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SOME RESULTS ON THE TOPOLOGY OF FUZZY METRIC TYPE SPACES

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Abstract. In this study, we investigate the concept of fuzzy metric type spaces. We show that $s < Kt$ implies $M(x, y, s) \leq M(x, y, t)$. After emphasizing the fact that $M(x, y, \_)$ may not be nondecreasing for a fuzzy metric type space, we prove that intersection of two open sets is open. We give examples to show that open balls are not necessarily open and closed balls are not necessarily closed. Moreover, we show that these spaces are sequential, Fréchet and weakly first countable.

Key Words: Fuzzy metric type spaces, sequential, Fréchet, weakly first countable.
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1. Introduction and Preliminaries

In the literature there are different definitions of fuzzy metric spaces ([2, 4, 8, 10, 11]).

In [13], Saadati introduced and studied the concept of fuzzy metric type space which is a generalization of fuzzy metric space introduced by George and Veeramani [2].

Definition 1.1 ([3]). A binary operation $* : [0, 1] \times [0, 1] \longrightarrow [0, 1]$ is called a continuous t-norm if $*$ satisfies the following conditions;
1) $*$ is associative and commutative,
2) $*$ is continuous,
3) $a * 1 = a$ for all $a \in [0, 1]$,
4) $a * b \leq c * d$ whenever $a \leq c$ and $b \leq d$, $a, b, c, d \in [0, 1]$.

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Definition 1.2 ([13])**.** A 4-tuple $(X, M, *, K)$ is called a fuzzy metric type space and M is called fuzzy metric type on X if X is an arbitrary (non-empty) set, $*$ is a continuous t-norm, and M is a fuzzy set on $X \times X \times (0, \infty)$, satisfying the following conditions for each $x, y, z \in X$ and $t, s > 0$,
1) $M(x, y, t) > 0$,
2) $M(x, y, t) = 1$ if and only if $x = y$,
3) $M(x, y, t) = M(y, x, t)$,
4) $M(x, y, t) * M(y, z, s) \leq M(x, z, K(t + s))$ for some constant $K \geq 1$,
5) $M(x, y, .) : (0, \infty) \to [0, 1]$ is continuous.

These spaces can be thought as fuzzy extension of metric type spaces.

Definition 1.3 ([6, 14, 9])**.** An ordered triple (X, D, K) is called metric type space and D is called metric type on X if X is a nonempty set, $K \geq 1$ is a given real number and $D{:}X \times X \to [0, \infty)$ satisfies the following conditions for all $x, y, z \in X$
1) $D(x, y) = 0$ if and only if $x = y$,
2) $D(x, y) = D(y, x)$,
3) $D(x, z) \leq K[D(x, y) + D(y, z)]$.

For a given fuzzy metric type space $(X, M, *, K)$, in [13], open and closed balls were defined as

$$B(x, r, t) = \{y \in X : M(x, y, t) > 1 - r\}$$

and

$$B[x, r, t] = \{y \in X : M(x, y, t) \geq 1 - r\}$$

for $t > 0$, with center $x \in X$ and radius $0 < r < 1$ respectively and it was shown that M induces a topology τ_M on X where

$$\tau_M = \{A \subset X : x \in A \iff \exists t > 0 \textit{ and } 0 < r < 1 \textit{ such that } B(x, r, t) \subset A\}.$$

In Example 2.3 and 2.4, we will show that open balls are not necessarily open and closed balls are not necessarily closed.

Definition 1.4 ([13])**.** Let $(X, M, *, K)$ be a fuzzy metric type space.
i) A sequence $\{x_n\}$ in X is said to be convergent to x if and only if $M(x_n, x, t) \to 1$ as $n \to \infty$, for each $t > 0$.
ii) $\{x_n\}$ is called a Cauchy sequence if for each $0 < \varepsilon < 1$ and $t > 0$, there exits $n_0 \in \mathbb{N}$ such that $M(x_n, x_m, t) > 1 - \varepsilon$ for each $n, m \geq n_0$.
iii) $(X, M, *, K)$ is said to be complete if every Cauchy sequence is convergent .

