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SIMULATION OF SAMPLE PATHS AND DISTRIBUTION OF LIU INTEGRALS

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Abstract. Uncertain process and uncertain integral are important contents of uncertainty theory. In this paper, we study some types of uncertain integral named Liu integral. Here, we obtained the analytical solution of Liu integral by integration methods. As a new work, we simulated the sample paths and distribution of Liu integral. The simulation method run on some new examples and results are shown.

Key Words: Liu process, Liu integral, Sample path, Simulation method, Partitioned rand.
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1. Introduction

Considering integration play very important roles in the fundamental theory of mathematics. Many problems are difficult to be described exactly and randomness, fuzziness and uncertainty appear frequently in like systems. For modelling these systems, it is better to use uncertainty theory. Liu [5] introduced uncertainty theory and improved it [6] according to being normal, monotonic, selfdual, countable subadditive and product measure axioms. Liu introduced the concepts of uncertain variable and uncertain distribution.
As a generalization of stochastic integral and fuzzy integral, uncertain integral was introduced by Liu [3]. Later, Liu [5] deduced the linear property of uncertain integral and presented the fundamental theorem

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of uncertain calculus, then the formulas of chain rule, change of variables, and integration by parts were derived.
Few people have worked on Liu integrals. You and Xiang [7] present some useful formulas of uncertain integral such as nonnegativity, monotonicity. the main purpose of this paper is to study the Liu integral. Applying uncertain integral, uncertain differential equation has been studied by some researches (see [8, 4]). Here, we solved Liu integrald by analytical methods and Also, we used simulation approach to solve Liu integrals.
The remainder of this paper is structured as follows. Section 2, is intended to introduce some concepts of uncertain theory. In section 3, Liu integral is considered and explained analytical solution. In section 4, some integration methods is stated. Simulation method is validated in section 5. Some new examples for more explanation are brought in every sections.

2. Definitions and Preliminaries

In this section, we present some essential concept of uncertain theory which is needed in this paper. For more details, you can see [5].
Definition 1. Assume that $\mathcal{L}$ be a σ-algebra on a nonempty set Γ. Then, each element $\Lambda \in \mathcal{L}$ is an event. Uncertain measure $\mathcal{M}$ is a function from $\mathcal{L}$ to $[0,1]$, that is, it allocates to each event Λ a number $\mathcal{M}\{\Lambda\}$ which determines the degree of belief that Λ will happen. So, Liu [5] presented following four axioms:
Axiom 1 *(Normality)* $\mathcal{M}\{\Gamma\} = 1$ for the universal set Γ.
Axiom 2 *(Self-Duality)* $\mathcal{M}\{\Lambda\} + \mathcal{M}\{\Lambda^c\} = 1$ for every event $\Lambda \in \mathcal{L}$.
Axiom 3 *(Countable Subadditivity)* For every countable sequence of events $\Lambda_1, \Lambda_2, ...,$ we have

$$\mathcal{M}\{\bigcup_{i=1}^{\infty} \Lambda_i\} \leq \sum_{i=1}^{\infty} \mathcal{M}\{\Lambda_i\}. \tag{2.1}$$

In this case, the triple $(\Gamma, \mathcal{L}, \mathcal{M})$ is called an uncertainty space.
Axiom 4 *(Product Axiom)* The product uncertain measure $\mathcal{M}$ is an uncertain measure satisfying

$$\mathcal{M}\{\prod_{k=1}^{\infty} \Lambda_k\} = \bigwedge_{k=1}^{\infty} \mathcal{M}_k\{\Lambda_k\},$$

where Λ_k are arbitrarily chosen events $\mathcal{L}_k$ for $k = 1, 2, ...$
Definition 2. An uncertain variable ξ is a measurable function from

an uncertainty space $(\Gamma, \mathcal{L}, \mathcal{M})$ to the set of real numbers, i.e., for any Borel set B of real numbers, the set

$$\{\xi \in B\} = \{\gamma \in \Gamma | \xi(\gamma) \in B\}, \tag{2.2}$$

is an event.

Definition 3. Let ξ be an uncertain variable. Then its uncertainty distribution is defined by

$$\Phi(x) = \mathcal{M}\{\xi \le x\}, \quad x \in \Re. \tag{2.3}$$

Liu showed that a function $\Phi^{-1} : (0, 1) \to \Re$ is an inverse uncertainty distribution if and only if it is a continuous and strictly increasing function.

