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FUZZY WEAKLY IRREDUCIBLE IDEALS OF A RING

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Abstract. In this paper, we introduce the concept of a fuzzy weakly irreducible ideal of a commutative ring R with identity. This concept is a generalization of the concept of a fuzzy strongly irreducible ideal. Also, relationships between fuzzy maximal, fuzzy quasi primary and fuzzy weakly irreducible ideals in a ring are proved.

Key Words: Fuzzy weakly irreducible ideal, fuzzy quasi primary ideal, fuzzy prime ideal.
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1. INTRODUCTION

The concept of a fuzzy set, introduced by Zadeh [9], was applied to generalize some concepts from algebra. In [6], Rosenfeld has defined fuzzy subgroupiod and fuzzy subgroups. Liu [3] introduced fuzzy ideals and Mukherjee and Sen [5] introduced fuzzy prime ideals in a ring. Malik and Moderson [4] have studied fuzzy maximal, fuzzy prime and fuzzy primary ideals in rings. Dixit et. al. [1] have studied fuzzy rings. In [2], Kumar studied fuzzy irreducible ideals in rings. Later, Shah and Saeed [8] have introduced strongly primary fuzzy ideals and strongly irreducible fuzzy ideals in a unitary commutative ring. In [7], Samiei and Moghimi introduced weakly irreducible ideals in rings.

The purpose of this paper is to introduce and study fuzzy analogues of the results in [7]. In this paper we introduce the concept of a fuzzy

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weakly irreducible and a fuzzy quasi primary ideal and prove some results.

2. PRELIMINARIES

Throughout this paper R denotes a commutative ring with identity. For the definition of a fuzzy set, its image, the intersection of two fuzzy sets, we refer to Zadeh [9].
We recall some definitions and results.

Definition 2.1. (Zadeh [9]) Let ' $\cdot$ ' be a binary composition in S and let μ, σ be any two fuzzy subsets of S. Then, product $\mu \cdot \sigma$ is defined as follows:

$$(\mu \cdot \sigma)(x) = \begin{cases} sup_{x=y\cdot z}(min(\mu(y), \sigma(z))), & \text{where } y,\ z \in S, \\ 0, & \text{if x is not expressible as } x = y \cdot z \text{ for all } y,\ z \in S. \end{cases}$$

Definition 2.2. (Rosenfeld [6]) Let f be any function from a set S to a set T. Let μ be a fuzzy subset of S. Then, μ is called f-invariant if the following condition holds:

$$f(x) = f(y) \text{ implies } \mu(x) = \mu(y), \text{ where } x, y \in S.$$

Definition 2.3. (Liu [3]) A fuzzy subset μ of a ring R is called a fuzzy ideal of R if, for all $x,\ y \in R$, the following conditions are satisfied:

(i) $\mu(x - y) \geq min(\mu(x), \mu(y))$.
(ii) $\mu(xy) \geq max(\mu(x), \mu(y))$.

Definition 2.4. (Mukherjee and Sen [5]) A fuzzy ideal μ of a ring R, is called fuzzy prime if, for any two fuzzy ideals σ and θ of R, $\sigma \cdot \theta \subseteq \mu$ implies that either $\sigma \subseteq \mu$ or $\theta \subseteq \mu$.

Definition 2.5. (Kumar [2, Definition 4.1.13, p. 60]) Let μ be a fuzzy ideal of a ring R. The fuzzy nil radical of μ, denoted by $\sqrt{\mu}$, is defined by $(\sqrt{\mu})(x) = t$, whenever $x \in \sqrt{\mu_t}$, $x \notin \sqrt{\mu_s}$ *for all* $s > t$.

Remark 2.6. The following observations can be proved by using Definition 2.5.

