Journal of Hyperstructures 6 *(1) (2017), 28-39.*
ISSN: 2322-1666 print/2251-8436 online

PRODUCT OF DERIVATIONS ON TRIANGULAR BANACH ALGEBRAS

MOHAMMAD HOSSEIN SATTARI* AND HAMID SHAFIEASL

Abstract. In this paper, we give a necessary and sufficient condition for the product of two derivations on the triangular Banach algebra to be a derivation. We also study the case where the product of derivations is commutative.

Key Words: Banach bimodule, Derivations, Triangular Banach algebras.
2010 Mathematics Subject Classification: 16W25, 47B47, 46H25, 47L35.

1. Introduction and Preliminaries

Let $\mathfrak{A}$ be an algebra and $D : \mathfrak{A} \to \mathfrak{A}$ be a linear map such that

$$D(ab) = aD(b) + D(a)b, \qquad (a, b \in \mathfrak{A});$$

in this case, D is called derivation. For every Banach algebra $\mathfrak{A}$, let $\mathfrak{Der}(\mathfrak{A})$ denote the linear space of all continuous derivations on $\mathfrak{A}$. Consider the following Lie bracket

$$\begin{cases} [\cdot,\cdot] : \mathfrak{Der}(\mathfrak{A}) \times \mathfrak{Der}(\mathfrak{A}) \to \mathfrak{Der}(\mathfrak{A}) \\ [D, C] = DC - CD \end{cases}.$$

Received: 24 April 2016, Accepted: 8 June 2016. Communicated by Nasrin Eghbali;
∗Address correspondence to M. H. Sattari; E-mail: sattari@azaruniv.edu

This research has been supported by Azarbaijan Shahid Madani University.

In the theory of derivations, product of derivations has an important role that is investigated by several authors. One of the thing we want to find some condition under that product of two derivations can commutes. For example, if we set

$$\mathcal{Z}(\mathfrak{A}) = \{a \in \mathfrak{A} : ab = ba, \ b \in \mathfrak{A}\}$$

and $D \in \mathfrak{Der}(\mathfrak{A})$ such that $D(\mathfrak{A}) \subseteq \mathcal{Z}(\mathfrak{A})$, then for all derivations $\mathrm{ad}_x : \mathfrak{A} \to \mathfrak{A}$ ($\mathrm{ad}_x(a) = ax - xa$) with $x \in \mathfrak{A}$, we have $[D, \mathrm{ad}_x] = \mathrm{ad}_{D(x)} = 0$; or, $[\mathrm{ad}_x, \mathrm{ad}_y] = 0$, for $x, y \in \mathfrak{A}$ if and only if $xy - yx \in \mathcal{Z}(\mathfrak{A})$.
Set $\mathfrak{A} = \mathbb{C} \oplus \mathbb{C}$, which is a Banach algebra with multiplication

$$(a_1, b_1)(a_2, b_2) = (a_1 a_2, a_1 b_2), \quad (a_1, a_2, b_1, b_2 \in \mathbb{C}),$$

and norm $\|(a, b)\| = |a| + |b|$. Obviously $\mathcal{Z}(\mathfrak{A}) = \{(0, 0)\}$, and consequently $[\mathrm{ad}_x, \mathrm{ad}_y] = 0$ for $x, y \in \mathfrak{A}$ if and only if $xy = yx$.
In [5], Kamowitz and Scheinberg proved that product of two derivations $D_1(f) = xf * \mu_1, \ D_2(f) = xf * \mu_2$, on $L^1(0, 1)$ commutes (or, $[D_1, D_2] = 0$) if $\mu_1 = c\mu_2$ on $[0, 1 - b)$, where c is a constant and b is the largest number such that $|\mu_2|[0, b) = 0$

The section 2, deals with characterizing those derivations D, C of $\mathfrak{T}$ for which $[D, C] = 0$, where $\mathfrak{T}$ is a triangular Banach algebra.

Considering derivations, we would like to find whether $DC \in \mathfrak{Der}(\mathfrak{A})$ for $D, C \in \mathfrak{Der}(\mathfrak{A})$ is the case, under some appropriate conditions? For instance Creedon in [1] proved that if the product of two derivations on a semiprime algebra is derivation, then the product is zero. In the section 3, we obtain a condition for which product of two derivations on $\mathfrak{T}$ is a derivation, where $\mathfrak{T}$ is a triangular Banach algebra.

