Journal of Hyperstructures 10*(1)(2021), 47-62.*
ISSN: 2322-1666 print/2251-8436 online

FUZZY SOFT BI-INTERIOR IDEALS OVER Γ−SEMIRINGS

MARAPUREDDY MURALI KRISHNA RAO

Abstract. In this paper, we introduce the notion of fuzzy soft bi-interior ideals over Γ−semirings and study some of their algebraical properties.bi-interior ideal

Key Words: bi-interior ideal, fuzzy soft bi-interior ideal, Γ−semiring.
2010 Mathematics Subject Classification: Primary: 16Y60, Secondary: 16Y99, 03E72.

1. Introduction

The notion of Γ−ring was introduced by Nobusawa [42] as a generalization of ring in 1964. Sen [44] introduced the notion of Γ−semigroup in 1981. The notion of ternary algebraic system was introduced by Lehmer [12] in 1932, Lister [13] introduced ternary ring. The notion of a Γ−semiring was introduced by Murali Krishna Rao [17-23] in 1995, not only generalizes the notion of semiring and Γ−ring but also the notion of ternary semiring. The notion of a semiring is an algebraic structure with two associative binary operations where one distributes over the other, was first introduced by Vandiver [47] in 1934 but semirings had appeared earlier in studies on the theory of ideals of rings. Herniksen [5] defined $k-$ideals in semirings to obtain analogous of ring results for semiring. The notion of ideals was introduced by Dedekind for the theory of algebraic numbers, was generalized by E. Noether for

Received: 29 October 2019, Accepted: 13 October 2020. Communicated by Yuming Feng;
∗Address correspondence to Marapureddy Murali Krishna Rao; E-mail: mmarapureddy@gmail.com

associative rings. The one and two sided ideals introduced by her, are still central concepts in ring theory. We know that the notion of a one sided ideal of any algebraic structure is a generalization of notion of an ideal. The quasi ideals are generalization of left and right ideals whereas the bi-ideals are generalization of quasi ideals. Iseki [6,7,8] introduced the concept of quasi ideal for a semiring. Quasi ideals in $\Gamma-$semirings studied by Jagtap and Pawar[9] studied ideals in semirings. As a further generalization of ideals, Steinfeld [45] first introduced the notion of quasi ideals for semigroups and then for rings. We know that the notion of the bi-ideal in semirings is a special case of (m, n) ideal introduced by S. Lajos.The concept of bi-ideals was first introduced by R.A. Good and D.R. Hughes[4] for a semigroup. Lajos and Szasz [10,11] introduced the concept of bi-ideals for rings. Semirings play an important role in studying matrices and determinants. Murali Krishna Rao [24-31] introduced the notion of left (right) bi-quasi ideal, bi-interior ideal and bi- quasi-interior ideal of semiring, Γ-semiring,Γ-semigroup and studied their properties. The theory of fuzzy sets introduced by Zadeh [48,49] is the most appropriate theory for dealing with uncertainty. The concept of fuzzy subgroup was introduced by Rosenfeld [43]. D. Mandal [15] studied fuzzy ideals and fuzzy interior ideals in an ordered semiring. N. Kuroki studied fuzzy interior ideals in semigroups. K.L. N. Swamy and U. M. Swamy [46] studied fuzzy prime ideals in rings in 1988. Molodtsov [16] was introduced the concept of soft set theory as a new mathematical tool for dealing with uncertainties,only partially resolves the problem is that objects in universal set often does not precisely satisfy the parameters associated to each of the elements in the set. Then Maji et al. [14]extended soft set theory to fuzzy soft set theory. Aktas and Cagman defined the soft set and soft groups. Majumdar and Samantha extended soft sets to fuzzy soft set. Acar et al. [1], gave the basic concept of soft ring. Jayanth Ghosh et al. [3] initiated the study of fuzzy soft rings and fuzzy soft ideals. Feng et al. [2] studied soft semirings by using the soft set theory. M.Murali Krishna Rao et al. [32-41] introduced and studied fuzzy ideal, fuzzy soft ideals, fuzzy soft ordered $\Gamma-$semiring, fuzzy soft ideal,fuzzy soft bi-ideal,fuzzy soft quasi-ideal and fuzzy soft interior ideal over ordered $\Gamma-$Semirings. In this paper, we introduce the notion of fuzzy soft bi-interior ideals over $\Gamma-$semirings and study some of their algebraical properties.

2. Preliminaries

In this section, we recall some definitions introduced by the pioneers in this field earlier.

Definition 2.1. Let $(M,+)$ and $(\Gamma,+)$ be commutative semigroups. Then we call M as a $\Gamma-$semiring,if there exists a mapping $M\times\Gamma\times M\rightarrow M$, (x,α,y) is written as as $x\alpha y$ such that it satisfies the following axioms for all $x,y,z\in M$ and $\alpha,\beta\in\Gamma$

(i) $x\alpha(y+z)=x\alpha y+x\alpha z$
(ii) $(x+y)\alpha z=x\alpha z+y\alpha z$
(iii) $x(\alpha+\beta)y=x\alpha y+x\beta y$
(iv) $x\alpha(y\beta z)=(x\alpha y)\beta z$.

Definition 2.2. A $\Gamma-$semiring M is said to be commutative $\Gamma-$semiring if $x\alpha y=y\alpha x$, for all $x,y\in M$ and $\alpha\in\Gamma$.

Definition 2.3. Let M be a $\Gamma-$semiring. An element $1\in M$ is said to be unity if for each $x\in M$ there exists $\alpha\in\Gamma$ such that $x\alpha 1=1\alpha x=x$.

Definition 2.4. A fuzzy subset $\mu:S\rightarrow [0,1]$ is non-empty if μ is not the constant function.

Definition 2.5. For any two fuzzy subsets λ and μ of S, $\lambda\subseteq\mu$ means $\lambda(x)\leq\mu(x)$ for all $x\in S$.

