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AN INFINITE FAMILY OF FINITE 2-GROUPS WITH FIXED CO-CLASS

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Abstract. In [1] six pro-2-groups of finite and fixed coclasses is studied. Infinite sequences of finite 2-groups arising from this pro-2-groups also investigated. One of the five infinite sequences associated with the pro-2-group $S = \langle a, u \mid a^2 = u^4, (u^2)^a = u^{-2} \rangle$ is the sequence

$$G_j(1,2) = \langle a, u \mid a^2u^{-4}(u^{-1}a)^{2^j}, (u^2)^a u^2 (u^{-1}a)^{2^{j+1}}, (u^{-1}a)^{2^{j+2}} \rangle.$$

In this paper we calculate the orders of the members of this infinite family.

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1. Introduction

Presentation of a group is introducing the group by a set of generators and a sufficient set of relations between the generators, that is as a factor group of a free group. For a group G it is denoted by $G = \langle X \mid R \rangle$ in which X is the set of its generators and R is the set of relations. Such an expression of a group provides a short description of its associated group. A group may has many presentations. A presentation $\langle X \mid R \rangle$ is called finite presentation if the cardinals of X and R are both finite. We refer to [5] for background and an introduction into the theory of group presentations. A parameterized presentation for an infinite family of finite groups (in particular for finite p-groups) is of much interest in

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group presentation theory, so that one can defines an infinite family of finite groups with only one presentation. In this paper we show that the parameterized presentation

$$G(n) = \langle x, y \mid x^2y^{-4}w^{2^n} = 1, x^{-1}y^2xy^2w^{2^{n+1}} = 1, w^{2^{n+2}} = 1\rangle,$$

in which $w = y^{-1}x$, defines an infinite family of finite 2-groups and for every positive integer n the order of $G(n)$ is 2^{n+5}.

Infinite pro-2-groups and their presentations have been studied in [1], [2] and [4]. The groups $G(n)$ is a coclass family associated with the infinite pro-2-group $S = \langle a, u \mid a^2 = u^4, (u^2)^a = u^{-2}\rangle$ of coclass 3.

For a p-group G of order p^n and nilpotency class c, the coclass of G is the number $n - c$. In [1] it is shown that all the members of this family have the same coclass 3 and therefore $G(n)$ has nilpotency class $n + 2$.

Throughout the paper $|G : H|$ denotes the index of the subgroup H in a group G, $[x, y]$ used for the commutator $x^{-1}y^{-1}xy$ and $Z(G)$ used for the center of the group G. We use Todd-Coxeter coset enumeration algorithm in the form as given in [3].

2. Preliminaries

We make some small changes in the presentation of $G(n)$ and in the obtained group we find some relations to help us to prove the main theorem of the paper.

Lemma 2.1. *Let $n \geq 2$ be an integer and let $G = \langle x, y \mid x^2y^{-4}(yx)^{2^n} = 1, x^{-1}y^2xy^2(x^{-1}y)^{2^{n+1}} = 1, (y^{-1}x)^{2^{n+2}} = 1\rangle$. Then $(y^{-1}x)^{2^{n+1}} \in Z(G)$.*

Proof. Consider the subgroup $H = \langle a = x^2, b = y^2, c = (yx)^2, d = (y^{-1}x)^2\rangle$ of G. By using the modified Todd-Coxeter coset enumeration we find a presentation for H. Defining $1.x = 2$ completes the table of the generator $a = x^2$ and we obtain the bonus $2.x = a.1$. By defining $1.y = 3$ we have $3.y = b.1$. Now by defining $3.x = 4$, the table of the generator c completes to give us the bonus $4.y = ca^{-1}.2$ and from the table of the generator d we get the bonus $2.y = ad^{-1}b^{-1}.4$. Now the 4th row of the table of the first relation of G gives us $4.x = t.3$ in which $t = c^{-2^{n-1}}(cd^{-1}b^{-1})^2$. Now all the tables are complete. We deduce that $|G : H| = 4$, and we have the following presentation for H,