Definition 1.1. Let $\mathfrak{A}$ and $\mathfrak{B}$ be two Banach algebras and $\mathfrak{M}$ be a Banach $\mathfrak{A}$-$\mathfrak{B}$-bimodule. Define

$$\begin{pmatrix} \mathfrak{A} & \mathfrak{M} \\ 0 & \mathfrak{B} \end{pmatrix} = \left\{ \begin{pmatrix} a & m \\ 0 & b \end{pmatrix} : \ a \in \mathfrak{A}, b \in \mathfrak{B}, m \in \mathfrak{M} \right\},$$

and we shall simplify the notation throughout this paper by writing $\mathfrak{T}$ for $\begin{pmatrix} \mathfrak{A} & \mathfrak{M} \\ 0 & \mathfrak{B} \end{pmatrix}$. The linear space $\mathfrak{T}$ with the standard product of matrices is an algebra. In addition, let

$$\left\| \begin{pmatrix} a & m \\ 0 & b \end{pmatrix} \right\| = \|a\| + \|m\| + \|b\|, \qquad (a \in \mathfrak{A}, b \in \mathfrak{B}, m \in \mathfrak{M});$$

which turns $\mathfrak{T}$ into a Banach algebra and it is called *triangular Banach algebra*.

Proposition 1.2. *Let $\mathfrak{A}$ be a unital Banach algebra and let $\mathfrak{B}$ be a Banch algebra with bounded approximate identity $\{e_\alpha\}_{\alpha \in \Lambda}$. Let $\mathfrak{M}$ be an essential Banach $\mathfrak{A}$-$\mathfrak{B}$-bimodule (that is, $\overline{\mathfrak{A}\mathfrak{M}} = \overline{\mathfrak{M}\mathfrak{B}} = \mathfrak{M}$) and $\mathfrak{T}$ be as Definition 1.1 and $D : \mathfrak{T} \to \mathfrak{T}$ be a continuous derivation. Then, there exist $m_D \in \mathfrak{M}$, continuous derivations $D_1 : \mathfrak{A} \to \mathfrak{A}$, $D_2 : \mathfrak{B} \to \mathfrak{B}$ and a continuous linear map $M_D : \mathfrak{M} \to \mathfrak{M}$ and a net $\{z_\alpha\}_{\alpha \in \Lambda} \subset \mathfrak{B}$ such that for each $a \in \mathfrak{A}, b \in \mathfrak{B}, m \in \mathfrak{M}$ we have*

(1) $\lim_\alpha z_\alpha b = \lim_\alpha b z_\alpha = 0$,
(2) $D(\begin{pmatrix} 1_{\mathfrak{A}} & 0 \\ 0 & 0 \end{pmatrix}) = \begin{pmatrix} 0 & m_D \\ 0 & 0 \end{pmatrix}$,
(3) $D(\begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix}) = \begin{pmatrix} D_1(a) & am_D \\ 0 & 0 \end{pmatrix}$,
(4) $D(\begin{pmatrix} 0 & 0 \\ 0 & e_\alpha \end{pmatrix}) = \begin{pmatrix} 0 & -m_D e_\alpha \\ 0 & z_\alpha \end{pmatrix}$,
(5) $D(\begin{pmatrix} 0 & 0 \\ 0 & b \end{pmatrix}) = \begin{pmatrix} 0 & -m_D b \\ 0 & D_2(b) \end{pmatrix}$,
(6) $D(\begin{pmatrix} 0 & m \\ & 0 \end{pmatrix}) = \begin{pmatrix} 0 & M_D(m) \\ 0 & 0 \end{pmatrix}$,
(7) $M_D(a \cdot m) = D_1(a) \cdot m + a \cdot M_D(m)$,
(8) $M_D(m \cdot b) = M_D(m) \cdot b + m \cdot D_2(b)$.

Proof. See Propositions 2.1 and 2.2 of [3]. □

Remark 1.3. Throughout this paper, we use the assumptions and notations of Proposition 1.2, unless stated otherwise.