Definition 2.6. Let f and g be fuzzy subsets of $\Gamma-$semiring M. Then $f\circ g, f+g, f\cup g, f\cap g$, are defined by

$$f\circ g(z)=\begin{cases}\sup\limits_{z=x\alpha y}\{min\{f(x),g(y)\}\},\\ 0,\quad \text{otherwise.}\end{cases};$$

$$f+g(z)=\begin{cases}\sup\limits_{z=x+y}\{min\{f(x),g(y)\}\},\\ 0,\quad \text{otherwise}\end{cases}$$

$$f\cup g(z)=max\{f(z),g(z)\}\ ;\ f\cap g(z)=min\{f(z),g(z)\}$$

$x,y\in M,\alpha\in\Gamma,$ for all $z\in M$.

Definition 2.7. Let M be a $\Gamma-$semiring.A fuzzy subset μ of M is said to be fuzzy $\Gamma-$subsemiring of M if it satisfies the following conditions
(i) $\mu(x+y)\geq min\ \{\mu(x),\mu(y)\}$
(ii) $\mu(x\alpha y)\geq min\ \{\mu(x),\mu(y)\}$, for all $x,y\in M,\alpha\in\Gamma$.

Definition 2.8. A function $f : R \to M$ where R and M are $\Gamma-$semirings is said to be $\Gamma-$semiring homomorphism if $f(a+b) = f(a) + f(b)$ and $f(a\alpha b) = f(a)\alpha f(b)$ for all $a, b \in R, \alpha \in \Gamma$.

Definition 2.9. Let A be a non-empty subset of M. The characteristic function of A is a fuzzy subset of M, defined by

$$\chi_{_A}(x) = \begin{cases} 1, & \text{if } x \in A; \\ 0, & \text{if } x \notin A. \end{cases}$$

Definition 2.10. A fuzzy subset μ of $\Gamma-$semiring M is called a fuzzy left (right) ideal of M if for all $x, y \in M, \alpha \in \Gamma$ it satisfies the following conditions
(i) $\mu(x+y) \geq min\{\mu(x), \mu(y)\}$
(ii) $\mu(x\alpha y) \geq \mu(y)$ $(\mu(x))$, for all $x, y \in M, \alpha \in \Gamma$.

Definition 2.11. A fuzzy subset μ of $\Gamma-$semiring M is called a fuzzy ideal of M if it satisfies the following conditions
(i) $\mu(x+y) \geq min\{\mu(x), \mu(y)\}$
(ii) $\mu(x\alpha y) \geq max\ \{\mu(x), \mu(y)\}$, for all $x, y \in M, \alpha \in \Gamma$.

Definition 2.12. Let U be an initial Universe set and E be the set of parameters. Let $P(U)$ denotes the power set of U. A pair (f, E) is called soft set over U where f is a mapping given by $f : E \to P(U)$.

Definition 2.13. For a soft set (f, A), the set $\{x \in A \mid f(x) \neq \emptyset\}$ is called Support of (f, A) denoted by $Supp(f, A)$. If Supp$(f, A) \neq \emptyset$ then (f, A) is called non null soft set.

Definition 2.14. Let U be an initial Universe set and E be the set of parameters. Let $A \subseteq E$. A pair (f, A) is called fuzzy soft set over U where f is a mapping given by $f : A \to I^U$ where I^U denotes the collection of all fuzzy subsets of U.

Definition 2.15. Let $(f, A), (g, B)$ be fuzzy soft sets over U then (f, A) is said to be a fuzzy soft subset of (g, B) denoted by $(f, A) \subseteq (g, B)$ if $A \subseteq B$ and $f(a) \subseteq g(a)$ for all $a \in A$.

Definition 2.16. Let $(f, A), (g, B)$ be fuzzy soft sets. The intersection of fuzzy soft sets (f, A) and (g, B) is denoted by $(f, A) \cap (g, B) = (h, C)$ where $C = A \cup B$ is defined as

$$h_c = \begin{cases} f_c, & \text{if } c \in A \setminus B; \\ g_c, & \text{if } c \in B \setminus A; \\ f_c \cap g_c, & \text{if } c \in A \cap B. \end{cases}$$

Definition 2.17. Let $(f, A), (g, B)$ be fuzzy soft sets over U. Then "(f, A) and (g, B) is denoted by "$(f, A) \wedge (g, B)$" is defined by $(f, A) \wedge (g, B) = (h, C)$ where $C = A \times B$. $h_c(x) = min\ \{f_a(x), g_b(x)\}$ for all $c = (a, b) \in A \times B$ and $x \in U$.

Definition 2.18. Let S be a$\Gamma-$semiring and E be a parameter set and $A \subseteq E$. Let f be a mapping given by $f : A \rightarrow\ [0, 1]^S$ where $[0, 1]^S$ denotes the collection of all fuzzy subsets of S. Then (f, A) is called a fuzzy soft left(right) ideal over S if and only if for each $a \in A$, the corresponding fuzzy subset $f_a : S \rightarrow\ [0, 1]$ is a fuzzy left(right) ideal of S. i.e., (i) $f_a(x+y) \geq\ \min\ \{f_a(x), f_a(y)\}$ (ii) $f_a(x\alpha y) \geq f_a(y)(f_a(x))$,for all $x, y \in S, \alpha, \beta \in \Gamma$.

Definition 2.19. Let S be a $\Gamma-$semiring and E be a parameter set and $A \subseteq E$. Let f be a mapping given by $f : A \rightarrow\ [0, 1]^S$ where $[0, 1]^S$ denotes the collection of all fuzzy subsets of S. Then (f, A) is called a fuzzy soft ideal over S if and only if for each $a \in A$, the corresponding fuzzy subset $f_a : S \rightarrow\ [0, 1]$ is a fuzzy ideal of S. i.e., (i) $f_a(x + y) \geq \min\ \{f_a(x), f_a(y)\}$ (ii) $f_a(x\alpha y) \geq\ \max\ \{f_a(x), f_a(y)\}$,for all $x, y \in S, \alpha, \beta \in \Gamma$.