Example 1. An uncertain variable ξ is called normal $\mathcal{N}(e, \sigma)$ if it has a normal uncertainty distribution

$$\Phi(x) = (1 + \exp(\frac{\pi(e - x)}{\sqrt{3}\sigma}))^{-1}, \quad x \in \Re. \tag{2.4}$$

The inverse uncertainty distribution of a normal uncertain variable $\mathcal{N}(e, \sigma)$ is

$$\Phi^{-1}(\alpha) = e + \frac{\sqrt{3}\sigma}{\pi} \ln \frac{\alpha}{1 - \alpha}, \quad \alpha \in (0, 1). \tag{2.5}$$

Example 2. An uncertain variable ξ is called lognormal if it has a lognormal uncertainty distribution

$$\Phi(x) = (1 + \exp(\frac{\pi(e - \ln x)}{\sqrt{3}\sigma}))^{-1}, \quad x \ge 0. \tag{2.6}$$

The inverse uncertainty distribution of a lognormal uncertain variable $\mathcal{LOGN}(e, \sigma)$ is

$$\Phi^{-1}(\alpha) = \exp(e + \frac{\sqrt{3}\sigma}{\pi} \ln \frac{\alpha}{1 - \alpha}), \quad \alpha \in (0, 1). \tag{2.7}$$

Definition 4. Let ξ be an uncertain variable. Then the expected value of ξ is defined by

$$E[\xi] = \int_0^{+\infty} \mathcal{M}\{\xi \ge r\} dr - \int_{-\infty}^0 \mathcal{M}\{\xi \le r\} dr, \tag{2.8}$$

provided that at least one of the two integrals is finite. Also, we can rewrite as $E[\xi] = \int_0^1 \Phi^{-1}(\alpha) d\alpha$.

If expected value of uncertain variable ξ is finite, then the variance of ξ is defined as

$$Var[\xi] = E[(\xi - e)^2] = \int_0^1 (\Phi^{-1}(\alpha) - e)^2 d\alpha.$$

3. Liu Integral

A sequence of uncertain variables indicated by time is called an uncertain process. More precisely, if T be an index set and $(\Gamma, \mathcal{L}, \mathcal{M})$ be an uncertainty space, then any measurable function from $T \times (\Gamma, \mathcal{L}, \mathcal{M})$ to the set of real numbers is an uncertain process, so that the set $\{X_t \in B\}$ is an event, for each $t \in T$ and any Borel set B of real numbers.
In 2009, Liu [4] defined a type of process whose increments are independent and normal uncertain variables, as below.
Definition 5. Liu process C_t is an uncertain process which has the following conditions:
(i) $C_0 \equiv 0$ and almost all sample paths are Lipschitz continuous,
(ii) C_t has stationary and independent increments,
(iii) every increment $C_{t+s} - Cs, s < t$ is a normal uncertain variable with zero mean and variance t^2.
The distribution of Liu process is

$$\Phi_t(x) = (1 + \exp(\frac{-\pi x}{\sqrt{3}t}))^{-1}, \quad x \in \Re. \tag{3.1}$$

The inverse uncertainty distribution of Liu process is

$$\Phi_t^{-1}(\alpha) = \frac{t\sqrt{3}}{\pi} \ln \frac{\alpha}{1-\alpha}, \quad \alpha \in (0, 1). \tag{3.2}$$

Liu Integral was introduced by Liu [5] that allows us to integrate from the uncertain process with the Liu process. The solution of Liu integral is a uncertain process.
Definition 6. Let X_t is an uncertain process and C_t a Liu process. We consider every partition of $[a, b]$ as $a = t_1 < t_2 < ... < t_{k+1} = b$ and set $\Delta = \max_{1 \le i \le k} |t_{i+1} - t_i|$. Then a Liu integral of X_t with respect to C_t is defined as follow

$$\int_a^b X_t dC_t = \lim_{\Delta \to 0} \sum_{i=1}^{k} X_{t_i}.(C_{t_{i+1}} - C_{t_i}), \tag{3.3}$$

provided that above limit exists almost surely and be finite. Therefore, the uncertain process X_t is integrable.