Example 1.4. Let $\mathfrak{A}$ be a commutative unital Banach algebra, $\mathfrak{B} = \mathfrak{A}$, $\mathfrak{M}$ be an essential Banach $\mathfrak{A}$-bimodule and $\mathfrak{T}$ be as Definition 1.1. In [6] Singer and Wermer showed that the range of a continuous derivation on a commutative Banach algebra is contained in the radical, in particular, that if the algebra is semisimple, there are no non-zero continuous derivations. Hence, if $D : \mathfrak{T} \to \mathfrak{T}$ is a derivation, then $D_1 = D_2 = 0$ and

$$M_D(a \cdot m) = a \cdot M_D(m), \quad M_D(a \cdot m) = M_D(m) \cdot a,$$

for all $a \in \mathfrak{A}, m \in \mathfrak{M}$. Moreover, if $\mathfrak{M} = \mathfrak{A}$ then by setting $a_0 = M_D(1_{\mathfrak{A}})$ we have

$$M_D(a) = aa_0 \qquad (a \in \mathfrak{A}).$$

2. Commuting of derivations

First we show that the product of derivations is not commutative necessarily through the following example.

Example 2.1. Suppose that $\mathfrak{A}$ and $\mathfrak{B}$ are two unital Banach algebras, $\mathfrak{M}$ is a essential Banach $\mathfrak{A}$-$\mathfrak{B}$-bimodule, $\lambda, \gamma \in \mathbb{C}$, $v, u \in \mathfrak{M}$ and

$$V = \begin{pmatrix} 1_{\mathfrak{A}} & v \\ 0 & \lambda 1_{\mathfrak{B}} \end{pmatrix}, \quad U = \begin{pmatrix} 1_{\mathfrak{A}} & u \\ 0 & \gamma 1_{\mathfrak{B}} \end{pmatrix}.$$

For all $a \in \mathfrak{A}, m \in \mathfrak{M}, b \in \mathfrak{B}$, we have

$$\begin{aligned} \mathrm{ad}_V(\begin{pmatrix} a & m \\ 0 & b \end{pmatrix}) &= \begin{pmatrix} a & av + \lambda m \\ 0 & \lambda b \end{pmatrix} - \begin{pmatrix} a & m + vb \\ 0 & \lambda b \end{pmatrix} \\ &= \begin{pmatrix} 0 & av - vb + (\lambda - 1)m \\ 0 & 0 \end{pmatrix}, \end{aligned}$$

and consequently

$$\mathrm{ad}_U(\mathrm{ad}_V(\begin{pmatrix} a & m \\ 0 & b \end{pmatrix})) = \begin{pmatrix} 0 & (\gamma - 1)(av - vb + (\lambda - 1)m) \\ 0 & 0 \end{pmatrix}.$$

Suppose that $[\mathrm{ad}_V, \mathrm{ad}_U] = 0$. Then

$$(\gamma - 1)(av - vb + (\lambda - 1)m) = (\lambda - 1)(au - ub + (\gamma - 1)m).$$

We now deal with the following cases:
case 1: If $\lambda = 1$, then $\mathrm{ad}_U \, \mathrm{ad}_V = 0 = \mathrm{ad}_V \, \mathrm{ad}_U$.
case 2: If $\gamma = 1$, then again we have $\mathrm{ad}_U \, \mathrm{ad}_V = 0 = \mathrm{ad}_V \, \mathrm{ad}_U$.
case 3: If $\lambda \neq 1, \gamma \neq 1$, then one can find $c \in \mathbb{C}$ such that $v = cu$.
The condition $v = cu$, in case 3 inspires us to construct an example of derivations where commuting of product is not the case.
For instance, consider $\mathfrak{T}_2 \hat{\otimes} \mathbb{M}_2$ and set

$$U = \begin{pmatrix} \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} & \begin{pmatrix} 1 & 2 \\ 0 & 0 \end{pmatrix} \\ 0 & \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \end{pmatrix}, \quad V = \begin{pmatrix} \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} & \begin{pmatrix} 0 & 2 \\ 0 & 1 \end{pmatrix} \\ 0 & \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \end{pmatrix}.$$

As $\begin{pmatrix} 1 & 2 \\ 0 & 0 \end{pmatrix}$ is not aligned with $\begin{pmatrix} 0 & 2 \\ 0 & 1 \end{pmatrix}$, we obtain $[\mathrm{ad}_U, \mathrm{ad}_V] \neq 0$.