Definition 2.20. Let S be a $\Gamma-$semiring and E be a parameter set and $A \subseteq E$. Let f be a mapping given by $f : A \rightarrow\ [0, 1]^S$ where $[0, 1]^S$ denotes the collection of all fuzzy subsets of S. Then (f, A) is called a fuzzy soft bi- ideal over S if and only if for each $a \in A$, the corresponding fuzzy subset $f_a : S \rightarrow\ [0, 1]$ is a fuzzy bi- ideal of S. i.e., (i) $f_a(x + y) \geq \min\ \{f_a(x), f_a(y)\}$ (ii) $f_a(x\alpha y\beta z) \geq\ \max\ \{f_a(x), f_a(z)\}$ for all $x, y \in S, \alpha, \beta \in \Gamma$.,for all $x, y \in S, \alpha, \beta \in \Gamma$.

Definition 2.21. Let S be a $\Gamma-$semiring and E be a parameter set and $A \subseteq E$. Let f be a mapping given by $f : A \rightarrow\ [0, 1]^S$ where $[0, 1]^S$ denotes the collection of all fuzzy subsets of S. Then (f, A) is called a fuzzy soft interor ideal over S if and only if for each $a \in A$, the corresponding fuzzy subset $f_a : S \rightarrow\ [0, 1]$ is a fuzzy interior ideal of S. i.e., (i) $f_a(x+y) \geq\ \min\ \{f_a(x), f_a(y)\}$ (ii) $f_a(x\alpha y\beta z) \geq\ \max\ \{f_a(y)\}$for all $x, y \in S, \alpha, \beta \in \Gamma$.

Definition 2.22. Let S be a $\Gamma-$semiring and E be a parameter set and $A \subseteq E$. Let f be a mapping given by $f : A \rightarrow\ [0, 1]^S$ where $[0, 1]^S$ denotes the collection of all fuzzy subsets of S. Then (f, A) is called a fuzzy soft quasi ideal over S if and only if for each $a \in A$, the corresponding fuzzy subset $f_a : S \rightarrow\ [0, 1]$ is a fuzzy quasi ideal of S. i.e.A fuzzy subset f_a of $\Gamma-$semiring S is called a fuzzy quasi ideal if

(i) $f_a(x+y) \geq \min(f_a(x), f_a(y))$ (ii)$f_a \circ \chi_S \wedge \chi_S \circ f_a \subseteq f_a$

Definition 2.23. Let $(f, A), (g, B)$ be fuzzy soft ideals over a $\Gamma-$semiring S. The product (f, A) and (g, B) is defined as $((f \circ g), C)$ where $C = A \cup B$ and

$$(f \circ g)_c(x) = \begin{cases} f_c(x), & \text{if } c \in A \setminus B; \\ g_c(x), & \text{if } c \in B \setminus A; \\ \underset{x=a\alpha b}{Sup} \{min\{f_c(a), g_c(b)\}\}, & \text{if } c \in A \cap B. \end{cases}$$

for all $c \in A \cup B$ and $x \in S, \alpha \in \Gamma$.

3. Fuzzy soft bi-interior ideal

In this section, the concept of fuzzy soft bi-interior ideal of $\Gamma-$semiring and study somo of their properties.

Definition 3.1. A non-empty subset B of a $\Gamma-$semiring M is said to be bi-interior ideal of M if B is a $\Gamma-$subsemiring of M and $M\Gamma B\Gamma M \cap B\Gamma M\Gamma B \subseteq B$.

Definition 3.2. A $\Gamma-$semiring M is said to be bi-interior simple $\Gamma-$semiring if M has no bi-interior ideals other than M itself.

In the following theorem, we mention some important properties and we omit the proofs since proofs are straight forward.

Theorem 3.3. *Let M be a $\Gamma-$semiring.Then the following are hold*

1. *Every left ideal is a bi-interior ideal of M.*
2. *Every right ideal is a bi-interior ideal of M.*
3. *Every quasi ideal is a bi-interior ideal of M.*
4. *If A and B are bi-interior ideals of M,then $A\Gamma B$ and $B\Gamma A$ are bi-interior ideals of M.*
5. *Every ideal is a bi-interior ideal of M.*
6. *If B is a bi-interior ideal of M, then $B\Gamma M$ and $M\Gamma B$ are bi-interior ideals of M.*

Definition 3.4. A fuzzy subset μ of a semiring M is called a fuzzy bi-interior ideal if $\mu(x+y) \geq \min\{\mu(x), \mu(y)\}$ for all $x, y \in M$. $\chi_M \circ \mu \circ \chi_M \cap \mu \circ \chi_M \circ \mu \subseteq \mu$

Example 3.5. Let Q be the set of all rational numbers,

$$M = \left\{ \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \mid a, b, c \in Q \right\}$$

and $M = \Gamma$. A ternary operation is defined as the usual matrix multiplication and $A = \left\{ \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \mid a, 0 \neq b \in Q \right\}$. Then M is a $\Gamma-$ semigroup and A is a bi-interior ideal but not a bi-ideal of semigroup M. Define $\mu : M \to [0,1]$ by

$$\mu(x) == \begin{cases} 1 & \text{if } x \in A, \\ 0, & \text{otherwise.} \end{cases}$$

Then μ is a fuzzy bi-interior ideal of M.

Definition 3.6. Let S be $\Gamma-$semiring and E be a parameter set and $A \subseteq E$. Let μ be a mapping given by $\mu : A \to \ [0,1]^S$ where $[0,1]^S$ denotes the collection of all fuzzy subsets of S. Then (μ, A) is called a fuzzy soft bi-interiori ideal over S if and only if for each $a \in A$, the corresponding fuzzy subset ideal of S. i.e.A fuzzy subset μ_a of $\Gamma-$semiring S is called a fuzzy bi-interiori ideal if
(i)$\mu_a(x + y) \geq \min(\mu_a(x), \mu_a(y))$
(ii) $\chi_M \circ \mu_a \circ \chi_M \cap \mu_a \circ \chi_M \circ \mu_a \subseteq \mu_a$.