(i) $\sqrt{\mu}(0) = \mu(0)$.
(ii) $\mu \subseteq \sqrt{\mu}$.
(iii) $\sqrt{\sqrt{\mu}} = \sqrt{\mu}$.
(iv) If μ is constant, then $\sqrt{\mu} = \mu$.
(v) $\sqrt{\mu}(x) = sup\{t \mid x \in \sqrt{\mu_t}\}$.
(vi) $Im(\sqrt{\mu}) \subseteq Im\mu$, if card $Im\ \mu < \infty$.

Definition 2.7. (Kumar [2, Definition 4.2.1, p. 62]) A fuzzy ideal μ of a ring R is called fuzzy maximal if, $Im(\mu) = \{1, \alpha\}$, where $\alpha \in [0, 1[$ and the level ideal $\mu_t = \{x \in R \mid \mu(x) = 1\}$ is a maximal ideal of R.

Example 2.8. Define a fuzzy ideal μ of the ring $\mathbb{Z}_{27}$ as follows.

$$\mu(x) = \begin{cases} 1, & \text{if } x \in \ \{0, 3, 6, 9, 12, 15, 18, 21, 24\}, \\ 0.75, & \text{otherwise.} \end{cases}$$

Then μ is a fuzzy maximal ideal of $\mathbb{Z}_{27}$, as $Im(\mu) = \{1, 0.75\}$ and the level ideal $\mu_t = \{x \in R \mid \mu(x) = 1\} = \{0, 3, 6, 9, 12, 15, 18, 21, 24\}$ is a maximal ideal of $\mathbb{Z}_{27}$.

Remark 2.9. (Kumar [2, p. 63]) If μ is a fuzzy maximal ideal of a ring R, then the following statements hold:

(i) μ is fuzzy prime.
(ii) $\sqrt{\mu} = \mu$.

Definition 2.10. (Kumar [2, Definition 6.1.1, p. 87]) A fuzzy ideal μ of a ring R is called fuzzy irreducible if it is not an intersection of two fuzzy ideals of R properly containing μ; otherwise μ is called fuzzy reducible.

Definition 2.11. (Kumar [2, Definition 5.1.1, p. 73]) A fuzzy ideal μ of a ring R is called fuzzy primary if, for any two fuzzy ideals σ and θ of R, the conditions $\sigma \cdot \theta \subseteq \sqrt{\mu}$ and $\sigma \nsubseteq \mu$ together imply that $\theta \subseteq \sqrt{\mu}$.

Definition 2.12. (Shah and Saeed [8]) A nonconstant fuzzy ideal μ of a ring R is said to be fuzzy strongly irreducible if for each pair of fuzzy ideals θ and σ of R, if $\theta \cap \sigma \subseteq \mu$, then either $\theta \subseteq \mu$ or $\sigma \subseteq \mu$.

Lemma 2.13. *(Rosenfeld [6]) Let f be any function from a set S to a set S'; μ, θ be any two fuzzy subsets of S; and μ', θ' be any two fuzzy subsets of S', then the following statements hold:*

(i) $f(f^{-1}(\mu')) = \mu'$, $\mu \subseteq f^{-1}(f(\mu))$.
(ii) $f^{-1}(f(\mu)) = \mu$, *provided that μ is f-invariant.*
(iii) $\mu \subseteq \theta \Rightarrow f(\mu) \subseteq f(\theta)$.
(iv) $\mu' \subseteq \theta' \Rightarrow f^{-1}(\mu') \subseteq f^{-1}(\theta')$.

3. FUZZY WEAKLY IRREDUCIBLE IDEALS OF A RING

As a generalization of the concept of a strongly irreducible ideal, Samiei and Moghimi [7] introduced the concept of a weakly irreducible ideals in a commutative ring.

Definition 3.1. [7] A proper ideal I of R is called weakly irreducible provided that for each pair of ideals A and B of R, $A \cap B \subseteq I$ implies that either $A \subseteq \sqrt{I}$ or $B \subseteq \sqrt{I}$.

We define a fuzzy analogue of this concept as follows.

