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ON FUZZY PRIME AND FUZZY SEMIPRIME IDEALS OF ≤–HYPERGROUPOIDS

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Abstract. We deal with an hypergroupoid endowed with a relation denoted by "≤", we call it ≤–hypergroupoid. We prove that a nonempty subset A of a ≤–hypergroupoid H is a prime (resp. semiprime) ideal of H if and only if the characteristic function f_A is a fuzzy prime (resp. fuzzy semiprime) ideal of H.

Key Words: Hypergroupoid, ≤–Hypergroupoid, Left Ideal, Fuzzy Left Ideal, Prime (Semiprime) Ideal, Fuzzy Prime (Fuzzy Semiprime) Ideal.

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1. Introduction

A characterization of prime and semiprime ideals of groupoids in terms of fuzzy subsets has been considered in [1], and similar characterizations hold for ordered groupoids in general. Fuzzy sets in ordered groupoids have been first considered in [2]. In the present paper we examine the results in [1] in case of an hypergroupoid H endowed with a relation denoted by "≤" (not an ordered relation, as so no compatible with the multiplication of H in general). As a consequence, our results hold for ordered hypergroupoids as well.

An *hypergroupoid* is a nonempty set H with an hyperoperation

$$\circ : H \times H \to \mathcal{P}^*(H) \mid (a, b) \to a \circ b$$

on H and an operation

$$* : \mathcal{P}^*(H) \times \mathcal{P}^*(H) \to \mathcal{P}^*(H) \mid (A, B) \to A * B$$

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on $\mathcal{P}^*(H)$ (induced by the operation of H) such that

$$A * B = \bigcup_{(a,b)\in A\times B} (a \circ b)$$

for every $A, B \in \mathcal{P}^*(H)$ ($\mathcal{P}^*(H)$ is the set of all nonempty subsets of H). An hypergroupoid can be also denoted by $(H, \circ)$ as the operation "$*$" depends on "$\circ$". A nonempty subset A of an hypergroupoid H is called a *left* (resp. *right*) *ideal* of H if $H * A \subseteq A$ (resp. $A * H \subseteq A$). It is called an ideal of H if it is both a left and a right ideal of H. If H is an hypergroupoid then, for every $x, y \in H$, we have $\{x\} * \{y\} = x \circ y$. The following proposition, though clear, plays an essential role in the theory of hypergroupoids.

Proposition 1. *Let $(H, \circ)$ be an hypergroupoid, $x \in H$ and $A, B \in \mathcal{P}^*(H)$. Then we have the following:*
(1) *$x \in A * B \Longleftrightarrow x \in a \circ b$ for some $a \in A$, $b \in B$.*
(2) *If $a \in A$ and $b \in B$, then $a \circ b \subseteq A * B$.*

Proposition 2. *Let $(H, \circ)$ be an hypergroupoid. If A is a left (resp. right) ideal of H, then for every $h \in H$ and every $a \in A$, we have $h \circ a \subseteq A$ (resp. $a \circ h \subseteq A$). "Conversely", if A is a nonempty subset of H such that $h \circ a \subseteq A$ (resp. $a \circ h \subseteq A$) for every $h \in H$ and every $a \in A$, then the set A is a left (resp. right) ideal of H.*

2. Main results

Definition 3. By a *≤–hypergroupoid* we mean an hypergroupoid H endowed with a relation denoted by "$\leq$".

Definition 4. Let H be a ≤–hypergroupoid. A nonempty subset A of H is called a *left* (resp. *right*) *ideal* of H if
(1) $H * A \subseteq A$ (resp. $A * H \subseteq A$) and
(2) if $a \in A$ and $H \ni b \leq a$, then $b \in A$.
A subset of H which is both a left ideal and a right ideal of H is called an *ideal* of H. A nonempty subset A of H is called a *subgroupoid* of H if $A * A \subseteq A$.
Clearly, every left ideal, right ideal or ideal of H is a subgroupoid of H.