Theorem 3.7. *Let M be a $\Gamma-$semiring, E be a parameterset and $A \subseteq E$. If (μ, A) is a fuzzy soft left ideal over M then (μ, A) is a fuzzy soft bi-interior ideal over M*

Proof. Suppose (μ, A) is a fuzzy soft left ideal over M. Then, for each $a \in A$, μ_a is a fuzzy left ideal of M and $x \in M, \alpha, \beta \in \Gamma$. Then

$$\begin{aligned} \chi_M \circ \mu_a(x) &= \sup_{x=c\alpha b} \{\min\{\chi_M(c), \mu_a(b)\}\} \\ &= \sup_{x=c\alpha b} \{\min\{1, \mu_a(b)\}\} \\ &= \sup_{x=c\alpha b} \{\mu_a(b)\} \\ &\leq \sup_{x=c\alpha b} \{\mu_a(cb)\} \\ &= \sup_{x=c\alpha b} \{\mu_a(x)\} \\ &= \mu(x) \end{aligned}$$

$$\Rightarrow\ \chi_M \circ \mu_a(x) \leq \mu(x).$$

$$\begin{aligned}\mu \circ \chi_M \circ \mu_a(x) &= \sup_{x=u\alpha v\beta s} \{\min\{\mu_a(u), \chi_M \circ \mu_a(v\beta s)\}\}\\ &\leq \sup_{x=u\alpha v\beta s} \{\min\{\mu_a(u), \mu_a(v\beta s)\}\}\\ &= \mu(x).\end{aligned}$$

Hence

$$\begin{aligned}\chi_M \circ \mu_a \circ \chi_M \cap \mu_a \circ \chi_M \circ \mu_a(x) &= \min\{\chi_M \circ \mu_a \circ \chi_M(x), \mu_a \circ \chi_M \circ \mu_a(x)\}\\ &\leq \min\{\chi_M \circ \mu_a \circ \chi_M(x), \mu_a(x)\} \leq \mu_a(x).\end{aligned}$$

Therefore $\chi_M \circ \mu_a \circ \chi_M \cap \mu_a \circ \chi_M \circ \mu_a(x) \subseteq \mu_a$.
Therefore μ_a is a fuzzy soft bi-interior ideal of M. Hence(μ, A) is a fuzzy soft bi-interior ideal over M □

Corollary 3.8. *Let M be a $\Gamma-$semiring and E be a parameterset and $A \subseteq E$. If (μ, A) is a fuzzy soft right ideal over M then (μ, A) is a fuzzy soft bi-interior ideal over M*

Theorem 3.9. *Let M be a $\Gamma-$semiring and μ be a non-empty fuzzy subset of M . A fuzzy subset μ is a fuzzy bi-interior ideal of a $\Gamma-$semiring M if and only if the level subset μ_t of μ is a bi-interior ideal of a $\Gamma-$semiring M for everyt $\in [0, 1]$, where $\mu_t \neq \phi$.*

Theorem 3.10. *Let M be a $\Gamma-$semiring and μ be a non-empty fuzzy subset of M . A fuzzy subset μ is a fuzzy bi-interior ideal of a $\Gamma-$semiring M if and only if the level subset μ_t of μ is a bi-interior ideal of a $\Gamma-$semiring M for every $t \in [0, 1]$, where $\mu_t \neq \phi$.*

Proof. Let M be a $\Gamma-$semiring and μ be a non-empty fuzzy subset of M. Suppose μ is a fuzzy bi-interior ideal of the $\Gamma-$semiring M, $\mu_t \neq \phi, t \in [0, 1]$ and $a, b \in \mu_t$, Then

$$\begin{aligned}&\mu(a) \geq t, \mu(b) \geq t\\ \Rightarrow &\mu(a + b) \geq \min\{\mu(a), \mu(b)\} \geq t\\ \Rightarrow &a + b \in \mu_t.\end{aligned}$$

Let $x \in M\Gamma\mu_t\Gamma M \cap \mu_t\Gamma M\Gamma\mu_t$. Then $x = b\alpha a\beta u = c\gamma d\delta e$, where $b, u, d \in M, a, c, e \in \mu_t.\alpha, \beta, \gamma, \delta \in \Gamma$. Then $\chi_M \circ \mu \circ \chi_M(x) \geq t$ and $\mu \circ \chi_M \circ \mu(x) \geq t$ $\Rightarrow \mu(x) \geq t$
Therefore $x \in \mu_t$. Hence μ_t is a bi-interior ideal of M. Conversely suppose

that μ_t is a bi-interior ideal of $\Gamma-$semiring M, for all $t \in Im(\mu)$.
Let $x, y \in M, \mu(x) = t_1, \mu(y) = t_2$ and $t_1 \geq t_2$. Then $x, y \in \mu_{t_2}$.

$$\Rightarrow x + y \in \mu_{t_2} \text{ and } xy \in \mu_{t_2}$$
$$\Rightarrow \mu(x + y) \geq t_2 = \min\{t_1, t_2\} = \min\{\mu(x), \mu(y)\}$$
$$\text{Therefore } \mu(x + y) \geq t_2 = \min\{\mu(x), \mu(y)\}.$$

We have $M\Gamma\mu_t\Gamma M \cap \mu_t\Gamma M\Gamma\mu_t \subseteq \mu_t$, for all $t \in Im(\mu)$.
Suppose $t = \min\{Im(\mu)\}$. Then $M\Gamma\mu_t\Gamma M \cap \mu_t\Gamma M\Gamma\mu_t \subseteq \mu_t$.
Therefore $\chi_M \circ \mu \circ \chi_M \cap \mu \circ \chi_M \circ \mu \subseteq \mu$.
Hence μ is a fuzzy bi-interior ideal of the $\Gamma-$semiring M. □

Corollary 3.11. *Let M be a $\Gamma-$semiring and $A = [0, 1]$. A fuzzy subset μ is a fuzzy bi-interior ideal of a $\Gamma-$semiring M if and only if (μ, A) is a fuzzy soft bi-interior ideal over M then (μ, A) is a fuzzy soft bi-interior ideal over M*

Theorem 3.12. *Let I be a non-empty subset of a Γ-semiring M and χ_I be the characteristic function of I. Then I is a bi-interior ideal of Γ-semiring M if and only if χ_I is a fuzzy bi-interior ideal of a Γ-semiring M.*

Proof. Let I be a non-empty subset of the Γ-semiring M and χ_I be the characteristic function of I. Suppose I is a bi-interior ideal of the $\Gamma-$semiring M. Obviously χ_I is a fuzzy $\Gamma-$subsemiring of M. We have $M\Gamma I\Gamma M \cap I\Gamma M\Gamma I \subseteq I$. Then

$$\chi_M \circ \chi_I \circ \chi_M \cap \chi_I \circ \chi_M \circ \chi_I \subseteq \chi_I = \chi_{M\Gamma I\Gamma M \cap I\Gamma M\Gamma I} \subseteq \chi_I.$$