The components obtained in Proposition 1.2 leads us to study derivations over triangular Banach algebras in terms of commuting.

Theorem 2.2. *Let $D, C : \mathfrak{T} \to \mathfrak{T}$ be two continuous derivations. Then $DC = CD$ if and only if $D_1C_1 = C_1D_1$, $D_2C_2 = C_2D_2$, $M_DM_C = M_CM_D$ and $M_D(m_C) = M_C(m_D)$.*

Proof. By using Proposition 1.2, we have the following statements

$$DC(\begin{pmatrix} a & m \\ 0 & b \end{pmatrix}) = \begin{pmatrix} D_1C_1(a) & s \\ 0 & D_2C_2(b) \end{pmatrix},$$
$$CD(\begin{pmatrix} a & m \\ 0 & b \end{pmatrix}) = \begin{pmatrix} C_1D_1(a) & t \\ 0 & C_2D_2(b) \end{pmatrix},$$

where

$$s = C_1(a) \cdot m_D - m_D \cdot C_2(b) + M_D(a \cdot m_C - m_C \cdot b) + M_DM_C(m),$$

and

$$t = D_1(a)\cdot m_C - m_C\cdot D_2(b) + M_C(a\cdot m_D - m_D\cdot b) + M_C M_D(m).$$

Now if $DC = CD$, then we must have

$$\begin{aligned} &C_1(a)\cdot m_D - m_D\cdot C_2(b) + M_D(a\cdot m_C - m_C\cdot b) + M_D M_C(m) = \\ &D_1(a)\cdot m_C - m_C\cdot D_2(b) + M_C(a\cdot m_D - m_D\cdot b) + M_C M_D(m). \end{aligned} \tag{2.1}$$

Bearing part 7 and 8 of Proposition 1.2 in mind, we have

$$\begin{aligned} &M_D(a\cdot m_C - m_C\cdot b) \\ &= D_1(a)\cdot m_C - m_C\cdot D_2(b) + a\cdot M_D(m_C) - M_D(m_C)\cdot b, \\ &M_C(a\cdot m_D - m_D\cdot b) \\ &= C_1(a)\cdot m_D - m_D\cdot C_2(b) + a\cdot M_C(m_D) - M_C(m_D)\cdot b. \end{aligned}$$

It follows from 2.1

$$\begin{aligned} &a\cdot M_D(m_C) - M_D(m_C)\cdot b + M_D M_C(m) \\ &= a\cdot M_C(m_D) - M_C(m_D)\cdot b + M_C M_D(m). \end{aligned}$$

Setting $a = 1_{\mathfrak{A}}$, and $b = e_\alpha$, for an arbitrary $\alpha\in\Lambda$, we get

$$\begin{aligned} &M_D(m_C) - M_D(m_C)\cdot e_\alpha + M_D M_C(m) \\ &= M_C(m_D) - M_C(m_D)\cdot e_\alpha + M_C M_D(m). \end{aligned}$$

Taking limit over α implies that

$$M_D M_C(m) = M_C M_D(m),$$

for all $m\in\mathfrak{M}$, and hence $M_D M_C = M_C M_D$. Also obviously

$$D_1 C_1 = C_1 D_1, \quad D_2 C_2 = C_2 D_2.$$

On the other hand, $CD(\begin{pmatrix} 1_{\mathfrak{A}} & 0 \\ 0 & 0 \end{pmatrix}) = DC(\begin{pmatrix} 1_{\mathfrak{A}} & 0 \\ 0 & 0 \end{pmatrix})$, yeild $M_D(m_C) = M_C(m_D)$.

The converse of theorem is straightforward. □

Remark 2.3. Let $\mathfrak{A}$ be a Banach algera and $\mathfrak{B} = \mathfrak{M} = \mathfrak{A}$ and $\mathfrak{T}$ be as definition 1.1. Suppose that $\mathfrak{T}_2$ denotes the algebra of 2×2 upper triangular matrices. Then, $\mathfrak{T} = \mathfrak{T}_2 \hat{\otimes} \mathfrak{A}$.