Definition 5. Let H be an hypergroupoid (or a ≤–hypergroupoid). A nonempty subset I of H is called a *prime subset* of H if
(1) $a, b \in H$ such that $a \circ b \subseteq I$ implies $a \in I$ or $b \in I$ and
(2) if $a, b \in H$, then $a \circ b \subseteq I$ or $(a \circ b) \cap I = \emptyset$.
The following are equivalent:

(1) $a, b \in H$, $a \circ b \subseteq I \Longrightarrow a \in I$ or $b \in I$.
(2) $\emptyset \neq A, B \subseteq H$, $A * B \subseteq I \Longrightarrow A \subseteq I$ or $B \subseteq I$.
Indeed: (1) $\Longrightarrow$ (2). Let $A, B \in \mathcal{P}^*(H)$, $A * B \subseteq I$ and $A \nsubseteq I$. Let $a \in A$ such that $a \notin I$ and $b \in B$. We have $a \circ b \subseteq A * B \subseteq I$. Then, by (1), $a \in I$ or $b \in I$.
(2) $\Longrightarrow$ (1). Let $a, b \in H$, $a \circ b \subseteq I$. Then $\{a\} * \{b\} = a \circ b \subseteq I$. By (2), we have $\{a\} \subseteq I$ or $\{b\} \subseteq I$, so $a \in I$ or $b \in I$.

By a prime ideal of H we clearly mean an ideal of H which is at the same time a prime subset of H.

Following Zadeh, any mapping $f : H \to [0, 1]$ of a $\leq$–hypergroupoid H into the closed interval $[0, 1]$ of real numbers is called a *fuzzy subset* of H or a (*fuzzy set* in H) and f_A (: the characteristic function of A) is the mapping

$$f_A : H \to \{0, 1\} \mid x \to f_A(x) = \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{if } x \notin A. \end{cases}$$

Definition 6. Let H be a $\leq$–hypergroupoid. A fuzzy subset f of H is called a *fuzzy left ideal* of H if
(1) $x \leq y \Rightarrow f(x) \geq f(y)$ and
(2) if $f(x \circ y) \geq f(y)$ for all $x, y \in H$.
With the property (2) we mean the following:
(2) if $x, y \in H$ and $u \in x \circ y$, then $f(u) \geq f(y)$.
A fuzzy subset f of H is called a *fuzzy right ideal* of H if
(1) $x \leq y \Rightarrow f(x) \geq f(y)$ and
(2) if $f(x \circ y) \geq f(x)$ for all $x, y \in H$.
With the property (2) we mean the following:
(2) if $x, y \in H$ and $u \in x \circ y$, then $f(u) \geq f(x)$.

A fuzzy subset of H is called a *fuzzy ideal* of H if it is both a fuzzy left ideal and a fuzzy right ideal of H. As one can easily see, a fuzzy subset f of H is a fuzzy ideal of H if and only if

$$f(x \circ y) \geq \max\{f(x), f(y)\} \text{ for all } x, y \in H$$

in the sense that

$$x, y \in H \text{ and } u \in x \circ y \text{ implies } f(u) \geq \max\{f(x), f(y)\}.$$

Proposition 7. *Let H be a $\leq$–hypergroupoid. If A is a left (resp. right) ideal of H, then the characteristic function f_A is a fuzzy left (resp. fuzzy right) ideal of H. "Conversely", if A is a nonempty subset of H such that f_A is a fuzzy left (resp. fuzzy right) ideal of H, then the set A is a left (resp. right) ideal of H.*

Proposition 8. *Let H be an $\leq$–hypergroupoid. If A is an ideal of H, then f_A is a fuzzy ideal of H. "Conversely", if A is a nonempty subset of H such that f_A is a fuzzy ideal of H, then the set A is an ideal of H.*

Definition 9. Let H be an hypergroupoid (or a $\leq$–hypergroupoid). A fuzzy subset f of H is called *fuzzy prime subset* of H if

$$f(x \circ y) \leq \max\{f(x), f(y)\} \text{ for all } x, y \in H$$

that is, if $x, y \in H$ and $u \in x \circ y$, then $f(u) \leq \max\{f(x), f(y)\}$.

By a fuzzy prime ideal of H we clearly mean a fuzzy ideal of H which is at the same time a fuzzy prime subset of H. So a fuzzy subset f of a $\leq$–hypergroupoid H is a fuzzy prime ideal of H if and only if the following assertions are satisfied:
(1) $x \leq y$ implies $f(x) \geq f(y)$ and
(2) $f(x \circ y) = \max\{f(x), f(y)\}$ for all $x, y \in H$
that is, if $x, y \in H$ and $u \in x \circ y$, then $f(u) = \max\{f(x), f(y)\}$.