Therefore χ_I is a fuzzy bi-interior ideal of the $\Gamma-$semiring M. Conversely suppose that χ_I is a fuzzy bi-interior ideal of M.
Then I is a $\Gamma-$ subsemiring of M. We have $\chi_M \circ \chi_I \circ \chi_M \cap \chi_I \circ \chi_M \circ \chi_I \subseteq \chi_I$
$\Rightarrow \chi_{M\Gamma I\Gamma M} \cap \chi_{I\Gamma M\Gamma I} \subseteq \chi_I$
$\Rightarrow \chi_{M\Gamma I\Gamma M \cap I\Gamma M\Gamma I} \subseteq \chi_I$

Therefore $M\Gamma I\Gamma M \cap I\Gamma M\Gamma I \subseteq I$. Hence I is a bi-interior ideal of the Γ-semiring M. □

Theorem 3.13. *If μ and λ are fuzzy bi-interior ideals of a $\Gamma-$semiring M, then $\mu \cap \lambda$ is a fuzzy bi-interior ideal of a $\Gamma-$semiring M.*

Proof. Let μ and λ be fuzzy bi-interior ideals of the $\Gamma-$semiring M and $x, y \in M, \alpha, \beta \in \Gamma$. Then

$$\begin{aligned}
\mu \cap \lambda(x+y) &= \min\{\mu(x+y), \lambda(x+y)\} \\
&\geq \min\{\min\{\mu(x), \mu(y)\}, \min\{\lambda(x), \lambda(y)\}\} \\
&= \min\{\min\{\mu(x), \lambda(x)\}, \min\{\mu(y), \lambda(y)\}\} \\
&= \min\{\mu \cap \lambda(x), \mu \cap \lambda(y)\} \\
\chi_M \circ \mu \cap \lambda(x) &= \sup_{x=a\alpha b}\{\min\{\chi_M(a), \mu \cap \lambda(b)\}\} \\
&= \sup_{x=a\alpha b}\{\min\{\chi_M(a), \min\{\mu(b), \lambda(b)\}\} \\
&= \sup_{x=a\alpha b}\{\min\{\min\{\chi_M(a), \mu(b)\}, \min\{\chi_M(a), \lambda(b)\}\} \\
&= \min\{\sup_{x=a\alpha b}\{\min\{\chi_M(a), \mu(b)\}, \sup_{x=a\alpha b}\{\min\{\chi_M(a), \lambda(b)\}\} \\
&= \min\{\chi_M \circ \mu(x).\chi_M \circ \lambda(x)\} \\
&= \chi_M \circ \mu \cap \chi_M \circ \lambda(x)
\end{aligned}$$

Therefore $\chi_M \circ \mu \cap \lambda = \chi_M \circ \mu \cap \chi_M \circ \lambda$.

$$\begin{aligned}
\mu \cap \lambda \circ \chi_M \circ \mu \cap \lambda(x) &= \sup_{x=a\alpha b\beta c}\{\min\{\mu \cap \lambda(a), \chi_M \circ \mu \cap \lambda(b\beta c)\}\} \\
&= \sup_{x=a\alpha b\beta c}\{\min\{\mu \cap \lambda(a), \chi_M \circ \mu \cap \chi_M \circ \lambda(b\beta c)\}\} \\
&= \sup_{x=a\alpha b\beta c}\{\min\{\min\{\mu(a), \lambda(a)\}, \min\{\chi_M \circ \mu(b\beta c), \chi_M \circ \lambda(b\beta c)\}\} \\
&= \sup_{x=a\alpha b\beta c}\{\min\{\min\{\mu(a), \chi_M \circ \mu(b\beta c)\}, \min\{\lambda(a), \chi_M \circ \lambda(b\beta c)\}\} \\
&= min\{\mu \circ \chi_M \circ \mu(x), \lambda \circ \chi_M \circ \lambda(x)\} \\
&= \mu \circ \chi_M \circ \mu \cap \lambda \circ \chi_M \circ \lambda(x).
\end{aligned}$$

Therefore $\mu \cap \lambda \circ \chi_M \circ \mu \cap \lambda = \mu \circ \chi_M \circ \mu \cap \lambda \circ \chi_M \circ \lambda$.
Similarly $\chi_M \circ \mu \cap \lambda \circ \chi_M = \chi_M \circ \mu \circ \chi_M \cap \chi_M \circ \lambda \circ \chi_M$. Hence $\chi_M \circ \mu \cap \lambda \circ \chi_M \cap \mu \cap \lambda \circ \chi_M \circ \mu \cap \lambda = (\chi_M \circ \mu \circ \chi_M) \cap (\mu \circ \chi_M \circ \mu) \cap (\chi_M \circ \lambda \circ \chi_M) \cap (\lambda \circ \chi_M \circ \lambda) \subseteq \mu \cap \lambda$.
Hence $\mu \cap \lambda$ is a fuzzy bi-interior ideal of M. □

Theorem 3.14. *Let (f, A) and (g, B) be fuzzy soft bi-interiors over $\Gamma-$semiring M. Then $(f, A) \cap (g, B)$ is a fuzzy soft bi-interior ideal over M.*

Proof. By definition 2.20, we have $(f, A) \cap (g, B) = (h, C)$ where $C = A \cup B$.