Example 2.4. Let $\mathbb{R}^+ = [0, +\infty)$, $\omega : \mathbb{R}^+ \to (0, +\infty)$ be a continuous function such that $\omega(s+t) \leq \omega(s)\omega(t)$, for every $s, t \in \mathbb{R}^+$ and $\omega(0) = 1$. Suppose that

$$C_0(\mathbb{R}^+, \omega) = \{f : \mathbb{R}^+ \to \mathbb{C} : \frac{f}{\omega} \in C_0(\mathbb{R}^+)\},$$

which is a Banach algebra under pointwise mutiplication and norm:

$$\|f\| = \sup_{t \in \mathbb{R}^+} \frac{|f(t)|}{\omega(t)}.$$

Denote by $M(\mathbb{R}^+, \omega)$ the space of all complex regular Borel measures μ on $\mathbb{R}^+$ such that $\|\mu\| = \int_0^{+\infty} w(t) d|\mu|(t) < \infty$. The Banach space $M(\mathbb{R}^+, \omega)$ can be identified with the dual of $C_0(\mathbb{R}^+, \omega)$ and with the convolution product

$$\int_0^{+\infty} \psi(x) d(\mu * \nu)(x) = \int_0^{+\infty} \int_0^{+\infty} \psi(x+y) d\mu(x) d\nu(y), \quad (\psi \in C_0(\mathbb{R}^+, \omega))$$

is a commutative Banach algebra. Consider Dirac measure $\delta_x \in M(\mathbb{R}^+, \omega)$, at point $x \in \mathbb{R}^+$. The algbera $M(\mathbb{R}^+, \omega)$ is unital with unit element δ_0. If also

$$\lim_{t \to +\infty} -\frac{\ln \omega(t)}{t} = \infty,$$

then $M(\mathbb{R}^+, \omega)$ is not semisimple (it is radical algebra).
Suppose that $\mathfrak{T} = \mathfrak{T}_2 \hat{\otimes} M(\mathbb{R}^+, \omega)$, $D : \mathfrak{T} \to \mathfrak{T}$ is a continuous derivaton and D_1, D_2, M_D induced operators on $M(\mathbb{R}^+, \omega)$ as Proposition 1.2. Using lemma 2.3 of [4] there are locally finite measures μ_1, μ_2 on $\mathbb{R}^+$ such that

$$D_i(\delta_a) = a\delta_a * \mu_i, \quad (a \in \mathbb{R}^+, i = 1, 2).$$

For every $a, b \in \mathbb{R}^+$

$$M_D(\delta_a * \delta_b) = \delta_a * M_D(\delta_b) + a\delta_a * \mu_1 * \delta_b$$
$$M_D(\delta_b * \delta_a) = M_D(\delta_b) * \delta_a + \delta_b * a\delta_a * \mu_2$$

and setting $b = 0$ we get

$$M_D(\delta_a) = \delta_a * M_D(\delta_0) + a\delta_a * \mu_1$$
$$M_D(\delta_a) = M_D(\delta_0) * \delta_a + a\delta_a * \mu_2$$

which implies that $\mu_1 = \mu_2$. This implies $D_1 = D_2$. Setting $\mu = M_D(\delta_0) + \mu_1$ implies

$$M_D(\delta_a) = \delta_a * \mu.$$

3. Product of derivations

That the product of two derivations on a semiprime Banach algebra is a derivation or not has been the subject of different papers in this area. The interested reader is referred to [1, 4]. We would like to study the case whether the product of derivations over triangular Banach algebras (which is non-semiprime) is a derivation or not.

Definition 3.1. Let $\mathfrak{A}, \mathfrak{B}$ be two Banach algebras and $\mathfrak{M}$ be a Banach $\mathfrak{A}$-$\mathfrak{B}$-bimodule. Fix $a \in \mathfrak{A}$ and $b \in \mathfrak{B}$ and define the *Rosenblum operator*

$$\begin{cases} \tau_{a,b} : \mathfrak{M} \to \mathfrak{M} \\ \tau_{a,b}(m) = a \cdot m - m \cdot b. \end{cases}$$