Definition 3.2. A nonconstant fuzzy ideal μ of a ring R is said to be fuzzy weakly irreducible if for each pair of fuzzy ideals σ and θ of R, if $\sigma \cap \theta \subseteq \mu$ then either $\sigma \subseteq \sqrt{\mu}$ or $\theta \subseteq \sqrt{\mu}$.

Example 3.3. Define fuzzy ideals μ, σ and θ of $\mathbb{Z}$ by,

$$\mu(x) = \begin{cases} 1, & \text{if } x \in <7^3>, \\ 0.6, & \text{if } x \in <7^2> \sim <7^3>, \\ 0.4, & \text{if } x \in <7> \sim <7^2>, \\ 0.1, & \text{if } x \in \mathbb{Z} \sim <7>. \end{cases}$$

Then,

$$\sqrt{\mu}(x) = \begin{cases} 1, & \text{if } x \in <7>, \\ 0.1, & \text{if} x \in \mathbb{Z} \sim <7>. \end{cases}$$

$$\sigma(x) = \begin{cases} 0.8, & \text{if } x \in <7>, \\ 0.1, & \text{if} x \in \mathbb{Z} \sim <7>. \end{cases}$$

$$\theta(x) = \begin{cases} 1, & \text{if } x \in <7^3>, \\ 0.6, & \text{if } x \in <7^2> \sim <7^3>, \\ 0.4, & \text{if } x \in <7> \sim <7^2>, \\ 0.2, & \text{if } x \in \mathbb{Z} \sim <7>. \end{cases}$$

Now here,

$$(\sigma \cap \theta)(x) = \begin{cases} 0.8, & \text{if } x \in <7^3>, \\ 0.6, & \text{if } x \in <7^2> \sim <7^3>, \\ 0.4, & \text{if } x \in <7> \sim <7^2>, \\ 0.1, & \text{if } x \in \mathbb{Z} \sim <7>. \end{cases}$$

Also, we observe that, $\sigma \cap \theta \subseteq \mu$ with $\theta \nsubseteq \sqrt{\mu}$, but $\sigma \subseteq \sqrt{\mu}$. Hence, μ is a fuzzy weakly irreducible ideal of ring $\mathbb{Z}$.

Theorem 3.4. *If μ is a fuzzy strongly irreducible ideal of a ring R, then μ is fuzzy weakly irreducible.*

Proof. Let σ and θ be any two fuzzy ideals of R such that $\sigma \cap \theta \subseteq \mu$. Since μ is a fuzzy strongly irreducible ideal,

$$\sigma \subseteq \mu \text{ or } \theta \subseteq \mu. \tag{3.1}$$

From Remark 2.6(2), $\mu \subseteq \sqrt{\mu}$ and so from (3.1), we conclude that either

$$\sigma \subseteq \sqrt{\mu} \text{ or } \theta \subseteq \sqrt{\mu}.$$

Thus, μ is a fuzzy weakly irreducible ideal of R. □

Remark 3.5. We note that the converse of Theorem 3.4 may not be true. The fuzzy ideal μ in Example 3.3 is fuzzy weakly irreducible but it is not a strongly irreducible ideal because neither $\theta \nsubseteq \mu$ nor $\sigma \nsubseteq \mu$.

Theorem 3.6. *If μ is a fuzzy weakly irreducible and a fuzzy prime (semiprime) ideal of R, then $\sqrt{\mu}$ is a fuzzy irreducible ideal of R.*