Theorem 1. Let $f(t)$ is a integrable function according to t. Then Liu integral $\int_0^s f(t)dC_t$ is a normal uncertain variable. That's mean

$$\int_0^s f(t)dC_t \sim \mathcal{N}\left(0, \int_0^s |f(t)|dt\right). \tag{3.4}$$

Proof. Since increments C_t are independent and stationary uncertain variables, then for every partition of $[0,s]$ as $0 < t_1 < t_2 < ... < t_{k+1} = s$, we have

$$\sum_{i=1}^{k} f_{t_i}.(C_{t_{i+1}} - C_{t_i}) \sim \mathcal{N}\left(0, \sum_{i=1}^{k} |f(t_i)|(t_{i+1} - t_i)\right)$$

and f is an integrable function, as $\Delta \to 0$, we have

$$\sum_{i=1}^{k} |f(t_i)|(t_{i+1} - t_i) \to \int_0^s |f(t)|dt$$

then

$$\int_0^s f(t)dC_t \sim \mathcal{N}\left(0, \int_0^s |f(t)|dt\right).$$

For more properties of Liu integral, you can see [7].

Now, we consider some examples and obtain their analytical solution by Theorem 1.

Example 3. We have from equation (3.3), for every partition of $[0,s]$ as $0 = t_1 < t_2 < ... < t_{k+1} = s$

$$\int_0^s C_t dC_t = \lim_{\Delta \to 0} \sum_{i=1}^{k} C_{t_i}(C_{t_{i+1}} - C_{t_i})$$

$$= \frac{1}{2} \lim_{\Delta \to 0} \sum_{i=1}^{k} ((C_{t_{i+1}}^2 - C_{t_i}^2) - (C_{t_{i+1}} - C_{t_i})^2)$$

$$= \frac{1}{2} \lim_{\Delta \to 0} [\sum_{i=1}^{k} (C_{t_{i+1}}^2 - C_{t_i}^2) - \sum_{i=1}^{k} (C_{t_{i+1}} - C_{t_i})^2]$$

$$= \frac{1}{2} {C_s}^2 - \frac{1}{2} \lim_{\Delta \to 0} \sum_{i=1}^{k} (C_{t_{i+1}} - C_{t_i})^2,$$

on the other hand, since

$$\lim_{\Delta \to 0} \sum_{i=1}^{k} (C_{t_{i+1}} - C_{t_i})^2 = \lim_{\Delta \to 0} \sum_{i=1}^{k} (C_{t_{i+1}} - C_{t_i})(C_{t_{i+1}} - C_{t_i})$$

$$= \lim_{\Delta\to 0}\sum_{i=1}^{k}(C_{t_{i+1}}(C_{t_{i+1}}-C_{t_i})-C_{t_i}(C_{t_{i+1}}-C_{t_i}))$$

$$= \int_0^s C_t dC_t - \int_0^s C_t dC_t = 0.$$

As a result, the analytical solution of the above Liu integral is obtained as follows

$$\int_0^s C_t dC_t = \frac{1}{2}C_s^2.$$

It follows from Theorem 1, that this Liu integral has following normal distribution

$$\int_0^s C_t dC_t \sim \mathcal{N}\left(0, \int_0^s |C_t| dt\right).$$

Example 4. We have from equation (3.3), for every partition of $[0, s]$ as $0 = t_1 < t_2 < ... < t_{k+1} = s$

$$\int_0^s \exp(t) dC_t = \lim_{\Delta\to 0}\sum_{i=1}^{k}\exp(t_i)(C_{t_{i+1}}-C_{t_i})$$

$$= \lim_{\Delta\to 0}\sum_{i=1}^{k}((\exp(t_{i+1})C_{t_{i+1}}-\exp(t_i)C_{t_i})-C_{t_{i+1}}(\exp(t_{i+1})-\exp(t_i)))$$

$$= \lim_{\Delta\to 0}\sum_{i=1}^{k}(\exp(t_{i+1})C_{t_{i+1}}-\exp(t_i)C_{t_i})-\lim_{\Delta\to 0}\sum_{i=1}^{k}C_{t_{i+1}}(\exp(t_{i+1})-\exp(t_i))$$

$$= \exp(s)C_s - \int_0^s C_t d(\exp(t)).$$

Then the analytical solution of this Liu integral is calculated as follow

$$\int_0^s \exp(t) dC_t = \exp(s)C_s - \int_0^s C_t \exp(t) dt.$$

It follows from Theorem 1, that above Liu integral has following normal distribution

$$\int_0^s \exp(t) dC_t \sim \mathcal{N}\left(0, \int_0^s |\exp(t)| dt\right) = \mathcal{N}(0, \exp(s)-1).$$