Proposition 10. *Let H be a $\leq$–hypergroupoid. If A is a prime ideal of H, then f_A is a fuzzy prime ideal of H. "Conversely", if A is a nonempty subset of H such that f_A is a fuzzy prime ideal of H, then A is a prime ideal of H.*

Proof. $\Longrightarrow$. Since A is an ideal of H, f_A is a fuzzy ideal of H. Let $x, y \in H$ and $u \in x \circ y$. Then $f_A(u) = \max\{f_A(x), f_A(y)\}$. Indeed: Let $x \circ y \subseteq A$. Since A is a prime ideal of H, we have $x \in A$ or $y \in A$. Then $f_A(x) = 1$ or $f_A(y) = 1$, and $\max\{f_A(x), f_A(y)\} = 1$. Since $u \in x \circ y \subseteq A$, we have $u \in A$. Then $f_A(u) = 1$, so $f_A(u) = \max\{f_A(x), f_A(y)\}$. Let $x \circ y \nsubseteq A$. Since A is a prime ideal of H, we have $(x \circ y) \cap A = \emptyset$. Since $u \in x \circ y$, we have $u \notin A$, so $f_A(u) = 0$. Since $x \circ y \nsubseteq A$ and A is an ideal of H, we have $x \notin A$ and $y \notin A$ (since $x \in A$ implies $x \circ y \subseteq A * H \subseteq A$ and $y \in A$ implies $x \circ y \subseteq H * A \subseteq A$ which is impossible). Then we have $f_A(x) = 0 = f_A(y)$, and $f_A(u) = \max\{f_A(x), f_A(y)\}$.
$\Longleftarrow$. Let f_A be a fuzzy prime ideal of H. Since f_A is a fuzzy ideal of H, A is an ideal of H. Let $x, y \in H$ such that $x \circ y \subseteq A$. Suppose $x \notin A$ and $y \notin A$. Then $f_A(x) = 0 = f_A(y)$. Take an element $u \in x \circ y$ ($x \circ y \neq \emptyset$). Since $u \in A$, we have $f_A(u) = 1$, so $f_A(u) \neq \max\{f_A(x), f_A(y)\}$ which is impossible. Thus we have $x \in A$ or $y \in A$. Let now $x, y \in H$ such that $x \circ y \nsubseteq A$. Then $(x \circ y) \cap A = \emptyset$. Indeed: Let $u \in (x \circ y) \cap A$. Since $u \in x \circ y$, by hypothesis, we have $f_A(u) = \max\{f_A(x), f_A(y)\}$. Since $u \in A$, we have $f_A(u) = 1$. Then $f_A(x) = 1$ or $f_A(y) = 1$, so $x \in A$ or $y \in A$. If $x \in A$, then $x \circ y \subseteq A * H \subseteq A$ (since A is an ideal of H),

which is impossible. If $y \in A$, then $x \circ y \subseteq H * A \subseteq A$ which again is impossible. Hence we have $(x \circ y) \cap A = \emptyset$. $\square$

Definition 11. Let H be an hypergroupoid (or a $\leq$–hypergroupoid). A nonempty subset I of H is called *semiprime subset* of H if

(1) if $a \in H$ such that $a \circ a \subseteq I$, then $a \in I$ and

(2) if $a \in H$, then $a \circ a \subseteq I$ or $(a \circ a) \cap I = \emptyset$.

The following are equivalent:

(1) if $a \in H$ such that $a \circ a \subseteq I$, then $a \in I$.

(2) if A is a nonempty subset of H such that $A * A \subseteq I$, then $A \subseteq I$.

By a semiprime ideal of H we mean an ideal of H which is at the same time a semiprime subset of H.

Definition 12. Let H be an hypergroupoid (or a $\leq$–hypergroupoid). A fuzzy subset f of H is called *fuzzy semiprime subset* of H if

$$f(x) \geq f(x \circ x) \text{ for every } x \in H$$

that is, if $x \in H$ and $u \in x \circ x$, then $f(x) \geq f(u)$.

Remark 13. If f is a fuzzy prime ideal of H and $a \in H$, then $f(a \circ a) = \max\{f(a), f(a)\} = f(a)$, so $f(a) \geq f(a \circ a)$, and f is fuzzy semiprime ideal of H. If f is a fuzzy ideal of H, then $f(a \circ a) \geq \max\{f(a), f(a)\} = f(a)$ for every $a \in H$.

By a fuzzy semiprime ideal of H we clearly mean a fuzzy ideal of H which is at the same time a fuzzy semiprime subset of H. So, a fuzzy subset f of H is a fuzzy semiprime ideal of H if and only if the following assertions are satisfied:

(1) $x \leq y$ implies $f(x) \geq f(y)$ and

(2) if $f(x \circ x) = f(x)$ for every $x \in H$

that is, if $x \in H$ and $u \in x \circ x$, then $f(u) = f(x)$.