Proof. Let σ and θ be two fuzzy ideals of R such that

$$\sigma \cap \theta \subseteq \sqrt{\mu}. \tag{3.2}$$

If μ is fuzzy prime, then it follows from the definition of $\sqrt{\mu}$, that $\mu = \sqrt{\mu}$.
If μ is a fuzzy semiprime ideal of R, then it follows from Theorem 4.3.7, p. 67 from Kumar [2], that $\mu = \sqrt{\mu}$. Hence from (3.2), $\sigma \cap \theta \subseteq \mu$.
As μ is a fuzzy weakly irreducible ideal of R we conclude that

$$\sigma \subseteq \sqrt{\mu} \text{ or } \theta \subseteq \sqrt{\mu}.$$

Thus, $\sqrt{\mu}$ is a fuzzy irreducible ideal of R. □

Definition 3.7. A nonconstant fuzzy ideal μ of R is said to be fuzzy quasi primary if for each pair of fuzzy ideals σ and θ of R, if $\sigma \cdot \theta \subseteq \mu$, then either $\sigma \subseteq \sqrt{\mu}$ or $\theta \subseteq \sqrt{\mu}$.

Example 3.8. Let $R = \mathbb{Z}_{12}$ and define fuzzy ideals μ, σ and θ as:

$$\mu(x) = \begin{cases} 0.9, & \text{if } x \in \{0, 2, 4, 6, 8, 10\}, \\ 0.3, & \text{otherwise.} \end{cases}$$

Then

$$\sqrt{\mu}(x) = \begin{cases} 0.9, & \text{if } x \in \{0, 2, 4, 6, 8, 10\}, \\ 0.3, & \text{otherwise.} \end{cases}$$

Also,

$$\sigma(x) = \begin{cases} 0.7, & \text{if } x \in \{0,4,8\}, \\ 0.1, & \text{otherwise.} \end{cases}$$

$$\theta(x) = \begin{cases} 1, & \text{if } x \in \{0,6\}, \\ 0.2, & \text{otherwise.} \end{cases}$$

Then

$$(\sigma \cdot \theta)(x) = \begin{cases} 0.7, & \text{if } x = 0, \\ 0.2, & \text{if } x \in \{4,8\}, \\ 0.1, & \text{otherwise.} \end{cases}$$

Here, $\sigma \cdot \theta \subseteq \mu$ and $\theta \nsubseteq \sqrt{\mu}$ but $\sigma \subseteq \sqrt{\mu}$.
Thus, μ is fuzzy quasi primary ideal of $\mathbb{Z}_{12}$.

Theorem 3.9. *If μ is a fuzzy quasi primary ideal of R, then μ is fuzzy weakly irreducible.*

Proof. Let $\sigma \cap \theta \subseteq \mu$. As $\sigma \cdot \theta \subseteq \sigma \cap \theta$, we get $\sigma \cdot \theta \subseteq \mu$.
Since μ is fuzzy quasi primary, either $\sigma \subseteq \sqrt{\mu}$ or $\theta \subseteq \sqrt{\mu}$.
Thus, μ is fuzzy weakly irreducible. □

Theorem 3.10. *Let f be a homomorphism from a ring R onto a ring R'. Let μ and μ' be fuzzy weakly irreducible ideals of R and R' respectively. Then the following statements hold:*

(i) *$f(\mu)$ is a fuzzy weakly irreducible ideal of R', provided that μ is f-invariant.*
(ii) *$f^{-1}(\mu')$ is a fuzzy weakly irreducible ideal of R, provided that every fuzzy ideal of R is f-invariant.*
(iii) *If each fuzzy ideal of R is f-invariant, then the mapping $\mu \to f(\mu)$ defines a one-to-one correspondence between the set of all f-invariant fuzzy weakly irreducible ideals of R and the set of all fuzzy weakly irreducible ideals of R'.*

Proof. (i): Let σ' and θ' be fuzzy ideals of R' such that $\sigma' \cap \theta' \subseteq f(\mu)$. As μ is f-invariant by Lemma 2.13, we get,

$$f^{-1}(\sigma' \cap \theta') \subseteq \mu. \tag{3.3}$$

For all $x \in R$ we have

$$\begin{aligned} f^{-1}(\sigma' \cap \theta')(x) &= (\sigma' \cap \theta')(f(x)), \\ &= \min(\sigma'(f(x)), \theta'(f(x)), \\ &= \min((f^{-1}(\sigma'))(x), (f^{-1}(\theta'))(x)), \\ &= (f^{-1}(\sigma') \cap f^{-1}(\theta'))(x). \end{aligned}$$

Hence $f^{-1}(\sigma' \cap \theta') = f^{-1}(\sigma') \cap f^{-1}(\theta')$.

