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SOME RESULTS ON PIT AND GPIT THEOREMS

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Abstract. In this paper we generalize the PIT and the $GPIT$ that can be used to study the heights of prime ideals in a general commutative Noetherian ring R and the dimension theory of such a ring and we use these generalizations to prove some useful results.

Key Words: PIT, GPIT, Noetherian rings, Local rings, Finitely generated module.
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1. Introduction

We assume throughout that R is a commutative Noetherian ring and M is a non-zero finitely generated R-module.

In this paper, we are going to generalize heights of prime ideals in R, and the dimension theory of such a ring. The start point will be Krull's Principal Ideal Theorem (PIT): this states that, if $a \in R$ is a non-unit of R and $P \in Spec(R)$ is a minimal prime ideal of the principal ideal (a), then $htP \leq 1$. From this, we are able to go on to prove Generalized Principal Ideal Theorem $GPIT$, which shows that, if I be a proper ideal of R which can be generated by n elements, then $htP \leq n$, for every minimal prime ideal P of I. A consequence is that each $Q \in Spec(R)$ has finite height, because Q is a minimal prime ideal of itself and every ideal of R is finitely generated.

There are consequences for local rings. If (R, J) is a local ring, ?then $dimR = htJ$, ?and so R has finite dimension. In fact, we know that

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$dimR$ is the least integer n for which there exists an $J-$primary ideal that can be generated by n elements.

In Theorem 2.4 and Corollary 2.5 we generalize the GPIT and the PIT and in Theorem 2.9, we prove the promised converse of Theorem 2.4.

Let I be an ideal of R. We recall that

$$htI = min\{htP | P \in Spec(R), I \subseteq P\}.$$

Let $a_1, ..., a_n \in R$. We know by $[1, 16.1]$, that $a_1, ..., a_n$ form an $M-$sequence of elements of R precisely when

(i) $M \neq (a_1, ..., a_n)M$, and

(ii) For each $i = 1, ..., n$, the element a_i is a non-zerodivisor on the $R-$module $\frac{M}{(a_1,...,a_{i-1})M}$.

For $P \in Supp(M)$, we know by $[4, Ex.17.15]$, that the $M-$height of P , denoted $ht_M P$, is defined by $dim_{R_P} M_P = dim(\frac{R_P}{Ann_{R_P}(M_P)})$. Let I be an ideal of R such that $M \neq IM$. We know by $[4, Ex.9.23]$ that there exists a prime ideal $P \in Supp(M)$ such that $I \subseteq P$ and we know by $[4, Ex.17.15]$, that the $M-$height of I, denoted $ht_M I$, is defined by

$$ht_M I = min\{ht_M P | P \in Supp(M), I \subseteq P\}.$$

If (R, J) is a local ring, then we show the $ht_M J$ by $dimM$.

We will denote the set of all prime ideals of R by $Spec(R)$ and the set of all maximal ideals of R by $Max(R)$.

2. Main Results

Remark 2.1. Let R be a commutative Noetherian ring, M be a non-zero finitely generated R-module and I be an ideal of R such that $M \neq IM$. The M-sequence $(a_i)_{i=1}^n$ is a maximal M-sequence in I if it is impassible to fined an element $a_{n+1} \in I$ such that $a_1, ..., a_{n+1}$ form an M-sequence of length $n + 1$. This is equivalent to the statement that $I \subseteq Zdv_R(\frac{M}{(a_1,...,a_n)M})$. Because, for every $b \in I$, we have $M \neq (a_1, ..., a_n, b)M$. There exists an M-sequence contained in I, for the empty M-sequence is one such. We know by [4, Thm. 16.13], that every two maximal $M-$sequence in I have the same length. The common length of all maximal $M-$sequences in I denoted by $grade_M I$. If $M = R$, then we show $grade_R I$ by $gradeI$. Also, every M-sequence in I can be extended to a maximal M-sequence in I and we have $grade_M(I) < \infty$, by [4, Prop. 16.10].

