{R}(\alpha) > 0 \\ \frac{d^n}{dx^n}[H(x)\frac{x^{\alpha+n-1}}{\Gamma(\alpha+n)}], & \mathcal{R}(\alpha) + n > 0; n \in \mathbb{N} \end{cases} \tag{2.7}$$

is the kernel distribution. For $\alpha = 0$ one can finds $K_+^0(x) = (\frac{d}{dx})H(x) = \delta(x)$ and $D_{0+}^0 = I$ as the identity operator. For the $\alpha = -n;\ n \in \mathbb{N}$, we have

$$K_+^{-n}(x) = \delta^{(n)}(x) \tag{2.8}$$

where $\delta^{(n)}$ is the n^{th} derivative of the δ distribution. The kernel distribution in equation (2.7) is given by:

$$K_+^\alpha(x) = \frac{d}{dx}[H(x)\frac{x^{-\alpha}}{\Gamma(1-\alpha)}] = \frac{d}{dx}K_+^{1-\alpha}(x) \tag{2.9}$$

for $0 < \alpha < 1$.

Now if $f \in C_0^\infty(\mathbb{R})'$ with $supp f \subset \mathbb{R}^+$, then

$$D_{0+}^\alpha f = D_{0+}^\alpha(If) = (K_+^{-\alpha} * K_+^0) * f = \delta^{(\alpha)} * f \tag{2.10}$$

for all $\alpha \in \mathbb{C}$. Also, the differentiation rule

$$D_{0+}^\alpha K_+^\beta = K_+^{\beta-\alpha} \tag{2.11}$$

holds for all β and $\alpha \in \mathbb{C}$. It contains

$$DK_+^\beta = K_+^{\beta-1}$$

for all $\beta \in \mathbb{C}$ as a special case [4].

3. The generalized Burr distribution

In probability and statistics theory, the Burr distribution is a continuous probability distribution for a non-negative random variable. It is also known as the Singh?Maddala distribution and is one of a number of different distributions which is sometimes called the generalized log-logistic distribution. It is most commonly used to model household income. The Burr distribution has probability density function

$$f_X(x) = \alpha\beta \frac{x^{\alpha-1}}{(1+x^\alpha)^{\beta+1}} \tag{3.1}$$

and its cumulative distribution function is

$$F_X(x) = 1 - (1+x^\alpha)^{-\beta} \tag{3.2}$$

such that $x > 0$, $\alpha > 0$ and $\beta > 0$. The PDF of the Burr distribution in terms of the $GCRV$ can be written as

$$f_X(x) = \frac{\beta\Gamma(\alpha+1)D_x\Phi_{\alpha+1}(x)}{(1+\Gamma(\alpha+1)\Phi_{\alpha+1}(x))^{\beta+1}}. \tag{3.3}$$

The PDF of this distribution by using equation (2.11) with $\beta = 0$ and equation (2.7), can be generalized as following definition.

Definition 3. Suppose that X be a random variable of Burr distribution as a two-parameter family of distribution functions, in which the parameter α and β are the shape parameters. The PDF of distribution of X can be defined as

$$f(x) = \begin{cases} \frac{\beta\Gamma(\alpha+1)K_+^\alpha(x)}{(1+\Gamma(\alpha+1)K_+^{\alpha+1}(x))^{\beta+1}}, & \alpha > 0, \\ \\ \frac{\beta\Gamma(\alpha+1)\delta^{(\alpha)}(x)}{(1+\Gamma(\alpha+1)K_+^{\alpha+1}(x))^{\beta+1}}, & -1 < \alpha \leq 0. \end{cases} \tag{3.4}$$

Now, we should show that $\int_0^\infty f_X(x)dx = 1$ for $-1 < \alpha \leq 0$.

$$\int_0^\infty f_X(x)dx = \beta\Gamma(\alpha+1)\int_0^\infty \frac{\delta^{(\alpha)}(x)}{(1+\Gamma(\alpha+1)K_+^{\alpha+1}(x))^{\beta+1}}dx, \tag{3.5}$$

or by using equation (1.8), we rewrite the above equation as following

$$= \beta\Gamma(\alpha+1)\int_0^\infty \frac{\delta^{(\alpha)}(x)}{(1+\Gamma(\alpha+1)I_0^1(\delta^{(\alpha)}(x)))^{\beta+1}}dx, \tag{3.6}$$

now, by formal substitution ?$u = 1 + \Gamma(\alpha+1)I_0^1(\delta^{(\alpha)}(x))$ which gives the result.