Hence from (3.3) we get,

$$f^{-1}(\sigma') \cap f^{-1}(\theta') \subseteq \mu.$$

Since μ is a fuzzy weakly irreducible ideal of R, either

$$f^{-1}(\sigma') \subseteq \sqrt{\mu} \text{ or } f^{-1}(\theta') \subseteq \sqrt{\mu}.$$

By Lemma 2.13, we get

$$\sigma' \subseteq f(\sqrt{\mu}) \text{ or } \theta' \subseteq f(\sqrt{\mu}).$$

Using Theorem 5.1.10, p. 77 from Kumar [2] and μ is f-invariant, we conclude that either

$$\sigma' \subseteq \sqrt{f(\mu)} \text{ or } \theta' \subseteq \sqrt{f(\mu)}.$$

This shows that $f(\mu)$ is a fuzzy weakly irreducible ideal of R'.

(ii): Let σ and θ be fuzzy ideals of R such that

$$\sigma \cap \theta \subseteq f^{-1}(\mu').$$

By Lemma 2.13, we get

$$f(\sigma \cap \theta) \subseteq \mu'.$$

Hence by Theorem 3.4.1, p. 44 from Kumar [2], we get

$$f(\sigma) \cap f(\theta) \subseteq \mu'.$$

Since μ' is a fuzzy weakly irreducible ideal of R', we get

$$f(\sigma) \subseteq \sqrt{\mu'} \text{ or } f(\theta) \subseteq \sqrt{\mu'}.$$

Hence by Lemma 2.13, either

$$\sigma \subseteq f^{-1}(\sqrt{\mu'}) \text{ or } \theta \subseteq f^{-1}(\sqrt{\mu'}).$$

Thus by Theorem 5.1.10, p. 77 from Kumar [2], either

$$\sigma \subseteq \sqrt{f^{-1}(\mu')} \text{ or } \theta \subseteq \sqrt{f^{-1}(\mu')}.$$

(iii): Follows (i) and (ii) and Lemma 2.13. □

Theorem 3.11. *If μ is a non-constant fuzzy weakly irreducible ideal of a ring R, then $1 \in Im(\mu)$.*

Proof. Assume that $\mu(0) < 1$.
Let $\mu(0) = t$ and $x \in R$ be such that $\mu(x) < t$.
Define fuzzy ideals σ and θ of R by $\sigma(x) = (\chi_{\mu_t})(x)$ and $\theta(x) = t$ for all $x \in R$.
Then we have

$$(\sigma \cap \theta)(x) = \begin{cases} t, & \text{if } x \in \mu_t, \\ 0, & \text{if } x \in R \sim \mu_t. \end{cases}$$

Clearly, $\sigma \cap \theta \subseteq \mu$.
Since $\sigma(0) = 1 > t = \mu(0) = (\sqrt{\mu})(0)$, we conclude that $\sigma \nsubseteq \sqrt{\mu}$.
Hence, as μ is a fuzzy weakly irreducible ideal, we get $\theta \subseteq \sqrt{\mu}$.
Since $\sqrt{\mu}$ is non-constant, there exists some $y \in R$ such that

$$\sqrt{\mu}(y) \neq \sqrt{\mu}(0) = \mu(0).$$

Since for any $x \in R$, $\mu(x) \leq \mu(0)$, we conclude that

$$(\sqrt{\mu})(y) < \sqrt{\mu}(0) = \mu(0) = t.$$

Thus,

$$(\sqrt{\mu})(y) < t = \theta(y).$$

This contradiction shows that $1 \in Im(\mu)$; whence $\mu(0) = 1$. □

Theorem 3.12. *Let μ be a nonconstant fuzzy ideal of R. Then the following statements are equivalent:*