$$H \cong \langle a, b, c, d \mid r_i, i = 1, ..., 9\rangle$$

in which

$$\begin{aligned}
r_1 &: ab^{-2}c^{2^{n-1}} = 1,\\
r_2 &: a^2(c^{-1}bd)^2a^{-1}\left(ad^{-1}b^{-1}tb\right)^{2^{n-1}} = 1,\\
r_3 &: tb^{-2}\left(bad^{-1}b^{-1}t\right)^{2^{n-1}} = 1,\\
r_4 &: d^{-1}b^{-1}cbd^{-2^n} = 1,\\
r_5 &: bad^{-1}b^{-1}ca^{-1}\left(t^{-1}ca^{-1}\right)^{2^n} = 1,\\
r_6 &: t^{-1}cd^{-1}b^{-1}tb\left(t^{-1}ca^{-1}\right)^{2^n} = 1,\\
r_7 &: cd^{-2^n-1} = 1,\\
r_8 &: d^{2^{n+1}} = 1,\\
r_9 &: (ac^{-1}t)^{2^{n+1}} = 1,
\end{aligned}$$

By the relation r_7 we have $[c,d] = 1$ and then r_4 gives us $[c,b] = 1$. Also by r_5 and r_6 the relation $bad^{-1}b^{-1}ca^{-1} = t^{-1}cd^{-1}b^{-1}tb$ or equivalently $bad^{-1}b^{-1}ca^{-1} = d^{2^n}$ holds in H. Translating this relation in the generators of G yields that $y^2xy^2x^{-1} = (y^{-1}x)^{2^{n+1}}$. Comparing this with the second relation of G yields that $[(y^{-1}x)^{2^{n+1}},\ x] = 1$ and therefore $[(y^{-1}x)^{2^{n+1}},\ y] = 1$ as $(y^{-1}x)^{2^{n+1}}$ is a power of $y^{-1}x$. Hence $(y^{-1}x)^{2^{n+1}} \in Z(G)$. □

Lemma 2.2. *Let G be as in Lemma 2.1. Then $|G| = 2^{n+5}$.*

Proof. By the second relation of G we have $(yx)^2 = (y^{-1}x)^{2^{n+1}+2}$ and using the third relation of G we get $(yx)^4 = (y^{-1}x)^4$. Hence $(yx)^{2^k} = (y^{-1}x)^{2^k}$ holds in G for $k \geq 2$. Set $N = \langle (y^{-1}x)^{2^{n+1}} \rangle$. By Lemma 2.1, N is a central subgroup of G of order 2. Therefore

$$\begin{aligned}
G/N &\cong \langle x, y \mid x^2y^{-4}(yx)^{2^n} = 1, x^{-1}y^2xy^2(x^{-1}y)^{2^{n+1}} = 1,\\
&\qquad (y^{-1}x)^{2^{n+2}} = 1, (y^{-1}x)^{2^{n+1}} = 1\rangle\\
&\cong \langle x, y \mid x^2y^{-4}(yx)^{2^n} = 1, x^{-1}y^2xy^2 = 1, (y^{-1}x)^{2^{n+1}} = 1\rangle\\
&\cong \langle x, y \mid x^2y^{-4}(y^{-1}x)^{2^n} = 1, x^{-1}y^2xy^2 = 1, (y^{-1}x)^{2^{n+1}} = 1\rangle
\end{aligned}$$

Let $n \geq 2$ and set

$$L := \langle x, y \mid x^2y^{-4}(y^{-1}x)^{2^n} = 1, x^{-1}y^2xy^2 = 1, (y^{-1}x)^{2^{n+1}} = 1\rangle.$$

Consider the subgroup $K = \langle a = x^2, b = y^{-1}x\rangle$ of L. Again using the modified Todd-Coxeter coset enumeration we find a presentation for K.