So, we succeed to represent the PDF of Burr distribution as a product of fractional derivation of Dirac delta function of shape parameter order. Morever, the parameter space extended from $(0,\infty)\times(0,\infty)$ to $(-1,\infty)\times(0,\infty)$.

4. The generalized Beta II distribution

The random variable X is said to have a Beta II distribution with parameters (α,β), $\alpha>0$ and $\beta>0$ and denoted as $X\sim B^{II}(a,b)$ if its probability density function is given by

$$f_X(x)=B(a,b)^{-1}x^{a-1}(1+x)^{-(a+b)},\qquad x>0. \tag{4.1}$$

The PDF of the Beta II distribution in terms of the $GCRV$ can be written as

$$f_X(x)=\frac{D_x\Phi_{a+1}(x)}{D_y\Phi_{a+b+1}(y)\Gamma(b)(a+b)},\qquad y=1+x \tag{4.2}$$

The PDF of this distribution by using equation (2.11) with $\beta=0$ and equation (2.7), can be generalized as following definition.

Definition 4. Suppose that X be a random variable of Beta II distribution as a tow-parameter family of distribution functions, in which the parameter a and b are the shape parameters. The PDF of distribution can be defined as

$$f(x)=\begin{cases}\frac{K_+^a(x)}{\Gamma(b)(a+b)K_+^{a+b}(y)}, & a>0,\ b>0,\\ \frac{\delta^{(a)}(x)}{\Gamma(b)(a+b)K_+^{a+b}(y)}, & -1<a\le 0,\ b>0,\ a+b>0,\\ \frac{\delta^{(a)}(x)}{\Gamma(b)(a+b)\delta^{(a+b)}(y)}, & -1<a\le 0,\ b>0,\ -1<a+b\le 0.\end{cases} \tag{4.3}$$

Now, we are showing that $\int_0^\infty f_X(x)dx=1$ for $-1<a\le 0$, $b>0$ and $a+b>0$. For this purpose, by the substitution $u=(1+x)^{-1}$ the integral

$$\int_0^\infty x^{a-1}(1+x)^{-(a+b)}dx \tag{4.4}$$

transforms to

$$\int_0^1 x^{b-1}(1-x)^{a-1}dx \tag{4.5}$$

then for this distribution, the form $\int_0^\infty f_X(x)dx$ in terms of the $GCRV$ will be as following

$$\int_0^\infty f_X(x)dx = \Gamma(a+b)\int_0^1 \delta^{(a)}(y)\frac{x^{b-1}}{\Gamma(b)}dx, \tag{4.6}$$

Now, by using the equation (1.7) with $a = 0$ and $t = 0$, we have

$$= \Gamma(a+b)\int_0^1 \frac{x^{b-1}}{\Gamma(b)}.\frac{(1-x)^{a-1}}{\Gamma(a)}dx, \; = \Gamma(a+b).\Gamma^{-1}(a+b), \tag{4.7}$$

which proves the result. This result is obtained with a similar way as in the case $-1 < a \leq 0$, $b > 0$ and $-1 < a + b \leq 0$.

So, we represent the PDF of Beta II distribution as a product of fractional derivation of Dirac delta function of shape parameter order. Moreover , the parameter space extended from $(0,\infty)\times(0,\infty)$ to $(-1,\infty)\times(-1,\infty)$.

5. The generalized Beta III distribution

The random variable X is said to have a Beta III distribution with parameters (a, b), $a > 0$ and $b > 0$ and denoted as $X \sim B^{III}(a, b)$ if its probability density function is given by

$$f_X(x) = 2^a B(a,b)^{-1} x^{a-1}(1-x)^{b-1}(1+x)^{-(a+b)}, \quad 0 < x < 1. \tag{5.1}$$

The PDF of the Beta III distribution in terms of the $GCRV$ can be written as

$$f_X(x) = \frac{2^a D_x\Phi_{a+1}(x)D_y\Phi_{b+1}(y)}{(a+b)D_z\Phi_{a+b+1}(z)}, \quad y = 1-x, \; z = 1+x. \tag{5.2}$$

The PDF of this distribution by using equation (2.11) with $\beta = 0$ and equation (2.7), can be generalized as following definition.