Proposition 14. *Let H be a $\leq$–hypergroupoid. If A is a semiprime ideal of H, then f_A is a fuzzy semiprime ideal of H. "Conversely", if A is a nonempty subset of H such that f_A is a fuzzy semiprime ideal of H, then A is a semiprime ideal of H.*

Proof. $\Longrightarrow$. Let A be a semiprime ideal of H. Since A is an ideal of H, f_A is a fuzzy ideal of H. Let $x \in H$ and $u \in x \circ x$. Then $f_A(u) = f_A(x)$. Indeed: Let $x \circ x \nsubseteq A$. Since A is a semiprime subset of H, we have $(x \circ x) \cap A = \emptyset$, so $u \notin A$, and $f_A(u) = 0$. On the other hand, since $x \circ x \nsubseteq A$ and A is an ideal of H, we have $x \notin A$, then $f_A(x) = 0$, so $f_A(u) = f_A(x)$. Let $x \circ x \subseteq A$. Then $u \in A$, so $f_A(u) = 1$. On the other hand, since A is a semiprime subset of H and $x \circ x \subseteq A$, we have $x \in A$,

so $f_A(x) = 1$. Then $f_A(u) = f_A(x)$, and f_A is a fuzzy semiprime ideal of H.
$\Longleftarrow$. Let f_A be a fuzzy semiprime ideal of H. Since f_A is a fuzzy ideal of H, the set A is an ideal of H. Let $x \in H$ such that $x \circ x \subseteq A$. Then $x \in A$. Indeed: Let $x \notin A$. Then $f_A(x) = 0$. Take an element $u \in x \circ x$ ($x \circ x \neq \emptyset$). Since $u \in A$, we have $f_A(u) = 1$. Since f_A is a fuzzy semiprime ideal of H, we have $f_A(u) = f_A(x)$ which is impossible. Thus we have $x \in A$. Let $x \in H$ such that $x \circ x \nsubseteq A$. Then $(x \circ x) \cap A = \emptyset$. Indeed: Let $u \in (x \circ x) \cap A$. Since $u \in x \circ x$, by hypothesis, we have $f_A(u) = f_A(x)$. Since $u \in A$, we have $f_A(u) = 1$, then $f_A(x) = 1$, and $x \in A$. Then $x \circ x \subseteq A * A \subseteq A$ (since A is a subgroupoid of H), which is impossible. Thus we have $(x \circ x) \cap A = \emptyset$, and A is a semiprime ideal of H. □

Remark 15. Let H be an hypergroupoid, f a fuzzy prime ideal of H and $x, y, z \in H$. Then we have the following:

(1) $f(x) = f(y)$, $f(z) = f(t)$, $u \in x \circ z$, $z \in y \circ t \Longrightarrow f(u) = f(z)$.
(2) $f(x) = f(y)$, $f(z) = f(t)$, $u \in z \circ x$, $z \in t \circ y \Longrightarrow f(u) = f(z)$.
(3) $f(x) \leq f(y)$, $f(z) \leq f(t)$, $u \in x \circ z$, $z \in y \circ t \Longrightarrow f(u) \leq f(z)$.
(4) $f(x) \leq f(y)$, $f(z) \leq f(t)$, $u \in z \circ x$, $z \in t \circ y \Longrightarrow f(u) \leq f(z)$.

In fact:
(1) Let $f(x) = f(y)$, $f(z) = f(t)$, $u \in x \circ z$, $z \in y \circ t$. Since f is a fuzzy prime ideal of H, we have $f(u) = \max\{f(x), f(z)\}$, $f(z) = \max\{f(y), f(t)\}$, then $f(u) = f(z)$.
(4) Let $f(x) \leq f(y)$, $f(z) \leq f(t)$, $u \in z \circ x$, $z \in t \circ y$. Since f is a fuzzy prime ideal of H, we have $f(u) = \max\{f(z), f(x)\}$, $f(z) = \max\{f(t), f(y)\}$, then $f(u) \leq f(z)$.
The proof of the rest is similar. □

The present paper is on fuzzy $\leq$–hypergroupoids (hypergroupoids). Interesting recent results on p-fuzzy hypergroups, p-fuzzy quasi-hypergroups and on p-fuzzy hypergraphs can be found in [3–5] of the References.

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