Here we want to emphasis that the definition of the convergence of a sequence given above is not equivalent to the definition of convergence in induced topology. Therefore we call it as M-convergence for the consistency and we will show that M-convergence implies convergence (see Proposition 2.5).

In [13], for a given metric type space (X, D, K), $(X, M_D, \cdot, K)$ was defined as standard fuzzy metric type space induced by D where $M_D(x, y, t) = \frac{t}{t+D(x,y)}$ and it was shown that the topology τ_D induced by D and the topology τ_{M_D} induced by M_D are the same .

In this study, we also prove that fuzzy metric type spaces are sequential, Fréchet and weakly first countable.

Definition 1.5 ([15, 5])**.** Let X be a topological space.
(1) A subset U of X is called sequentially open if each sequence $\{x_n\}$ in X converging to a point x in U is eventually in U, that is, there exists n_0 such that $x_n \in U$ for all $n \geq n_0$.
(2) A subset F of X is called sequentially closed if no sequence in F converges to a point not in F.
(3) X is called a sequential space if each sequentially open subset of X is open, equivalently, each sequentially closed subset of X is closed.
(4) X is called a Fréchet space if for all $A \subset X, \bar{A} = \{x \in X :$ there exists $\{x_n\} \subset A, \lim_{n \longrightarrow \infty} x_n = x\}$.

Proposition 1.6 ([16])**.** *A sequential space X is Fréchet if and only if every subspace of X is also a sequential space.*

Definition 1.7 ([1])**.** A topological space X is called weakly first countable if for each $x \in X$, there exists a family $\{N(x, n) : n \in \mathbb{N}\}$ of subsets of X such that the following are true:
(a) $x \in N(x, n + 1) \subset N(x, n)$ for each $n \in \mathbb{N}$.
(b) A subset U of X is open in the space X if and only if for each $x \in U$ there exists $n \in \mathbb{N}$ such that $N(x, n) \subset U$.
Such a family $\{N(x, n) : n \in \mathbb{N}\}$ is called a weak base at x in X.

Corollary 1.8 ([5])**.** *Every weakly first countable space is sequential.*

2. Results

Lemma 2.1. *Let $(X, M, *, K)$ be a fuzzy metric type space and $s \in X$. If $Ks < t$ then $M(x, y, s) \leq M(x, y, t)$.*

Proof. Let $t > 0$ such that $Ks < t$. Since $s < \frac{t}{K}$ we have $M(x, y, s) = M(x, y, s) * M\left(y, y, \frac{t}{K} - s\right) \leq M\left(x, y, K(s + \frac{t}{K} - s\right) = M(x, y, t)$. □

From Lemma 2.1, if $Ks < t$ we have $B(x,r,s) \subset B(x,r,t)$. But it may not be $B(x,r,s) \subset B(x,r,t)$ if $s < t \leq Ks$ which means $M(x,y,\_)$ may not be nondecreasing. Nevertheless Lemma 2.1 is enough to prove some topological properties of the fuzzy metric type space.

Since the above fact was ignored in the proof of iii-Proposition 1.8 in [13], here we prove it.

Theorem 2.2. *Intersection of two open sets in fuzzy metric type space $(X, M, *, K)$ is open.*

Proof. Let $A, B \subset X$ are open and $x \in A \cap B$. There exists r_1, r_2 with $0 < r_1, r_2 < 1$ and $t_1, t_2 > 0$ such that $B(x, r_1, t_1) \subset A$ and $B(x, r_2, t_2) \subset B$. Let $r = \min\{r_1, r_2\}, t < \min\{\frac{t_1}{K}, \frac{t_2}{K}\}$ and $y \in B(x,r,t)$. Since

$$M(x,y,t_1) \geq M(x,y,t) > 1 - r \geq 1 - r_1$$

and

$$M(x,y,t_2) \geq M(x,y,t) > 1 - r \geq 1 - r_2,$$

we have $B(x,r,t) \subset B(x,r_1,t_1)$ and $B(x,r,t) \subset B(x,r_2,t_2)$. Hence we have $B(x,r,t) \subset A \cap B$ means $A \cap B$ is open. □

In [13], it was shown that a fuzzy metric type space is Hausdorff (Proposition 1.9). But in the proof, open balls are used as open sets which is not true in general.