Theorem 2.2. *Let R be a commutative Noetherian ring, M be a non-zero finitely generated R-module and I be an ideal of R such that $M \neq IM$. Let $grade_M(I) = n$ and I generate by n elements. Then I can be generate by the elements of an M-sequence of length n.*

Proof. If $n = 0$, then there is nothing to prove. So we assume that $n > 0$. Suppose that $I = (a_1, ..., a_n)$. We show that there exists an M-sequence $(b_i)_{i=1}^{n}$ in I such that $b_n = a_n$ and for suitable elements $r_{ij} \in R$ $(1 \leq i \leq n-1$ and $i+1 \leq j \leq n)$, $b_i = a_i + \Sigma_{j=i+1}^{n} r_{ij} a_j$. We have $I = (a_1, ..., a_n) = (b_1, ..., b_n)$ and so the theorem will be proved.

Now, we construct $b_1, ..., b_n$ by an inductive process. We assume that $j \in \mathbf{N}$ with $1 \leq j \leq n$, and that we have constructed elements b_i of R for $1 \leq i < j$ with the required properties. This is certainly the case when $j = 1$. Set $J = (b_1, ..., b_{j-1})$. (for $j = 1$, set $J = (0)$ and other, similar, simplifications should be made in that case). Since $(b_i)_{i=1}^{n-1}$ is an M-sequence in I and $grade_M(I) = n > j-1$, we have $I \nsubseteq Zdv_R(\frac{M}{JM})$. Now, we show that $(a_j, a_{j+1}, ..., a_n) \nsubseteq Zdv_R(\frac{M}{JM})$. Suppose on the contrary, that $(a_j, ..., a_n) \subseteq Zdv_R(\frac{M}{JM})$.

Let $c \in I$. We have $c = s_1 a_1 + ... + s_n a_n$, where $s_i \in R, (1 \leq i \leq n)$. We have

$$\begin{gathered} a_1 = b_1 - \Sigma_{k=2}^{n} r_{1k} a_k \\ a_2 = b_2 - \Sigma_{k=3}^{n} r_{2k} a_k \\ \vdots \\ a_{j-1} = b_{j-1} - \Sigma_{k=j}^{n} r_{jk} a_k \end{gathered}$$

So there exist $t_1, ..., t_n \in R$ with $c = t_1 b_1 + ..., t_{j-1} b_{j-1} + t_j a_j + ... + t_n a_n$. Our supposition that $(a_j, ..., a_n) \subseteq Zdv_R(\frac{M}{JM})$ means that $t_j a_j + ... + t_n a_n \in Zdv_R(\frac{M}{JM})$. Thus $c - t_1 b_1 - ... - t_{j-1} b_{j-1} \in Zdv_R \in (\frac{M}{JM})$. So there exists $x + JM \in \frac{M}{JM}$ with $x \notin JM$ such that $c - t_1 b_1 - ... - t_{j-1} b_{j-1}(x + JM) = JM$. So $c(x + JM) = JM$ and so $c \in Zdv_R(\frac{M}{JM})$. Hence $I \subseteq Zdv_R(\frac{M}{JM})$, which is a contradiction. Therefore, $(a_j, ..., a_n) \nsubseteq Zdv_R(\frac{M}{JM})$.

We have $Zdv_R(\frac{M}{JM}) = \cup_{P \in Ass(\frac{M}{JM})} P$, by [4, Corollary 9.36]. Since R is Noetherian and $\frac{M}{JM}$ is finitely generated we have $|Ass(\frac{M}{JM})| < \infty$, by [2, Page72, Cor. 2]. Let $Ass(\frac{M}{JM}) = \{P_1, ..., P_t\}$. So $Zdv_R(\frac{M}{JM}) = \cup_{i=1}^{t} P_i$. Thus $(a_j, ...a_n) \nsubseteq \cup_{i=1}^{t} P_i$ and so $(a_j) + (a_{j+1}, ..., a_n) \nsubseteq \cup_{i=1}^{t} P_i$. So there exists $b'_j \in (a_{j+1}, ..., a_n)$ with $a_j + b'_j \notin Zdv_R(\frac{M}{JM})$, by [4,Theorem 3.64]. There exist $r_{jj+1}, ..., r_{jn} \in R$ such that $b'_j = r_{jj+1} a_{j+1} + ... + r_{jn} a_n$.