(i) *μ is fuzzy quasi primary.*
(ii) *$\sqrt{\mu}$ is fuzzy weakly irreducible.*
(iii) *$\sqrt{\mu}$ is fuzzy prime.*

Proof. (i) $\Rightarrow$ (iii): Follows from the definition.
(iii) $\Rightarrow$ (i): Assume that $\sqrt{\mu}$ is a fuzzy prime ideal of R.
Let $\sigma \cdot \theta \subseteq \mu$. Since $\mu \subseteq \sqrt{\mu}$ we get $\sigma \cdot \theta \subseteq \sqrt{\mu}$ and so

$$\sigma \subseteq \sqrt{\mu} \text{ or } \theta \subseteq \sqrt{\mu}.$$

Hence, μ is fuzzy quasi primary.
(i) $\Rightarrow$ (ii): Let μ be fuzzy quasi primary ideal of R.
Then $\sqrt{\mu}$ is fuzzy prime and so μ is fuzzy weakly irreducible.
(ii) $\Rightarrow$ (i): Assume that $\sqrt{\mu}$ is a fuzzy weakly irreducible and $\sigma \cdot \theta \subseteq \mu$.
Then

$$\sigma \cap \theta \subseteq \sqrt{\sigma \cap \theta} = \sqrt{\sigma \cdot \theta} \subseteq \sqrt{\mu}.$$

This implies that

$$\sigma \subseteq \sqrt{\mu} \text{ or } \theta \subseteq \sqrt{\mu}.$$

Thus, μ is fuzzy quasi primary. □

Example 3.13. Let R be the polynomial ring $F[x]$, where F is a field. Define a fuzzy ideal μ of R by,

$$\mu(a) = \begin{cases} 1, & \text{if } a \in <x^4>, \\ 0.6, & \text{if } a \in <x^2> \sim <x^4>, \\ 0.4, & \text{if } a \in \ R \ \sim <x^2>. \end{cases}$$

Then $F_\mu = \{<x^4>,\ <x^2>,\ R\}$.
As $\sqrt{<x^4>} = \sqrt{<x^2>} = <x>$, we have $F_{\sqrt{\mu}} = \{<x>,\ R\}$.
Hence, $\sqrt{\mu}$ is given by,

$$(\sqrt{\mu})(a) = \begin{cases} 1, & \text{if } a \in <x>, \\ 0.4 & \text{if } a \in \ R \ \sim <x>. \end{cases}$$

It follows from Theorem 1.2.49 from [2], that $\sqrt{\mu}$ is a fuzzy prime ideal of R. Hence by Theorem 3.12 we conclude that μ is fuzzy quasi primary. Thus by Theorem 3.9, μ is a fuzzy weakly irreducible ideal of R.

Theorem 3.14. *Let μ be a fuzzy weakly irreducible ideal of R. Then μ is fuzzy prime if and only if $\mu = \sqrt{\mu}$.*

Proof. If μ is fuzzy prime, then it follows from the definition of $\sqrt{\mu}$, that $\mu = \sqrt{\mu}$.
Conversely, suppose that for some fuzzy ideals σ and θ of R, $\sigma \cdot \theta \subseteq \mu$. Then

$$\sigma \cap \theta \subseteq \sqrt{\sigma \cap \theta} = \sqrt{\sigma \cdot \theta} \subseteq \sqrt{\mu} = \mu.$$