Theorem 3.2. *Let $D, C : \mathfrak{T} \to \mathfrak{T}$ be two continuous derivations. Then, DC is a derivation if and only if D_1C_1, D_2C_2 are derivations and the following holds for all $a, u \in \mathfrak{A}, b, u \in \mathfrak{B}, m, n \in \mathfrak{M}$*

$$\begin{aligned} &C_1(a) \cdot [M_D(n) - \tau_{u,v}(m_D)] + D_1(a) \cdot [M_C(n) - \tau_{u,v}(m_C)] \\ &= [\tau_{a,b}(m_D) + M_D(m)] \cdot C_2(v) + [\tau_{a,b}(m_C) + M_C(m)] \cdot D_2(v) \end{aligned} \tag{3.1}$$

Proof. Suppose that DC is a derivation. Then for each $a, u \in \mathfrak{A},\ b, v \in \mathfrak{B},\ m, n \in \mathfrak{M}$ we have

$$
\begin{aligned}
&DC(\begin{pmatrix} a & m \\ 0 & b \end{pmatrix}\begin{pmatrix} u & n \\ 0 & v \end{pmatrix}) \\
&= DC(\begin{pmatrix} a & m \\ 0 & b \end{pmatrix})\begin{pmatrix} u & n \\ 0 & v \end{pmatrix} + \begin{pmatrix} a & m \\ 0 & b \end{pmatrix} DC(\begin{pmatrix} u & n \\ 0 & v \end{pmatrix}) \\
&= \begin{pmatrix} D_1C_1(a) & s \\ 0 & D_2C_2(b) \end{pmatrix}\begin{pmatrix} u & n \\ 0 & v \end{pmatrix} + \begin{pmatrix} a & m \\ 0 & b \end{pmatrix}\begin{pmatrix} D_1C_1(u) & t \\ 0 & D_2C_2(v) \end{pmatrix} \\
&= \begin{pmatrix} D_1C_1(a)u & D_1C_1(a)\cdot n + s\cdot v \\ 0 & D_2C_2(b)v \end{pmatrix} + \begin{pmatrix} aD_1C_1(u) & a\cdot t + m\cdot D_2C_2(v) \\ 0 & bD_2C_2(v) \end{pmatrix},
\end{aligned}
$$

where

$$s = C_1(a)\cdot m_D - m_D\cdot C_2(b) + M_D(a\cdot m_C - m_C\cdot b) + M_DM_C(m),$$
$$t = C_1(u)\cdot m_D - m_D\cdot C_2(v) + M_D(u\cdot m_C - m_C\cdot v) + M_DM_C(n).$$

On the other hand, we have

$$
\begin{aligned}
DC(\begin{pmatrix} a & m \\ 0 & b \end{pmatrix}\begin{pmatrix} u & n \\ 0 & v \end{pmatrix}) &= DC(\begin{pmatrix} au & a\cdot n + m\cdot v \\ 0 & bv \end{pmatrix}) \\
&= \begin{pmatrix} D_1C_1(au) & r \\ 0 & D_2C_2(bv) \end{pmatrix},
\end{aligned}
$$

where

$$r = C_1(au)\cdot m_D - m_D\cdot C_2(bv) + M_D(au\cdot m_C - m_C\cdot bv) + M_DM_C(a\cdot n + m\cdot v).$$

Hence, D_1C_1, D_2C_2 are derivations. Furthermore

$$
\begin{aligned}
&D_1C_1(a)\cdot n + C_1(a)\cdot m_D\cdot v - m_D\cdot C_2(b)v + M_D(a\cdot m_C - m_C\cdot b)\cdot v \\
&+M_DM_C(m)\cdot v + aC_1(u)\cdot m_D - a\cdot m_D\cdot C_2(v) + a\cdot M_D(u\cdot m_C - m_C\cdot v) \\
&+a\cdot M_DM_C(n) + m\cdot D_2C_2(v) \\
&= C_1(au)\cdot m_D - m_D\cdot C_2(bv) + M_D(au\cdot m_C - m_C\cdot bv) \\
&+M_DM_C(a\cdot n + m\cdot v)
\end{aligned}
$$