Case (i) : $h_c = f_c$ if $c \in A \setminus B$. Then h_c is a fuzzy bi- ideal of S since (f, A) is a fuzzy soft bi-interior ideal over M.
Case (ii) : If $c \in B \setminus A$ then $h_c = g_c$. Therefore h_c is a fuzzy bi-interior Ideal of M since $(g, B)$is a fuzzy soft bi-interior ideal over M.
Case (iii) : If $c \in A \cap B$, and $x, y \in M, \alpha \in \Gamma$ then $h_c = f_c \cap g_c$ and Hence by Theorem [3.17],h_c is a fuzzy bi-interior ideal of M. Thus $(f, A) \cap (g, B)$ is a fuzzy soft bi- ideal over M. □

Theorem 3.15. *If μ and λ are fuzzy bi-interior ideals of semiring M then $\mu \cup \lambda$ is a fuzzy bi-interior ideal of a $\Gamma-$semiring M.*

Proof. Let μ and λ be fuzzy bi-interior ideals of $\Gamma-$semiring M. Then
$(\mu \cup \lambda)(x + y) = \max\{\mu(x + y), \lambda(x + y)\}$

$$\geq \max\{\max\{\mu(x), \mu(y)\}, \max\{\lambda(x), \lambda(y)\}\}$$
$$= \max\{\max\{\mu(x), \lambda(x)\}, \max\{\mu(y), \lambda(y)\}\}$$
$$= \max\{\mu \cup \lambda(x), \mu \cup \lambda(y)\}$$
$$\chi_M \circ \mu \cup \lambda(x)$$
$$= \sup_{x=a\alpha b} \max\{\chi_M(a), \mu \cup \lambda(b)\}$$
$$= \sup_{x=a\alpha b} \max\{\chi_M(a), \max\{\mu(b), \lambda(b)\}\}$$
$$= \sup_{x=a\alpha b} \max\{\max\{\chi_M(a), \mu(b)\}, \max\{\chi_M(a), \lambda(b)\}\}$$
$$= \max\{\sup_{x=a\alpha b} \max\{\chi_M(a), \mu(b)\}, \sup_{x=a\alpha b} \max\{\chi_M(a), \lambda(b)\}\}$$
$$= \max\{\chi_M \circ \mu(x).\chi_M \circ \lambda(x)\}$$
$$= \chi_M \circ \mu \cup \chi_M \circ \lambda(x)$$

Thus
$\chi_M \circ \mu \cup \lambda = \chi_M \circ \mu \cup \chi_M \circ \lambda.$
$\mu \cup \lambda \circ \chi_M \circ \mu \cup \lambda(x) = \sup_{x=a\alpha b\beta c} \max\{\mu \cup \lambda(a), \chi_M \circ \mu \cup \lambda(b\beta c)\}$
$= \sup_{x=a\alpha b\beta c} \max\{\mu \cup \lambda(a), \chi_M \circ \mu \cup \chi_M \circ \lambda(b\beta c)\}$
$= \sup_{x=a\alpha b\beta c} \max\{\max\{\mu(a), \lambda(a)\},$
$\max\{\chi_M \circ \mu(b\beta c), \chi_M \circ \lambda(b\beta c)\}\}$
$= \sup_{x=a\alpha b\beta c} \max\{\max\{\mu(a), \chi_M \circ \mu(b\beta c)\}, \max\{\lambda(a), \chi_M \circ \lambda(b\beta c)\}\} =$
$\max\{\sup_{x=a\alpha b\beta c} \max\{\mu(a), \chi_M \circ \mu(b\beta c)\}, \sup_{x=a\alpha b\beta c} \max\{\lambda(a), \chi_M \circ \lambda(b\beta c)\}\}$
$= \max\{\mu \circ \chi_M \circ \mu(x), \lambda \circ \chi_M \circ \lambda(x)\} = \mu \circ \chi_M \circ \mu \cup \lambda \circ \chi_M \circ \lambda(x).$
Therefore

$\mu \cup \lambda \circ \chi_M \circ \mu \cup \lambda = \mu \circ \chi_M \circ \mu \cup \lambda \circ \chi_M \circ \lambda$. Similarly $\chi_M \circ \mu \cup \lambda \circ \chi_M = \chi_M \circ \mu \circ \chi_M \cup \chi_M \circ \lambda \circ \chi_M$. Hence $\chi_M \circ \mu \cup \lambda \circ \chi_M \cap \mu \cup \lambda \circ \chi_M \circ \mu \cup \lambda = (\chi_M \circ \mu \circ \chi_M) \cap (\mu \circ \chi_M \circ \mu) \cup (\chi_M \circ \lambda \circ \chi_M) \cap (\lambda \circ \chi_M \circ \lambda) \subseteq \mu \cup \lambda$.

Hence $\mu \cup \lambda$ is a fuzzy bi-interior ideal of M. □

Corollary 3.16. *Let (f, A) and (g, B) be fuzzy soft bi-interiors over $\Gamma-$semiring M. Then $(f, A) \cup (g, B)$ is a fuzzy soft bi-interior ideal over M.*

Theorem 3.17. *Let (f, A) and (g, B) be fuzzy soft bi-interior ideals over$\Gamma-$semiring S. Then $(f, A) \wedge (g, B)$ is a fuzzy soft bi-interior ideal over S.*

Proof. By Definition 2.22, $(f, A) \wedge (g, B) = (h, C)$ where $C = A \times B$. Let $c = (a, b) \in C = A \times B$ and $x, y \in S, \alpha \in \Gamma$. Then

$$\begin{aligned} h_c(x+y) &= f_a(x+y) \wedge g_b(x+y) \\ &= min\{f_a(x+y), g_b(x+y)\} \\ &\geq min\{min\{f_a(x), f_a(y)\}, min\{g_b(x), g_b(y)\}\} \\ &= min\{min\{f_a(x), g_b(x)\}, min\{f_a(y), g_b(y)\}\} \\ &= min\{f_a \wedge g_b(x), f_a \wedge g_b(y)\} \\ &= min\{h_c(x), h_c(y)\} \end{aligned}$$

Hence by Theorem [3.17],h_c is a fuzzy bi-interior ideal of M. Hence h_c is a fuzzy soft bi-interior ideal over S. Therefore $(h, A \times B)$ is a fuzzy soft bi-interior ideal over S. □

Similarly, we can prove this following theorem.

Theorem 3.18. *Let (f, A) and (g, B) be fuzzy soft bi-interiors over $\Gamma-$semiring M. Then the product (f, A) and (g, B) is a fuzzy soft bi-interior ideal over M.*

Definition 3.19. A fuzzy set μ of an ordered $\Gamma-$semiring M is said to be normal fuzzy ideal if μ is a fuzzy ideal of M and $\mu(0) = 1$.