Example 2.3. Consider the metric type space given in Example 13 in [12]. Let $X = \{0, 1, \frac{1}{2}, \dots, \frac{1}{n}, \dots\}$ and

$$D(x,y) = \begin{cases} 0 & \text{, if } x = y \\ 1 & \text{, if } x \neq y \in \{0,1\} \\ |x-y| & \text{, if } x \neq y \in \{0\} \cup \{\frac{1}{2n} : n = 1, 2, \dots\} \\ 4 & \text{, otherwise.} \end{cases}$$

$(X, D, 4)$ is a metric type space and $B(1,2) = \{0,1\}$ is not open in τ_D ([17, 18]). Let $(X, M_D, \cdot, 4)$ be the standard fuzzy metric type spaces induced by D. $B(1, \frac{1}{3}, 3) = \{0,1\}$ is also not open in τ_{M_D} because $\tau_D = \tau_{M_D}$. Moreover M_D is not a fuzzy metric on X because

$$\begin{aligned} M(1,0,2).M(0,1/2,2) &= \frac{2}{2 + D(1,0)} \cdot \frac{2}{2 + D(0,1/2)} = \frac{8}{15} \\ &> \frac{4}{8} = \frac{4}{4 + D(1,1/2)} = M(1,1/2,4). \end{aligned}$$

Similarly in fuzzy metric type spaces, closed balls need not to be closed.

Example 2.4. Consider the metric type space given in Example 3.10 in [17]. Let $X = \{0, 1, \frac{1}{2}, \dots, \frac{1}{n}, \dots\}$ and

$$D(x,y) = \begin{cases} 0 & \text{, if } x = y \\ 1 & \text{, if } x \neq y \in \{0,1\} \\ |x-y| & \text{, if } x \neq y \in \{0\} \cup \{\frac{1}{2n} : n = 1, 2, \dots\} \\ \frac{1}{4} & \text{, otherwise.} \end{cases}$$

In [17], it was shown that $(X, D, 4)$ is a metric type and $B[1, 1/2] = X - \{0\}$ is not closed in τ_D. Let $(X, M_D, \cdot, 4)$ be the standard fuzzy metric type spaces induced by D. $B[1, \frac{1}{3}, 1] = X - \{0\}$ is also not closed in τ_{M_D} because $\tau_D = \tau_{M_D}$.

Proposition 2.5. *Let $(X, M, *, K)$ be a fuzzy metric type space and $\{x_n\} \subset X$. If $\{x_n\}$ is M-convergent, then $\{x_n\}$ is convergent.*

Proof. Let $\{x_n\}$ be M-convergent to $x \in X$ and $U \in \tau_M$ such that $x \in U$. We have $\lim_{n\to\infty} M(x, x_n, t) = 1$ for each $t > 0$ and $B(x, r_0, t_0) \subset U$ for some $r_0 \in (0, 1)$ and $t_0 > 0$. Then there exists n_0 such that $M(x, x_n, t_0) > 1 - r_0$ for all $n \geq n_0$. Therefore, $x_n \in B(x, r_0, t_0) \subset U$ for all $n \geq n_0$. This proves that $\{x_n\}$ is convergent. □

Proposition 2.6. *Let $(X, M, *, K)$ be a fuzzy metric type space. Then (X, τ_M) is sequential space.*

Proof. Let τ be the sequential topology on X and $U \in \tau$. Suppose to the contrary that $U \notin \tau_M$. Then there exists $x \in U$ such that $B(x, \frac{1}{n}, t) \not\subset U$ for all $n \in \mathbb{N}$ and $t > 0$. Choosing $x_n \in B(x, \frac{1}{n}, t)$ such that $x_n \notin U$ for all $n \in \mathbb{N}$, then $1 \geq M(x, x_n, t) > 1 - \frac{1}{n}$ for all $n \in \mathbb{N}$ and $t > 0$. This implies that $\lim_{n\to\infty} M(x, x_n, t) = 1$. Hence $\{x_n\}$ is M-convergent to x. By Proposition 2.5, $\{x_n\}$ is convergent to x in (X, τ_M). Since $U \in \tau$, there exists n_0 such that $x_n \in U$ for all $n \geq n_0$. It is a contradiction to $x_n \notin U$ for all $n \in \mathbb{N}$. □

It is well known that in a sequential space, continuity and sequentially continuity are equivalent. Hence the following is obvious.