This implies that $\sigma \cap \theta \subseteq \mu$.
As μ is a fuzzy weakly irreducible ideal of R, either

$$\sigma \subseteq \sqrt{\mu} \text{ or } \theta \subseteq \sqrt{\mu}.$$

But as $\mu = \sqrt{\mu}$, we conclude that either $\sigma \subseteq \mu$ or $\theta \subseteq \mu$.
Thus, μ is a fuzzy prime ideal of R. □

Theorem 3.15. *If μ is a fuzzy primary ideal of R, then μ is fuzzy quasi primary ideal of R.*

Proof. As μ is a fuzzy primary ideal of R, by Theorem 5.1.9, p. 76, from Kumar [2] we conclude that $\sqrt{\mu}$ is fuzzy prime. Hence by Theorem 3.12 we conclude that μ is a quasi primary ideal of R. □

Theorem 3.16. *For any fuzzy ideal μ of a commutative regular ring with unity the following statements are equivalent:*

(i) *μ is fuzzy maximal.*
(ii) *μ is fuzzy quasi primary.*
(iii) *μ is fuzzy weakly irreducible.*

Proof. (i)⇒ (ii): Suppose that $\sigma \cdot \theta \subseteq \mu$ for some fuzzy ideals σ and θ of R.
As μ is fuzzy maximal, it follows from Remark 2.9 that μ is fuzzy prime. Therefore, either

$$\sigma \subseteq \sqrt{\mu} \text{ or } \theta \subseteq \sqrt{\mu}.$$

Thus μ is a fuzzy quasi primary ideal of R.
(ii)⇒ (iii): Proved in Theorem 3.9.
(iii)⇒ (ii): Let σ and θ be two fuzzy ideal of R such that $\sigma \cdot \theta \subseteq \mu$.
Since R is a regular ring, $\sigma \cdot \theta = \sigma \cap \theta$ and so we get $\sigma \cap \theta \subseteq \mu$.
But μ is fuzzy weakly irreducible and hence either $\sigma \subseteq \sqrt{\mu}$ or $\theta \subseteq \sqrt{\mu}$.
This implies that μ is fuzzy quasi primary.
(ii)⇒ (i): Let μ be an fuzzy ideal of R. Since R is a commutative regular ring with unity, it follows from Theorem 4.4.3, p. 71 from Kumar [2] that μ is fuzzy semiprime.
This implies, by Theorem 4.3.7 p. 67 from Kumar [2], that $\mu = \sqrt{\mu}$.
As μ is fuzzy quasi primary, it follows from $\mu = \sqrt{\mu}$ and Theorem 3.12, that $\sqrt{\mu}$ is fuzzy prime.
Thus, μ is fuzzy maximal by Theorem 5.2.11 from [2]. □

Theorem 3.17. *For a ring R, consider the following statements:*

(i) *The radicals of any two fuzzy ideals of R are comparable.*
(ii) *The fuzzy prime ideals of R form a chain with respect to inclusion.*
(iii) *Every fuzzy ideal of R is fuzzy quasi primary.*
(iv) *Every fuzzy ideal of R is fuzzy weakly irreducible.*

Then the following implications hold.
(i) ⇒ (ii) ⇒ (iii) ⇒ (iv)

Proof. (i)⇒ (ii): Let σ and θ be fuzzy prime ideals of R such that

$$\sqrt{\sigma} \subseteq \sqrt{\theta} \text{ or } \sqrt{\theta} \subseteq \sqrt{\sigma}. \tag{3.4}$$

As σ and θ are fuzzy prime, therefore

$$\sigma = \sqrt{\sigma} \text{ or } \theta = \sqrt{\theta}.$$

Therefore, form (3.4) we conclude that either $\sigma \subseteq \theta$ or $\theta \subseteq \sigma$.
Thus (ii) holds.
(ii)$\Rightarrow$ (iii): From (ii) we conclude that the radical of every fuzzy ideal μ is a fuzzy prime ideal and so μ is fuzzy quasi primary.
(iii)$\Rightarrow$ (iv): Follows from Theorem 3.12. □

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