Definition 3.20. Let (f, A) be fuzzy soft ideal over a $\Gamma-$semiring S. Then (f, A) is said to be normal fuzzy soft $\Gamma-$semiring if f_a is normal fuzzy ideal of ordered $\Gamma-$semiring over S, for all $a \in A$.

Theorem 3.21. *If (f, A) is a fuzzy soft left ideal over $\Gamma-$semiring over S and for each $a \in A$, f_a^+ is defined by $f_a^+(x) = f_a(x) + 1 - f_a(0)$ for all $x \in S$ then (f^+, A) is a normal fuzzy soft bi-interior ideal over an ordered $\Gamma-$semiring over S and (f, A) is subset of (f^+, A).*

Proof follows from Theorem 3.17

Theorem 3.22. *Let M be a regular $\Gamma-$semiring and E be a parameterset and $A \subseteq E$. Then (I, A) is a fuzzy soft bi-interior ideal over M if and only if (f, A) is a fuzzy soft quasi ideal over M*

Proof. Let M be a regular $\Gamma-$ semiring. Suppose (μ, A) is a fuzzy soft bi-interior ideal over M. Then, for each $a \in A$, μ_a is a fuzzy bi-interior ideal of M. Let $x \in M, \alpha, \beta \in \Gamma$. Then

$\chi_M \circ \mu_a \circ \chi_M \cap \mu_M \circ \chi_M \circ \mu_a \subseteq \mu_a$. Suppose $\chi_M \circ \mu_a(x) > \mu_a(x)$. Since M is regular, there exist $y \in M, \alpha, \beta \in \Gamma$ such that $x = x\alpha y\beta x$. Then

$\mu_a \circ \chi_M \circ \mu_a(x) = \sup\limits_{x=x\alpha y\beta x} \{\min\{\mu_a(x), \chi_M \circ \mu_a(y\beta x)\}\ )$

$> \sup\limits_{x=x\alpha y\beta x} \{\min\{\mu_a(x), \mu_a(y\beta x)\}\}$

$= \mu(x)$

Which is a contradiction.Therefore $\mu_a \circ \chi_M \cap \chi_M \circ \mu_a \subseteq \mu_a$. Therefore μ_a is a fuzzy quasi ideal over M Hence(μ, A) is a fuzzy soft quasi ideal over M By Theorem [3.10], converse is true. □

Acknowledgments

The authors would like to thank the reviewers for their valuable comments and helpful suggestions forimprovement of the original manuscript

References

[1] D. D. Anderson and E. Smith, *Weakly prime ideals*, Houston J. Mathematics, **29** (2003), 831-840.

[2] D. F. Anderson, R. Levy and J. Shapirob, *Zero-divisor graphs, von Neumann regular rings, and Boolean algebras*, Journal of Pure and Applied Algebra, **180** (2003) 221-241.

[3] I. Beck, *Coloring of commutative rings*. J. Algebra, **116** (1988), 208-226.

[4] J. A. Huckaba, *Commutative rings with zero divisors*, New York: Dekker (1988).

[5] R. Y. Sharp, *Steps in commutative algebra*, Cambridge: Cambridge University Prees.

[6] U. Acar, F. Koyuncu and B. Tanay, *Soft sets and Soft rings*, Comput. and Math. with Appli.,**59** (2010), 3458-3463.

[7] F.Feng, Y.B. Jun and X. Zhao, *Soft semirings* , Comput. and Math. with Appli. **56** (2008), 2621-2628.

[8] J.Ghosh, B.Dinda and T.K. Samanta, *Fuzzy soft rings and Fuzzy soft ideals*, Int.J. P. App.Sc.Tech. **2(2)** (2011), 66-74.

[9] R. A. Good and D. R. Hughes, *Associated groups for a semigroup*, Bull. Amer. Math. Soc., **58** (1952), 624-625.

[10] M. Henriksen,*Ideals in semirings with commutative addition*, Amer. Math. Soc. Notices, **5** (1958), 321.
[11] K. Iseki, *Quasi-ideals in semirings without zero*, Proc. Japan Acad., **34** (1958), 79-84.
[12] K. Iseki, *Ideal theory of semiring*, Proc. Japan Acad., **32** (1956), 554-559.
[13] K. Iseki, *Ideal in semirings*, Proc. Japan Acad., **34** (1958), 29-31.
[14] R. D. Jagatap, Y.S. Pawar, *Quasi-ideals and minimal quasi-ideals in* $\Gamma-$*semirings*, Novi Sad J. Math., **39(2)** (2009), 79-87.
[15] S Lajos,*On the bi-ideals in semigroups*, Proc. Japan Acad., **45** (1969), 710-712.
[16] S. Lajos and F. A. Szasz, *On the bi-ideals in associative ring*, Proc. Japan Acad., **46** (1970), 505-507.
[17] H. Lehmer,*A ternary analogue of abelian groups*, Amer. J. of Math., **59** (1932), 329-338.
[18] G. Lister , *Ternary rings*, Trans. of Amer, Math. Soc., **154** (1971), 37-55.
[19] P.K.Maji, R. Biswas and A.R.Roy, *Fuzzy soft sets*, The J. of Fuzzy Math., **9(3)** (2001), 589-602.
[20] D. Mandal, *Fuzzy ideals and fuzzy interior ideals in ordered semirings*, Fuzzy info. and Engg.,**6** (2014), 101–114.
[21] D. Molodtsov,*Soft set theory-First results*, Comput. Math. Appl., **37** (1999), 19-31.
[22] M. Murali Krishna Rao, Γ-*semirings-I*, Southeast Asian Bulletin of Mathematics, **19** (1)(1995), 49-54.
[23] M. Murali Krishna Rao, Γ-*semirings-II*, Southeast Asian Bulletin of Mathematics,**21(3)** (1997), 281-287.
[24] M.Murali Krishna Rao, *The Jacobson radical of* Γ-*semiring*, South eastAsian Bulletin of Mathematics,**23** (1999), 127-134.
[25] M. Murali Krishna Rao, $\Gamma-$*field, Discussiones Mathematicae General Algebra and Applications*, **39** (2019), 125133 doi:10.7151/dmgaa.1303.
[26] M.Murali Krishna Rao,*Ideals in ordered*$\Gamma-$*semirings*, Discussiones Mathematicae General Algebra and Applications, **38** (2018), 47-68. doi:10.7151/dmgaa.1284
[27] M. Murali Krishna Rao, $\Gamma-$ *semiring with identity*, Discussiones Mathematicae General Algebra and Applications., **37** (2017), 189207,doi:10.7151/dmgaa.1276.
[28] M. Murali Krishna Rao and B. Venkateswarlu, *Regular* $\Gamma-$*incline and field* $\Gamma-$*semiring*, Novi Sad J. of Math., **45** (2), (2015), 155-171.
[29] M. Murali Krishna Rao, *A study of quasi-interior ideal of semiring*,Bull. Int. Math. Virtual Inst , Vol. **2** (2019), 287-300.
[30] M. Murali Krishna Rao, *A study of generalization of bi-ideal, quasi-ideal and interior ideal of semigroup*, Mathematica Morovica, Vol.**22**(2) (2018),103-115.
[31] Marapureddy Murali Krishna Rao,*A study of bi-quasi-interior ideal as a new generalization of ideal of generalization of semiring*,Bull. Int. Math. Virtual Inst , Vol. **8** (2018),519-535.
[32] M. Murali Krishna Rao,*bi-interior Ideals in semigroups, Discussiones Mathematicae General Algebra and Applications*, **38** (2018),69-7.8 doi:10.7151/dmgaa.1284.