Parts 7 and 8 of Proposition 1.2 imply

$$\begin{aligned}
&D_1C_1(a)\cdot n + C_1(a)\cdot m_D\cdot v - m_D\cdot C_2(b)v + a\cdot M_D(m_C)\cdot v\\
&+D_1(a)\cdot m_C\cdot v - m_C\cdot D_2(b)v - M_D(m_C)\cdot bv + M_DM_C(m)\cdot v\\
&+aC_1(u)\cdot m_D - a\cdot m_D\cdot C_2(v) + aD_1(u)\cdot m_C + au\cdot M_D(m_C)\\
&-a\cdot M_D(m_C)\cdot v - a\cdot m_C\cdot D_2(v) + a\cdot M_DM_C(n) + m\cdot D_2C_2(v)\\
&= C_1(a)u\cdot m_D + aC_1(u)\cdot m_D - m_D\cdot bC_2(v) - m_D\cdot C_2(b)v\\
&+aD_1(u)\cdot m_C + D_1(a)u\cdot m_C + au\cdot M_D(m_C) - M_D(m_C)\cdot bv\\
&-m_C\cdot D_2(b)v - m_C\cdot bD_2(v) + D_1C_1(a)\cdot n + C_1(a)\cdot M_D(n)\\
&+D_1(a)\cdot M_C(n) + a\cdot M_DM_C(n) + M_DM_C(m)\cdot v\\
&+M_C(m)\cdot D_2(v) + M_D(m)\cdot C_2(v) + m\cdot D_2C_2(v).
\end{aligned}$$

Hence

$$\begin{aligned}
&C_1(a)\cdot m_D\cdot v + D_1(a)\cdot m_C\cdot v - a\cdot m_D\cdot C_2(v) - a\cdot m_C\cdot D_2(v)\\
&= C_1(a)u\cdot m_D - m_D\cdot bC_2(v) + D_1(a)u\cdot m_C - m_C\cdot bD_2(v)\\
&+C_1(a)\cdot M_D(n) + D_1(a)\cdot M_C(n) + M_C(m)\cdot D_2(v) + M_D(m)\cdot C_2(v),
\end{aligned}$$

and 3.1 is obtained.
The converse is given by the similar fashion. □

Example 3.3. Let $\mathfrak{A}$ be a unital Banach algebra, $\mathfrak{B}$ be a Banach algebra with bounded approximate identity such that $\mathfrak{Der}(\mathfrak{B}) = \{0\}$, and consider triangular Banach algebra $\mathfrak{T} = \begin{pmatrix}\mathfrak{A} & \mathbb{C}\\ 0 & \mathfrak{B}\end{pmatrix}$, where $\mathbb{C}$ is an essential left $\mathfrak{A}$-module with $a\cdot\alpha = \varphi(a)\alpha$ for non-zero character φ of $\mathfrak{A}$ and similarly $\mathbb{C}$ is an essential right $\mathfrak{B}$-bimodule. Let $D, C : \mathfrak{T}\to\mathfrak{T}$ be continuous derivations, $M_D \neq 0, M_C \neq 0, m_D \neq 0, m_C \neq 0$, and $DC = CD$. For every $u \in \mathfrak{A}$ and $v \in \mathfrak{B}$, consider the Rosenblum operators $\tau_{u,v} : \mathbb{C}\to\mathbb{C}$. Hence there are non-zero elements $d_0, c_0, g(u, v) \in \mathbb{C}$ such that

$$M_D(n) = nd_0,\quad M_C(n) = nc_0,\quad \tau_{u,v}(n) = ng(u, v),\qquad (n\in\mathbb{C}).$$

Hence $D_2 = 0, C_2 = 0$, and so from 3.1, we have

$$C_1(a)\cdot[nd_0 - m_Dg(u, v))] = D_1(a)\cdot[m_Cg(u, v) - nc_0],\quad (n\in\mathbb{C}, a, u\in\mathfrak{A}, v\in\mathfrak{B}).$$

Choosing appropriate $u_0 \in \mathfrak{A}, v_0 \in \mathfrak{B}$ we get $g(u_0, v_0) = 1$ and by setting $n = 0$, we obtain that

$$D_1(a) = \frac{m_D}{m_C} C_1(a), \qquad (a \in \mathfrak{A}).$$

Moreover, by choosing appropriate $u_1 \in \mathfrak{A}, v_1 \in \mathfrak{B}$ we get $g(u_1, v_1) = 0$ and so

$$D_1(a) = \frac{d_0}{c_0} C_1(a), \qquad (a \in \mathfrak{A}).$$

If we set $\lambda := \frac{d_0}{c_0} = \frac{m_D}{m_C}$, then for all $a \in \mathfrak{A}, m \in \mathbb{C}, b \in \mathfrak{B}$

$$D(\begin{pmatrix} a & m \\ 0 & b \end{pmatrix}) = \begin{pmatrix} \lambda 1_{\mathfrak{A}} & 0 \\ 0 & 0 \end{pmatrix} C(\begin{pmatrix} a & m \\ 0 & b \end{pmatrix}).$$