Thus $a_j+r_{jj+1}a_{j+1}+...+r_{jn}a_n \notin Zdv_R(\frac{M}{JM})$. Set $b_j=a_j+r_{jj+1}a_{j+1}+...+r_{jn}a_n$. □

Remark 2.3. Let R be a commutative Noetherian ring, M be a non-zero finitely generated $R-$module and I be an ideal of R with $IM \neq M$. We know that $P \in Supp(\frac{M}{IM})$ if and only if $I+Ann(M) \subseteq P$, by [1, Page46, Ex19(vii)].

In Theorem 2.4, we generalize the GPIT.

Theorem 2.4. *Let R be a commutative Noetherian ring and M be a non-zero finitely generated $R-$module. Let $a_1,...,a_n \in R$ with $(a_1,...,a_n)M \neq M$. Then $ht_M P \leq n$, for every minimal ideal P in $Supp(\frac{M}{(a_1,...,a_n)M})$.*

Proof. Set $I=AnnM$ and $S=\frac{R}{I}$. So S is a commutative Noetherian ring and M is a non-zero finitely generated S-module. Also, $(a_1+I,...,a_n+I)M=(a_1,...,a_n)M \neq M$.

Let P be a minimal ideal in $Supp(\frac{M}{(a_1,...,a_n)M})$. We show that $\frac{P}{I}$ is a minimal ideal in $Supp(\frac{M}{(a_1+I,...,a_n+I)M})$. Since $(a_1,...,a_n) \subseteq P$ we have $(a_1+I,...,a_n+I) \subseteq \frac{P}{I}$. Also $Ann_S M=0$. So $\frac{P}{I} \in Supp(\frac{M}{(a_1+I,...,a_n+I)M})$, by Remark 2.3. Let $\frac{Q}{I} \in Supp(\frac{M}{(a_1+I,...,a_n+I)M})$ and $\frac{Q}{I} \subseteq \frac{P}{I}$. So $(a_1+I,...,a_n+I) \subseteq \frac{Q}{I}$, by Remark 2.3. Since $I+(a_1,...,a_n) \subseteq Q$ we have $Q \in Supp(\frac{M}{(a_1,...a_n)M})$, by Remark 2.3. Since P is a minimal ideal in $Supp(\frac{M}{(a_1,...,a_n)M})$ and $Q \subseteq P$ we have $P=Q$. Therefore, P is a minimal ideal in $Supp(\frac{M}{(a_1+I,...,a_n+I)M})$.

Now, we show that $ht_M P = ht_M \frac{P}{I}$. Let $ht_M P=t$. So there exists a chain of prime ideals $P_0 \subset P_! \subset ... \subset P_t = P$ such that $P_i \in Supp(M)$. So $I \subseteq P_i$, for all $i \in \{1,...,t\}$, by Remark 2.3. So $\frac{P_0}{I} \subset ... \subset \frac{P_t}{I} = \frac{P}{I}$ is a chain of prime ideals in $Supp_S(M)$. So $ht_M(\frac{P}{I}) \geq ht_M P$. If $\frac{P_0}{I} \subset ... \subset \frac{P_k}{I} = \frac{P}{I}$ be a chain in $Supp_S(M)$, then $I \subseteq P_i$, for all $i \in \{1,...,k\}$ and $P_0 \subset ... \subset P_k = P$ is a chain in $Supp_R(M)$ and so $ht_M P \geq ht_M \frac{P}{I}$. Therefore, $ht_M P = ht_M \frac{P}{I}$.