[33] M. Murali Krishna Rao, *bi-interior Ideals in* Γ- *semirings*, Discussiones Mathematicae General Algebra and Applications, **38,2** (2018),239-254 doi:10.7151/dmgaa.1284.

[34] M. Murali Krishna Rao and B. Venkateswarlu,*Right derivation of ordered* $\Gamma-$*semirings*,Discussiones Mathematicae General Algebra and Applications **36** (2016), 209221 doi:10.7151/dmgaa.1258

[35] M. Murali Krishna Rao, B. Venkateswarlu and N.Rafi,*Left bi-quasi-ideals of* Γ-*semirings*,Acia Pacific Journal of Mathematics, Vol.**4, No. 2** (2017), 144-153. ISSN 2357-2205.

[36] M. Murali Krishna Rao, B. Venkateswarlu and N.Rafi,$r-$*IDEALS IN* $\Gamma-$*INCLINE*,Acia Pacific Journal of Mathematics, Vol.**6, No. 1** (2019), 6:6 ISSN 2357-2205.

[37] M. Murali Krishna Rao,*Fuzzy soft* Γ- *semirings homomorphism*, Annals of Fuzzy Mathematics and Informatics,Volume**12, No. 4,** (October 2016), pp. 479- 489.

[38] M. Murali Krishna Rao and B. Venkateswarlu,*On generalized right derivations of* $\Gamma-$ *incline*,*Journal Of The International Mathematical Virtual Institute,* **6** (2016), 31-47.

[39] M. Murali Krishna Rao,*Fuzzy soft* $\Gamma-$*semiring and fuzzy soft* k *ideal over* $\Gamma-$*semiring*, Ann. Fuzzy Math. Inform.**9(2)** (2015), 12-25.

[40] M.Murali Krishna Rao.*Bi-quasi-ideals and fuzzy bi-quasi ideals of* Γ-*semigroups*, Bull. Int. Math. Virtual Inst., Vol. **7(2)** (2017), 231-242.

[41] M. Murali Krishna Rao, B. Venkateswarlu and N.Rafi,*r- Fuzzy Ideals of* Γ-*incline*, Annals of Fuzzy Mathematics and Informatics, **17(2)** (2019).

[42] M. Murali Krishna Rao,*Fuzzy prime ideals in ordered* Γ *-semirings*, Joul. Int. Math. Virtual Inst. **7** (2017) 85–99.

[43] M. Murali Krishna Rao,*T-fuzzy ideals in ordered* Γ *-semirings*, Annals of Fuzzy Mathematics and Informatics Volume **13, No. 2,** (February 2017), pp. 253 -276.

[44] M. Murali Krishna Rao and B. Venkateswarlu, $L-$ *fuzzy ideals in* Γ- *semirings*, Annals of Fuzzy Mathematics and Informatics,**10 (1)** (2015), 1-16.

[45] M. Murali Krishna Rao and B. Venkateswarlu,*Fuzzy soft k ideals over semirings and fuzzy soft semirings homomorphism*, Journal of Hyperstructures **4(2)** (2015), 93-116.

[46] M. Murali Krishna Rao and B. Venkateswarlu,*An intuitionistic normal fuzzy soft k -ideal over a* Γ *-semirings*, Annals of Fuzzy Mathematics and Informatics Volume **11, No. 3** (March 2016), pp. 361 -376.

[47] N. Nobusawa, *On a generalization of the ring theory*, *Osaka. J.Math.,* **1**(1964), 81 - 89.

[48] A. Rosenfeld,*Fuzzy groups* , J. Math. Anal. Appl. **35** (1971), 512-519.

[49] M. K. Sen, *On* $\Gamma-$*semigroup, Proc. of Inter. Con. of Alg. and its Appl.*, Decker Publication*, New York,* (1981), 301-308.

[50] O. Steinfeld,*Uher die quasi ideals*, Von halbgruppend Publ. Math., Debrecen, **4** (1956), 262 275.

[51] U. M. Swamy and K. L. N. Swamy, *Fuzzy prime ideals of rings*, Jour. Math. Anal. Appl., **134** (1988), 94-103.

[52] H. S. Vandiver, *Note on a simple type of algebra in which cancellation law of addition does not hold*, *Bull. Amer. Math. Soc.(N.S.),* **40** (1934), 914-920.

[53] L. A. Zadeh, *Fuzzy sets*, Information and Control **8** (1965), 338-353.
[54] L. A. Zadeh, *The concept of a linguistic variable and its application to approximate reasoning*, Inform. Sci. **8** (1975), 199-249.

Marapureddy Murali Krishna Rao
Department of Mathematics, University of GITAM SCHOOL OF TECHNOLOGY,
City Visakhapatnam, Country India
Email: mmarapureddy@gmail.com