According to [1], the product of two derivations in semiprime algebras is zero for the case that the product is a derivation. This yeilds us the following result:

Corollary 3.4. *Let $\mathfrak{A}$ and $\mathfrak{B}$ be as Proposition 1.2 and suppose that they are semiprime. Then, DC is a derivation if and only if $D_1C_1 = 0, D_2C_2 = 0$ and the following holds for all $a, u \in \mathfrak{A}, b, u \in \mathfrak{B}, m, n \in \mathfrak{M}$*

$$\begin{aligned} &C_1(a) \cdot [M_D(n) - \tau_{u,v}(m_D)] + D_1(a) \cdot [M_C(n) - \tau_{u,v}(m_C)] \\ &= [\tau_{a,b}(m_D) + M_D(m)] \cdot C_2(v) + [\tau_{a,b}(m_C) + M_C(m)] \cdot D_2(v). \end{aligned}$$

Corollary 3.5. *Let $\mathfrak{A}, \mathfrak{B}$ be two unital semisimple commutative Banach algebras, $\mathfrak{M}$ be an essential Banach $\mathfrak{A}$-$\mathfrak{B}$-bimodule and $\mathfrak{T}$ be as Definition 1.1 and let $D, C : \mathfrak{T} \to \mathfrak{T}$ be two continuous derivations. Then, DC is a derivation.*

Example 3.6. The group algebra $\ell^1(G)$ of commutative group G is a unital semisimple commutative Banach algebra and so $\ell^1(G)$ has the assumptions of Corollary 3.5.

Example 3.7. Suppose that $n \in \mathbb{N}$ and

$$\mathbb{D}_n = \left\{ \begin{pmatrix} \lambda_1 & 0 & \dots & 0 \\ 0 & \lambda_2 & \dots & 0 \\ & & & \\ 0 & 0 & \dots & \lambda_n \end{pmatrix} : \lambda_i \in \mathbb{C}, i = 1, 2, \dots n \right\}.$$

Set $\mathfrak{A} = \mathfrak{B} = \mathbb{D}_n$, and $\mathfrak{M} = \mathbb{M}_n$, where $\mathbb{M}_n$ is the Banach algebra of matrices with elements in $\mathbb{C}$. Suppose that $D, C : \mathfrak{T} \to \mathfrak{T}$ are two continuous derivations. Then, DC is a derivation.

References

[1] T. Creedon, *Product of derivations*, Proc. Edinb. Math. Soc., **41**(2) (1998), 407-410.

[2] H. G. Dales, *Banach algebra and automatic continuity*, London Math. Soc. Monogr. Ser., vol. 24. Clarendon Press, Oxford (2000).

[3] B. E. Forrest and L. W. Marcoux, *Derivations of triangular Banach algebras*, Indiana Univ. Math. J., **45** (1996), 441-462.

[4] F. Ghahramani, *Homomorphisms and derivations on weighted convolution algebras*, J. Lond. Math. Soc., **21**(2) (1980), 149-161.

[5] H. Kamowitz and S. Scheinberg, *Derivations and automorphisms of* $L^1(0,1)$, Trans. Amer. Math. Soc., **135** (1969), 415-427.

[6] I. M. Singer and J. Wermer, *Derivation on commutative normed algebra*, Math. Ann., **129** (1955), 260-264.

Mohammad Hossein Sattari

Department of Mathematics, Azarbaijan Shahid Madani University, P.O.Box 53751 71379, Tabriz, Iran

Email: `sattari@azaruniv.edu`

Hamid Shafieasl

Department of Mathematics, Azarbaijan Shahid Madani University, P.O.Box 53751 71379, Tabriz, Iran

Email: `h.shafieasl@azaruniv.ac.ir`