So we can assume that $AnnM=0$.

Also, we know that $ht_M \frac{P}{I} = ht_S \frac{P}{I}$ and $\frac{Q}{I}$ is a minimal ideal in $Supp(\frac{M}{(a_1+I,...,a_n+I)M})$ if and only if $\frac{Q}{I}$ is a minimal prime ideal over $(a_1+I,...,a_n+I)$, because $Ann_S M=0$. So without loss of generality we can assume that R is a commutative Noetherian ring and M is a non-zero finitely generated R-module and $a_1, ..., a_n \in R$ with $(a_1, ...a_n)M \neq M$ and P is a minimal prime ideal over $(a_1, ..., a_n)$. We must show that $htP \leq n$. Since $(a_1, ...a_n)M \neq M$ we have $(a_1, ..., a_n)$ is a proper ideal of R. so $htP \leq n$, by GPIT. □

Now, we have a generalization for the PIT in Corollary 2.5.

Corollary 2.5. *Let R be a commutative Noetherian ring and M be a non-zero finitely generated $R-$module. Let $a \in R$ with $(a)M \neq M$. Then $ht_M P \leq 1$, for every minimal ideal P in $Supp(\frac{M}{(a)M})$.*

Corollary 2.6. *Let R be a commutative Noetherian ring and M be a non-zero finitely generated $R-$module.Then $ht_M P < \infty$, for every $P \in Supp(M)$. So if (R, J) is a local ring, then $dimM < \infty$.*

Proof. We show that $PM \neq M$. If $PM = M$, then $M_P = PR_P M_P$. So $M_P = 0$, by Nakayama's lemma, a contradiction. So $PM \neq M$.

Since $P \in Supp(M)$ we have $AnnM \subseteq P$, by Remark 2.3, and hence $P \in Supp(\frac{M}{PM})$. Let $Q \in Supp(\frac{M}{PM})$ and $Q \subseteq P$. So $P + Ann(M) \subseteq Q$, by Remark 2.3. So $P = Q$. Hence P is a minimal ideal in $Supp(\frac{M}{PM})$. Since R is Noetherian, P is finitely generated. So $ht_M P < \infty$, by Theorem 2.4.

Let (R, J) be a local ring. Since $M \neq 0$ we have $AnnM \subseteq J$ and so $J \in SuppM$, by Remark 2.3. So $dimM = ht_M J < \infty$. □

Corollary 2.7. *Let R be a commutative Noetherian ring and M be a non-zero finitely generated $R-$module.*

(i) Let $P, Q \in Supp(M)$ with $P \subseteq Q$. Then $ht_M P \leq ht_M Q$, and $ht_M P = ht_M Q$ if and only if $P = Q$.

(ii) The ring R satisfies descending chain condition on $Supp(M)$.

Proof. (i) We know that $ht_M P < \infty$ by Lemma 2.6. Let $ht_M P = n$ and $P_0 \subset P_1 \subset ... \subset P_n = P$ be a chain of prime ideals in $Supp(M)$. If $P \neq Q$, then the chain $P_0 \subset P_1 \subset ... \subset P_n \subset Q$ in $Supp(M)$ shows that $ht_M Q \geq n + 1$. All the claims follow quickly from this.

(ii) Let $P_0 \supseteq P_1 \supseteq ...$ be a descending chain in $Supp(M)$. We have $ht_M P_0 < \infty$ by Corollary 2.6. So there exists an $n \in \mathbf{N} \cup \{0\}$ such that $P_i = P_n$, for every $i \geq n$. □

Lemma 2.8. *Let R be a commutative Noetherian ring and M be a non-zero finitely generated $R-$module. Let I be an ideal of R and $P \in Supp(M)$ with $I \subseteq P$. Suppose that $ht_M I = ht_M P$. Then P is a minimal ideal in $Supp(\frac{M}{IM})$.*