Corollary 2.7. *A map defined on a fuzzy metric type space is continuous if and only if it is sequentially continuous.*

In the following we show that fuzzy metric type spaces are Fréchet spaces.

Proposition 2.8. *Let $(X, M, *, K)$ be a fuzzy metric type space. Then (X, τ_M) is Fréchet space.*

Proof. Let A any arbitrary nonempty subset of X and $M_A(x, y, t) = M(x, y, t)$ for all $x, y \in A$ and for all $t > 0$. Then M_A is a fuzzy metric type on A. So $(A, M_A, *, K)$ is a fuzzy metric type space. This implies that (A, τ_{M_A}) is a sequential subspace of (X, τ_M) and by Proposition 1.6, (X, τ_M) is a Fréchet space. □

Since open balls $B(x, r, t)$ are not necessarily open for a fuzzy metric type space, they may not be a base at x. But in the following we give a weak base at x.

Theorem 2.9. *Let $(X, M, *, K)$ be a fuzzy metric type space. Then (X, τ_M) is weakly first countable.*

Proof. Let $x \in X$. We need to show that $\{N(x, n) = B(x, \dfrac{1}{n}, \dfrac{1}{(K+1)^n}) : n \in \mathbb{N}\}$ is a weak base at x.

(a) Let $y \in N(x, n+1) = B(x, \dfrac{1}{n+1}, \dfrac{1}{(K+1)^{n+1}})$. We need to show that $y \in N(x, n) = B(x, \dfrac{1}{n}, \dfrac{1}{(K+1)^n})$. Since $1 - \frac{1}{n+1} > 1 - \frac{1}{n}$ and $\frac{1}{(K+1)^{n+1}} K < \frac{1}{(K+1)^n}$, we have

$$M(x, y, \frac{1}{(K+1)^n}) > M(x, y, \frac{1}{(K+1)^{n+1}}) = 1 - \frac{1}{n+1} > 1 - \frac{1}{n}$$

which implies $y \in N(x, n)$. Hence $x \in N(x, n+1) \subset N(x, n)$ for each $n \in \mathbb{N}$.

(b) Let $U \in \tau_M$ such that $x \in U$. Since U is open, then there exists $r \in (0, 1)$ and $t > 0$ such that $B(x, r, t) \subset U$. Choose $n \in \mathbb{N}$ such that $\dfrac{1}{n} < r$ and $\dfrac{1}{(K+1)^n} K < t$. Now we need to show $B(x, \dfrac{1}{n}, \dfrac{1}{(K+1)^n}) \subset B(x, r, t)$. Let $z \in B(x, \dfrac{1}{n}, \dfrac{1}{(K+1)^n})$. Then $M(x, z, \dfrac{1}{(K+1)^n}) > 1 - \frac{1}{n} > 1 - r$. Since $\dfrac{1}{(K+1)^n} K < t$, we have

$1-r < M(x,z,\frac{1}{(K+1)^n}) \leq M(x,z,t)$. Hence $z \in B(x,r,t)$ which implies $B(x,\frac{1}{n},\frac{1}{(K+1)^n}) \subset B(x,r,t) \subset U$.
Consequently, $\{N(x,n) : n \in \mathbb{N}\}$ is a weak base at x. Hence (X,τ_M) is weakly first countable space. □

Since a fuzzy metric type space is weakly first countable space, by Corollary 1.8, we have following proposition again.

Proposition 2.10. *Let $(X,M,*,K)$ be a fuzzy metric type space. Then (X,τ_M) is sequential space.*

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