Defining $1.x = 2$ completes the tables of the generators $a = x^2$ and $b = y^{-1}x$. Now we have the bonuses $2.x = a.1$ and $2.y = ab^{-1}.1$. By defining $1.y = 3$ and $3.x = 4$, first row of the table of the relation $x^{-1}y^2xy^2 = 1$ gives us the bonus $4.y = b^2a^{-1}.2$ and third row of that gives $4.x = bab^{-1}.3$. Now the first row of the table of the relation $x^2y^{-4}(y^{-1}x)^{2^n} = 1$ completes to get $3.y = b^{2^n}ab^{-1}.4$. Now all the tables are compete and we deduce $|L : K| = 4$ and the following presentation for K

$$K \cong \langle a, b \mid s_i, i = 1, 2, 3, ..., 8\rangle$$

in which

$$\begin{aligned}
s_1 &: a^2b^{-1}a^{-1}b^{-2^n+1}a^{-1}(ab^{-1})^{2^n} = 1 \\
s_2 &: bab^{-1}a^{-1}b^{-2^n}(ab^{-1})^{2^n} = 1 \\
s_3 &: b^{-2^n}(ba^{-1}b^{-2^n})^{2^n} = 1 \\
s_4 &: a^{-1}b^{2^n}a(ab^{2^n-1}ab) = 1 \\
s_5 &: b^{2^n}(ab^{2^n-1}ab) = 1 \\
s_6 &: b^{2^{n+1}} = 1 \\
s_7 &: (ab^{-1})^{2^{n+1}} = 1 \\
s_8 &: (ba^{-1}b^{-2^n})^{2^{n+1}} = 1.
\end{aligned}$$

Comparing s_4 and s_5 yields that $[a, b^{2^n}] = 1$. Therefore s_5 could be written in the form $b^{2^{n+1}}ab^{-1}ab = 1$, which using s_6 gives us the relation $ab^{-1}ab = 1$ or equivalently $(ab^{-1})^2 = b^{-2}$. This relation together with s_2 and s_6 yields that $[a, b] = 1$ and $a^2 = 1$. Therefore K is abelian and

$$K \cong \langle a, b \mid a^2 = b^{2^{n+1}} = 1, [a, b] = 1\rangle.$$

Hence the order of L is $|L| = 4|K| = 2^{n+4}$ and consequently $|G| = |N||L| = 2^{n+5}$. □

3. Main Result

In this section we prove the main result of the paper. The key point of the proof is to show that the group G in Lemma 2.1 is isomorphic to the group $G(n)$.

Theorem 3.1. *Let $n \geq 2$ be an integer and let $G(n) = \langle x, y \mid x^2y^{-4}w^{2^n} = 1, x^{-1}y^2xy^2w^{2^{n+1}} = 1, w^{2^{n+2}} = 1\rangle$, in which $w = y^{-1}x$. Then $|G(n)| = 2^{n+5}$.*

Proof. Consider the group G in Lemma 2.1. By the second relation in G we see that the relation $yxy^2 = (y^{-1}x)^{2^{n+1}+1}$ holds in G and hence

$$(yx)^2 = (y^{-1}x)^{2^{n+1}+2}, \tag{3.1}$$

also holds in G. Now by third relation of G we have $(yx)^4 = (y^{-1}x)^4$ in G. Applying the Tietze transformations we have

$$\begin{aligned} G &\cong \langle x, y \mid x^2y^{-4}(yx)^{2^n} = 1, x^{-1}y^2xy^2(x^{-1}y)^{2^{n+1}} = 1, (y^{-1}x)^{2^{n+2}} = 1, \\ &\qquad (yx)^4 = (y^{-1}x)^4\rangle \\ &\cong \langle x, y \mid x^2y^{-4}(y^{-1}x)^{2^n} = 1, x^{-1}y^2xy^2(x^{-1}y)^{2^{n+1}} = 1, (y^{-1}x)^{2^{n+2}} = 1, \\ &\qquad (yx)^4 = (y^{-1}x)^4\rangle \\ &\cong \langle x, y \mid x^2y^{-4}(y^{-1}x)^{2^n} = 1, x^{-1}y^2xy^2(x^{-1}y)^{2^{n+1}} = 1, \\ &\qquad (y^{-1}x)^{2^{n+2}} = 1\rangle \quad (\text{by } 3.1) \end{aligned}$$

Therefore $G \cong G(n)$ and the result follows by Lemma 2.2. The order of $G(n)$ for $n = 1$ could be calculated separately by considering a suitable subgroup and using Todd-coxeter coset enumeration, however in this case the order of $G(1)$ is 64 and requires the obtained formula for the order of $G(n)$, $(n \geq 2)$. □

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