Proof. Suppose that P is not a minimal ideal in $Supp(\frac{M}{IM})$. Since $I + AnnM \subseteq P$ we have $P \in Supp(\frac{M}{IM})$, by Remark 2.3. So there exists a minimal ideal Q in $Supp(\frac{M}{IM})$ such that $Q \subset P$. Hence $ht_M Q < ht_M P$, by Corollary 2.7(i). Since $ht_M I = min\{ht_M P | P \in Supp(\frac{M}{IM})\}$, we have $ht_M I \leq ht_M Q < ht_M P$, which is a contradiction. □

We are now in a position to prove the promised converse of Theorem 2.4.

Theorem 2.9. *Let R be a commutative Noetherian ring and M be a non-zero finitely generated $R-$module. Let $P \in Supp(M)$ with $ht_M P = n$. Then there exists an ideal I of R which can be generated by n elements such that $I \subseteq P$ and $ht_M I = n$.*

Proof. We use induction on n. When $n = 0$, we just take $I = 0$ to fined an ideal with the stated properties. So suppose, inductively, that $n > 0$ and the claim has been proved for smaller values of n. Now there exists a chain $P_0 \subset P_1 \subset ... \subset p_{n-1} \subset P_n = P$ of $Supp(M)$. Note that $ht_M P_{n-1} = n-1$, because, $ht_M P_{n-1} < ht_M P$, by Corrolary 2.7(i), while $ht_M P_{n-1} \geq n - 1$, by virtue of the above chain. So we can apply the inductive hypothesis to P_{n-1}.

The conclusion is that there exists a proper ideal J of R which can be generated by $n-1$ elements, $a_1, ..., a_{n-1}$ and which is such that $J \subseteq P_{n-1}$ and $ht_M J = n - 1$. So we have P is a minimal ideal in $Supp(\frac{M}{JM})$, by Lemma 2.8. We have $Ass(\frac{M}{JM})$ is finite, by [2, Page72, Cor. 2] also minnimal elements of $Ass(\frac{M}{JM})$ and minimal elements of $Supp(\frac{M}{JM})$ are the same, by [2, Page75, Cor. of Prop. 7]. So minimal elements in $Supp(\frac{M}{JM})$ are finite. Note also that, in view of the Theorem 2.4, and the fact $ht_M J = n - 1$, $ht_M Q = n - 1$, for every minimal ideal Q in $Supp(\frac{M}{JM})$.

Let the other minimal ideals in $Supp(\frac{M}{JM})$, in addition to P_{n-1}, be $Q_1, ..., Q_t$. (In fact, t could be 0, but this does not affect the argument significantly.) We now use the Prime Avoidance Theorem to see that $P_1 \nsubseteq P_{n-1} \cup Q_1 \cup ... \cup Q_t$. If this were not the case, then either $P \subseteq P_{n-1}$ or $P \subseteq Q_i$, for some i with $i \in \{1, ..., t\}$, which are contradictions, by Corollary 2.7(i). Because, $ht_M P = n$ and $ht_M P_{n-1} = ht_M Q_1 = ... = ht_M Q_t = n - 1$. Therefore, there exists $a_n \in P \setminus (P_{n-1} \cup Q_1 \cup ... \cup Q_t)$.

Set $I := \Sigma_{i=1}^{n} Ra_i = J + Ra_n$. We show that I has all the desired properties. It is clear from its definition that I can be generated by n elements and that $I = J + Ra_n \subseteq P_{n+1} + P = P$. Now, we show that