We obtain its uncertain distribution as follow

$$\Phi_s(x) = (1+\exp(\frac{-\pi x}{\sqrt{3}(\exp(s)-1)}))^{-1}.$$

Also its inverse uncertain distribution is

$$(1+\exp(-\frac{\pi x}{\sqrt{3}(\exp(s)-1)}))^{-1}=\alpha$$

$$\alpha+\alpha\exp(-\frac{\pi x}{\sqrt{3}(\exp(s)-1)})=1$$

then

$$\Phi_s^{-1}(\alpha)=\frac{\sqrt{3}(\exp(s)-1)}{\pi}\ln\frac{\alpha}{1-\alpha},\qquad 0<\alpha<1.$$

4. Integration methods

Now, we present some methods for calculating analytical solution of Liu integral. These methods include change of variable and integration by Parts method.

Theorem 2. (Change of Variable) Let, f be a continuously differentiable function. Then for any $s>0$,

$$\int_0^s f'(C_t)dC_t=\int_{C_0}^{C_s} f'(c)dc.$$

that is

$$\int_0^s f'(C_t)dC_t=f(C_s)-f(C_0).$$

Proof. See [4].

Theorem 3. (Integration by Parts) Let X_t and Y_t are Liu process. Then

$$d(X_tY_t)=Y_tdX_t+X_tdY_t. \tag{4.1}$$

Proof. See [4].

Example 5. In order to illustrate the integration by parts, let us solve the following Liu integral

$$\int_0^s 2\exp(t)C_tdC_t.$$

For this purpose, define

$$X_t=\exp(t),\qquad Y_t={C_t}^2,$$

then

$$dX_t=\exp(t)dt,\qquad dY_t=2C_tdC_t.$$

It follows from Theorem 3 that

$$dZ_t=\exp(t){C_t}^2dt+2\exp(t)C_tdC_t,$$

$$Z_s - Z_0 = \int_0^s \exp(t) C_t^2 dt + \int_0^s 2\exp(t) C_t dC_t.$$

Since $Z_0 = 0 \quad Z_s = \exp(s){C_s}^2$, we have

$$\int_0^s 2\exp(t) C_t dC_t = \exp(s) C_s^2 - \int_0^s \exp(t) C_t^2 dt.$$

Also

$$\int_0^s 2\exp(t) C_t dC_t \sim \mathcal{N}\left(0, \int_0^s |2\exp(t) C_t| dt\right).$$

5. Simulation of Liu Integral

In this section, we intend to introduce an approach for solving of Liu integrals by simulation method. For this purpose, we used *rand* function, *partitioned rand* and *Halton* sequence. *Partitioned rand* is an approach based on *rand* function in Matlab. We divided (0,1) into n subsets and generated random numbers with uniform distribution in every subset. *Halton* sequence is also one of the quasi-random sequences for generating random numbers in (0,1) that used from a generator function to generate numbers. More descriptions about *Halton* sequence is in [2, 1]. For simulation Liu integral, first of all, sample paths are simulated that are showed the solution of Liu integral is an uncertain process. Then, we simulated the distribution of Liu integral to touch the normality of a Liu integral distribution.

5.1. **Simulation of sample paths of Liu Integral.** The algorithm of sample paths simulation is as follow:

Algorithm 1

1. *Select integer* n *and time* s,
2. *Set* $ds = \frac{s}{n}, \quad C_1 = dC_1, \quad sum_1 = 0$,
3. *Generate* $u_i \sim U(0,1), \quad for \quad i = 2, 3, ..., n$,
4. *Set* $dC_i = ds\frac{\sqrt{3}}{\pi} Ln\frac{u_i}{1-u_i}$,
5. *Set* $C_i = C_{i-1} + dC_i$,
6. *Set* $d_i = C_i - C_{i-1}$,
7. *Set* $sum_i = sum_{i-1} + f(t_i) d_i$.