Definition 5. Suppose that X be a random variable of Beta III distribution as a two-parameter family of distribution functions, in which the parameter a and b are the shape parameters. The PDF of distribution can be defined as

$$f(x)=\begin{cases}\frac{2^a K_+^a(x)K_+^b(y)}{(a+b)K_+^{a+b}(z)}, & a>0,\ b>0,\\ \frac{2^a \delta^{(a)}(x)K_+^b(y)}{(a+b)K_+^{a+b}(z)}, & -1<a\leq 0,\ \ b>0,\ \ a+b>0,\\ \frac{2^a \delta^{(a)}(x)K_+^b(y)}{(a+b)\delta^{(a+b)}(z)}, & -1<a\leq 0,\ \ b>0,\ \ -1<a+b\leq 0,\\ \frac{2^a K_+^a(x)\delta^{(b)}(y)}{(a+b)K_+^{a+b}(z)}, & a>0,\ \ a+b>0,\ \ -1<b\leq 0,\\ \frac{2^a K_+^a(x)\delta^{(b)}(y)}{(a+b)\delta^{(a+b)}(z)}, & a>0,\ \ -1<a+b\leq 0,\ \ -1<b\leq 0.\end{cases}$$

Now, we are showing that for $-1<\alpha\leq 0$, $b>0$ and $a+b>0$ we have $\int_0^\infty f_X(x)dx=1$. For this purpose, frist by the substitution, $u=(1-x)(1+x)^{-1}$ the integral

$$\int_0^1 2^a x^{a-1}(1-x)^{b-1}(1+x)^{-(a+b)}dx \tag{5.3}$$

transforms to

$$\int_0^1 x^{b-1}(1-x)^{a-1}dx \tag{5.4}$$

then for this distribution, the integral $\int_0^\infty f_X(x)dx$ in terms of the $GCRV$ will be as following

$$\int_0^\infty f_X(x)dx=\Gamma(a+b)\int_0^1 \delta^{(a)}(y)\frac{x^{b-1}}{\Gamma(b)}dx, \tag{5.5}$$

now, by using the equation (1.7) with $a=0$ and $t=0$ we have

$$=\Gamma(a+b)\int_0^1 \frac{x^{b-1}}{\Gamma(b)}.\frac{(1-x)^{a-1}}{\Gamma(a)}dx,\ =\Gamma(a+b).\Gamma^{-1}(a+b), \tag{5.6}$$

which is the result. This result obtained with a similar way as in the other cases.

So, we represent the PDF of Beta III distribution as a product of fractional derivation of Dirac delta function of shape parameter order. Also, the parameter space extended from $(0,\infty)\times(0,\infty)$ to $(-1,\infty)\times(-1,\infty)$.

6. Conclusion

In this paper, by rewriting some distributions like the Burr, Beta type II and Beta type III distributions in terms of the $GCRV$, we generalized them, i.e. the domain of parameter space for these distributions expanded and their PDFs also presented as a product of fractional derivation of Dirac delta function of shape parameter order.

Acknowledgments
The authors are thankful to the referee of this paper.

References

[1] M. Ganji, N. Eghbali and F. Gharari, *Some Fractional Special Functions and Fractional Moments*, Gen. Math. Notes. **18(2)**(2013) 120-127.
[2] M. Ganji and F. Gharari, *The Generalized Fourier Transform for the Generation of Complex Fractional Moments*, International Journal of Mathematical Archive (IJMA), **5(1)**(2014) 293-299.
[3] M. Ganji and F. Gharari, *The Generalized Random Variable Appears the Trace of Fractional Calculus in Statistics*, Applied Mathematics Information Sciences Letters, **3 (2)**(2015) 61-67.
[4] R. Hilfer, *Threefold introduction to fractional derivatives*, Wiley-VCH, Weinheim, (2008).
[5] A. Kilbas, M. H. Srivastava and J. J. Trujillo, *Theory and Application of Fractional Differential Equations,* North Holland Mathematics Studies 204, (2006).
[6] K.S. Miller and B. Ross, *An Introduction to the Fractional Calculus and Fractional Differential Equations*, John Wiley and Sons, New York, (1993).
[7] K. Oldham and J. Spanier *The Fractional Calculus, Theory and Application of Differentiation and Integration to Arbitrary Orders*, Academic Press, San Diego, CF,(1974).
[8] I. Podlubny, *Fractional differential equations*, San Diego: Academic Press, (1999).

Masoud Ganji
Department of Statistics, University of University of Mohaghegh Ardabili, P.O.Box 56199-11367, Ardabili, Iran
Email: mganji@uma.ac.ir

Fatemeh Gharari
Department of Statistics, University of University of Mohaghegh Ardabili, P.O.Box 56199-11367, Ardabili, Iran
Email: f.gharari@uma.ac.ir