$ht_M I = n$. Since $J \subseteq I \subseteq P$ and $ht_M J = n-1$ and $ht_M P = n$, we must have $ht_M I = n-1$ or $ht_M I = n$. Suppose that $ht_M I = n-1$. So there exists a minimal ideal P' in $Supp(\frac{M}{IM})$ with $ht_M P' = n-1$. Now, $J \subseteq I \subseteq P'$ and $ht_M J = n-1$. We have P' is a minimal ideal in $Supp(\frac{M}{JM})$, by Lemma 2.8. So $P' = P_{n-1}$ or $P' = Q_i$, for some $i \in \{1, ..., t\}$. But, we have $a_n \in I \subseteq P'$ and $a_n \notin P_{n-1}$ and $a_n \notin Q_i$, for all $i \in \{1, ..., t\}$. So $ht_M I = n$. □

Corollary 2.10. *With the same assumptions as in Theorem 2.9, we have P is a minimal ideal in $Supp(\frac{M}{IM})$.*

Proof. This is clear by Lemma 2.8 and Theorem 2.9. □

Lemma 2.11. *Let R be a commutative Noetherian ring, M be a non-zero finitely generated $R-$module and I be an ideal of R with $IM \neq M$. Then $Ann_{\frac{R}{I}} \frac{M}{IM} \subseteq rad(\frac{I+AnnM}{I})$.*

Proof. Let $r+I \in Ann_{\frac{R}{I}} \frac{M}{IM}$ and $M = (x_1, ..., x_n)$. So there exist $a_{ij} \in I$,$1 \leq i, j \leq n$, such that $rx_i = \Sigma_{j=1}^n a_{ij} x_j$. Let

$$A = \begin{pmatrix} r-a_{11} & -a_{12} & \cdots & -a_{1n} \\ -a_{21} & r-a_{22} & \cdots & -a_{2n} \\ \vdots & \ddots & & \\ -a_{n1} & \cdots & \cdots & r-a_{nn} \end{pmatrix}$$

We have $A \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}$. So $A^t A \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}$, where A^t is the transposed of A. Thus $(detA) I_n \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}$ and so $detA \in AnnM$. So we have $detA = r^n - \alpha$, for some $\alpha \in I$. Thus $r^n - detA \in I$. So $r^n + I \in \frac{AnnM+I}{I}$. Therefore, $r + I \in rad(\frac{AnnM+I}{I})$, by [3, Chap. 8, Thm. 2.6]. □

Corollary 2.12. *Let R be a commutative Noetherian ring and M be a non-zero finitely generated $R-$module. Let I be an ideal of R which can be generated by n elements and $P \in Supp(M)$ be such that $I \subseteq P$. Then*

$$ht_{\frac{M}{IM}} \frac{P}{I} \leq ht_M P \leq ht_{\frac{M}{IM}} \frac{P}{I} + n.$$

Proof. Let $ht_{\frac{M}{IM}}\frac{P}{I}=t$. So there exists a chain $\frac{P_0}{I}\subset\frac{P_1}{I}\subset...\subset\frac{P_t}{I}=\frac{P}{I}$ with $\frac{P_i}{I}\in Supp(\frac{M}{IM})$, for all $i\in\{1,...,t\}$. So $Ann_{\frac{R}{I}}(\frac{M}{IM})\subseteq\frac{P_i}{I}$, by Remark 2.3. But, it is easy to show that $\frac{Ann(M)+I}{I}\subseteq Ann_{\frac{R}{I}}(\frac{M}{IM})$. Thus, $\frac{Ann(M)+I}{I}\subseteq\frac{P_i}{I}$ and so $Ann(M)\subseteq P_i$, for all $i\in\{1,...,t\}$. So $P_i\in Supp(M)$, by Remark 4, and $P_0\subset P_1\subset...\subset P_t=P$ is a chain of $Supp(M)$ and so $ht_MP\geq t$. Therefore, $ht_MP\geq ht_{\frac{M}{IM}}\frac{P}{I}$.

Let $b_1,...,b_n$ generate I and $ht_{\frac{M}{IM}}\frac{P}{I}=t$. By Lemma 2.8 and Theorem 2.9, there exist $a_1,...,a_t\in R$ such that $\frac{P}{I}$ is a minimal ideal in $Supp(\frac{\frac{M}{IM}}{\frac{(a_1,...a_t)+I}{I}\frac{M}{IM}})$.