We run algorithm 1 for examples 3 to 5 with $n = 2000, 10000, 30000, 50000$ times iterations and $s = 1$ by *rand* function and *partitioned rand*. Figures 1 to 3 show the simulated sample paths for examples 3 to 5.

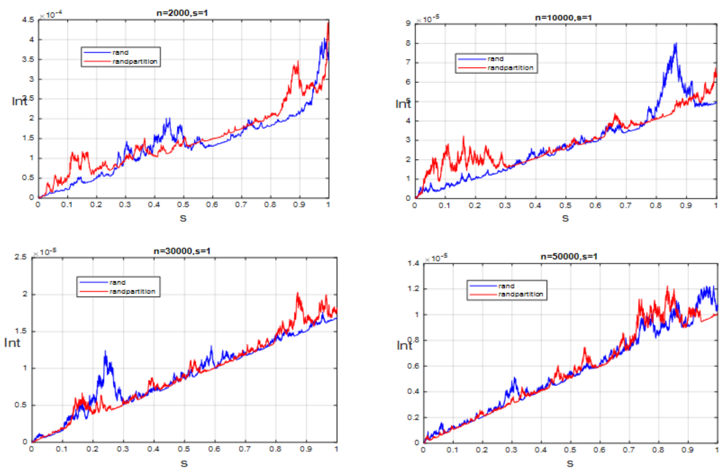

Figure 1:The simulation of sample paths of Liu integral for example3

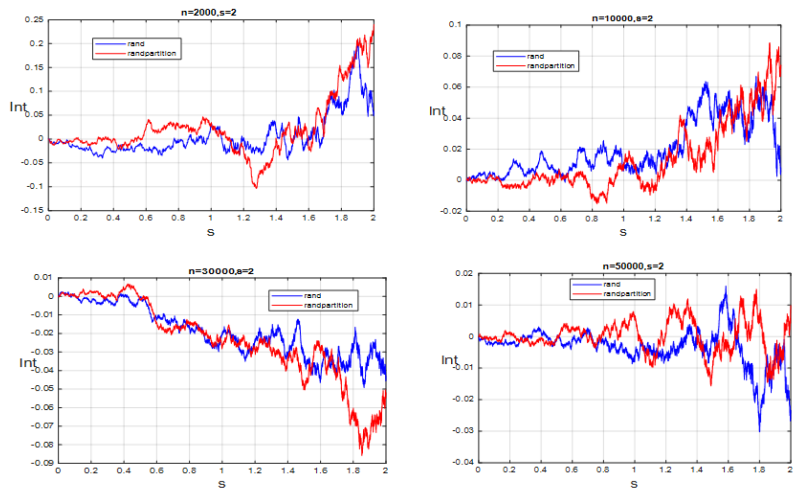

Figure 2:The simulation of sample paths Liu integral for example 4

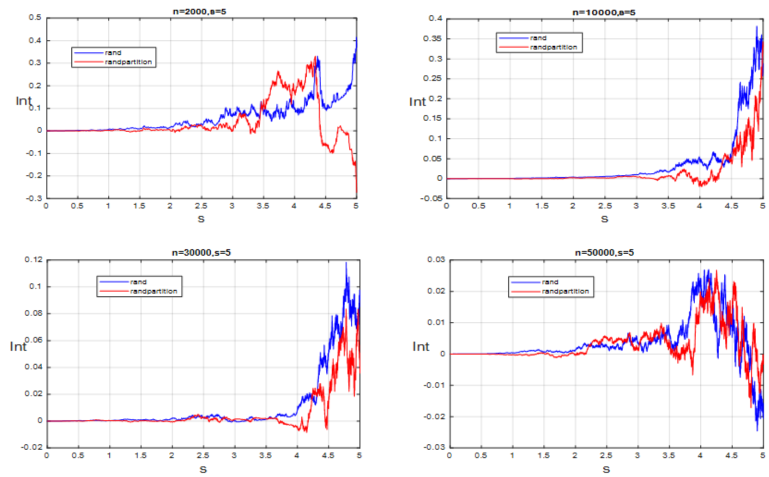

Figure 3:The simulation of sample paths Liu integral for example 5

5.2. **Simulation of distribution of Liu Integral.** The algorithm of distribution simulation of Liu Integral is as follow:

Algorithm 2 We run algorithm 2 for examples 3 to 5 with $n = 10000$

1. $Select \quad integer \quad n \quad and \quad time \quad s,$
2. $Generate \quad u_i \sim U(0,1), \quad for \quad i = 2, 3, ..., n,$
3. $Calculate \quad the \quad variance \quad V_i = |\int_0^{\frac{s}{n}} f(t)dt|,$
4. $Set \quad D_i = V_n \frac{\sqrt{3}}{\pi} Ln \frac{u_i}{1-u_i}.$

times iterations and $s = 1$ by *rand* function, *partitioned rand* and *Halton* sequence. Figures 4 to 6 show the simulated distribution of Liu integral for examples 3 to 5.

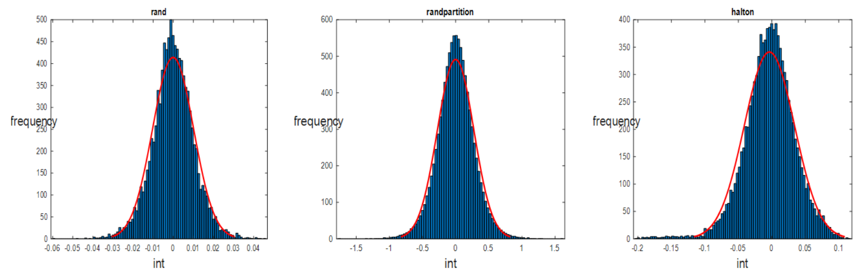

Figure 4: The simulation of distribution of Liu integral for example 3

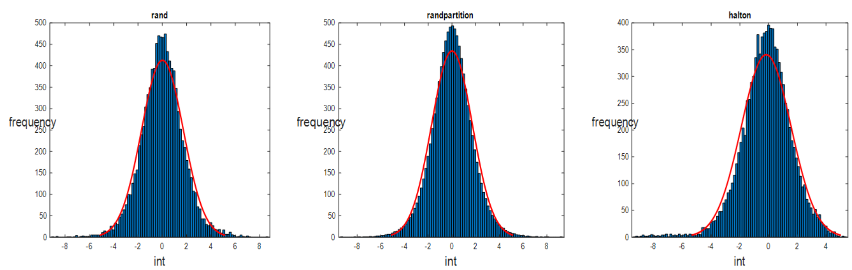

Figure 5: The simulation of distribution of Liu integral for example 4

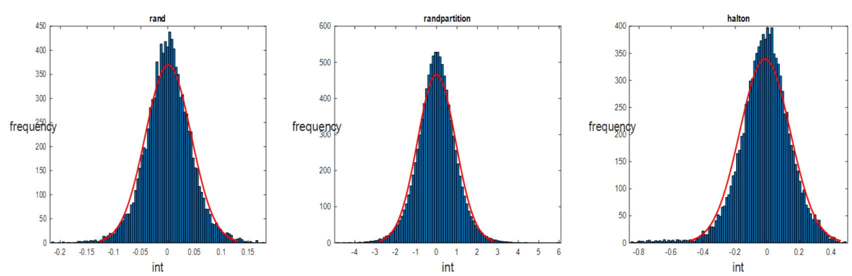

Figure 6: The simulation of distribution of Liu integral for example 5

It is clear from figures 4 to 6 that drawn histograms using *partitioned rand* have more regular and monolith histogram than *rand* function and *Halton* sequence. Also, when the number of repetitions increase, curves with *partitioned rand* fit more on normal curve. On the other hands, the normality of the distribution of Liu integral is obvious in these figures.

6. Conclusion

In this paper, we studied Liu integral and some method for obtaining its solution. Some integration methods was checked out and some new examples were solved. The important point is that we solved Liu integral by simulating of sample paths and uncertain distribution of Liu integrals. The normality of Liu integral distribution in Theorem 1 was palpable and obvious by simulation of uncertain distribution of Liu integral. Also, we found out when we use *partitioned rand* method, Liu integral distribution has more normal behaviour than when run algorithm based on original *rand* function or *Halton* sequence and the figures in the article confirmed this fact.

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