Set $J:=(a_1,...,a_t)$. We show that P is a minimal ideal in $Supp(\frac{M}{(I+J)M})$. First we have $\frac{J+I}{I}+Ann_{\frac{R}{I}}\frac{M}{IM}\subseteq\frac{P}{I}$. So $Ann_{\frac{R}{I}}\frac{M}{IM}\subseteq\frac{P}{I}$ and we know that $\frac{Ann_RM}{I}\subseteq Ann_{\frac{R}{I}}\frac{M}{IM}$. So $\frac{AnnM}{I}\subseteq\frac{P}{I}$ and so $AnnM\subseteq P$. Also, we have $J+I\subseteq P$. So $(J+I)+AnnM\subseteq P$. Therefore, P is a minimal ideal in $Supp(\frac{M}{(J+I)M})$, by Remark 2.3.

Let $P'\in Supp(\frac{M}{(J+I)M})$ with $P'\subseteq P$. So $(J+I)+AnnM\subseteq P'$. Thus, $\frac{I+AnnM}{I}\subseteq\frac{P'}{I}$. So $rad(\frac{I+AnnM}{I})\subseteq\frac{P'}{I}$ and so $Ann_{\frac{R}{I}}\frac{M}{IM}\subseteq\frac{P'}{I}$, by Lemma 2.11. Thus we have $\frac{I+J}{I}+Ann_{\frac{R}{I}}\frac{M}{IM}\subseteq\frac{P'}{I}$ and so $\frac{P'}{I}\in Supp(\frac{\frac{M}{IM}}{\frac{I+J}{I}\frac{M}{IM}})$, by Remark 2.3.

Since $\frac{P'}{I}\subseteq\frac{P}{I}$ and $\frac{P}{I}$ is a minimal ideal in $Supp(\frac{\frac{M}{IM}}{\frac{I+J}{I}\frac{M}{IM}})$ we have $\frac{P'}{I}=\frac{P}{I}$ and so $P'=P$. Therefore, P is a minimal ideal in $Supp(\frac{M}{(I+J)M})$. So $ht_MP\leq t+n$, by Theorem 2.4. □

Proposition 2.13. *Let R be a commutative Noetherian ring and M be a non-zero finitely generated $R-$module. Let $a_1,...,a_n$ be an $M-$sequence of elements of R and $I=(a_1,...,a_n)$. Then $ht_MI=n$.*

Proof. Since $(a_i)_{i=1}^n$ is an M-sequence we have $IM\neq M$. So there exists $x\in M\setminus IM$. So $a_ix\in IM$, for all $i\in\{1,...,n\}$. Thus $I\subseteq Zdv_R(\frac{M}{IM})$. So $(a_i)_{i=1}^n$ is a maximal M-sequence of elements of I, by Remark 2.1, and so $grade_MI=n$.

We know that $grade_MI\leq dim_{R_P}M_P=ht_MP$, for all $P\in Supp(M)$ with $I\subseteq P$, by [4 , 16.31]. So $grade_MI\leq min\{ht_MP|I\subseteq P\in Supp(M)\}=ht_MI$. Thus, $n=grade_MI\leq ht_MI$. We have

$$\begin{aligned} ht_M I &= min\{ht_M P | I \subseteq P \in Supp(M)\} \\ &= min\{ht_M P | I + Ann(M) \subseteq P\} \\ &= min\{ht_M P | P \in Supp(\frac{M}{IM})\} \\ &= min\{ht_M P | P \in min\{Supp(\frac{M}{IM})\}\} \end{aligned}$$

by Remark 2.3. Since $IM \neq M$ we have $ht_M P \leq n$, for every minimal ideal P in $Supp(\frac{M}{IM})$, by Theorem 2.4. So $ht_M I \leq n$. Therefore $ht_M I = n$. □

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