Journal of Hyperstructures 10 *(1) (2021), 22-37.*
ISSN: 2322-1666 print/2251-8436 online

ON PRIMARY HYPERIDEALS OF TERNARY HYPERSEMIRING

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ABSTRACT. In this article, we introduce the notions of radical of hyperideals and primary hyperideals of a ternary hypersemiring. We obtain some important properties of radical of hyperideals and primary hyperideals on a particular class of hyperideals, called $\mathcal{C}$-ternary hyperideals in ternary hypersemirings. We also generalize the concept of prime and primary avoidance theorem in ternary hypersemirings for $\mathcal{C}$-ternary hyperideals.

Key Words: Ternary hypersemiring, Primary hyperideals, Prime hyperideals, $\mathcal{C}$-ternary hyperideals.

2010 Mathematics Subject Classification: 20N99, 20N20, 16Y99.

1. Introduction

The theory of hyperstructures is a well established branch of classical algebraic theory. The hyperstructure theory was first introduced by the French mathematician, F. Marty [10] in 1934. Since then, algebraic hyperstructures have been investigated by many mathematicians with numerous applications in both pure and applied sciences. Algebraic hyperstructures are a suitable generalization of classical algebraic structures. The concept of multiplicative hyperring was initiated by R. Rota [12] in 1982. In [11], Procesi and Rota introduced and studied the prime hyperideals in multiplicative hyperrings. R. Ameri, A. Kordi and S. Sarka-Mayerova introduced the notion of coprime hyperideals

Received:07 January 2021, Accepted: 20 May 2021. Communicated by Sarka Hoskova-Mayerova;

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in multiplicative hypersemiring [1]. In recent years, the theory of hyperstructures is further developed by many researchers (see [2, 3, 14]). The notion of ternary algebraic system was introduced by D. H. Lehmer [9]. In 2003, Dutta and Kar introduced the notion of ternary semiring [5], which is a generalization of the ternary ring introduced by Lister [8]. The class of multiplicative ternary hyperring was introduced by Md Salim, T.K. Dutta and T. Chandra [13] in 2015. After that, in 2018, N. Tamang and M. Mandal [16] defined and studied ternary hypersemiring, which is a generalization of the concept of multiplicative ternary hyperring and ternary semiring as well.

The objective of this paper is to introduce and study radical of hyperideals and primary hyperideals in ternary hypersemiring. In Section 2, we recall some essential preliminaries so as to use them in the sequel. In Section 3, we introduce the notions of $\mathcal{C}$-ternary hyperideal, radical hyperideals and primary hyperideal and study some of their properties. Next, we prove the prime avoidance theorem (*cf. Theorem 3.33*) for ternary hypersemiring. Lastly, using the technique of efficient covering we prove the primary avoidance theorem (*cf. Theorem 3.36*) and an extended version of primary avoidance theorem (*cf. Theorem 3.37*) for ternary hypersemiring.

2. Preliminaries

In this section, we review some definitions and results which will be used later.

Definition 2.1. [4] Ternary hyperoperation on a set A is a map $\circ : A \times A \times A \to P^*(A)$, where $P^*(A)$ is the collection of all subsets of A.

Definition 2.2. [16] A ternary hypersemiring $(S, +, \circ)$ is an additive commutative semigroup $(S, +)$, endowed with a ternary hyperoperation '$\circ$' such that the following conditions hold:

(i) $(a \circ b \circ c) \circ d \circ e = a \circ (b \circ c \circ d) \circ e = a \circ b \circ (c \circ d \circ e)$;
(ii) $(a + b) \circ c \circ d \subseteq a \circ c \circ d + b \circ c \circ d$;
(iii) $a \circ (b + c) \circ d \subseteq a \circ b \circ d + a \circ c \circ d$;
(iv) $a \circ b \circ (c + d) \subseteq a \circ b \circ c + a \circ b \circ d$; for all $a, b, c, d \in S$.

A ternary hypersemiring $(S, +, \circ)$ is said to be commutative if for all $a_1, a_2, a_3 \in S$, $a_1 \circ a_2 \circ a_3 = a_{\sigma(1)} \circ a_{\sigma(2)} \circ a_{\sigma(3)}$, where σ is a permutation of $\{1, 2, 3\}$. If the inclusions in the Definition 2.2(ii), (iii) and (iv) are replaced by equalities, then the ternary hypersemiring is called a strongly distributive ternary hypersemiring.

Definition 2.3. [16] Let $(S,+,\circ)$ be a ternary hypersemiring. An element $0 \in S$ is called a zero element or absorbing zero or simply zero of S if $0 \in 0\circ x\circ y = x\circ 0\circ y = x\circ y\circ 0$ for all $x, y \in S$ (strongly absorbing zero if $0\circ x\circ y = x\circ 0\circ y = x\circ y\circ 0 = \{0\}$).

Definition 2.4. [16] An additive subsemigroup T of a ternary hypersemiring $(S,+,\circ)$ is called a ternary subhypersemiring if $t_1 \circ t_2 \circ t_3 \subseteq T$ for all $t_1, t_2, t_3 \in T$.

Definition 2.5. [16] Let $(S,+,\circ)$ be a ternary hypersemiring. A finite subset $\epsilon = \{(e_i; f_i); i = 1, 2....n\}$ of $S\times S$ is called a left (lateral or right) identity set of S if for any $a \in S$, $a \in \Sigma_{i=1}^{n} e_i \circ f_i \circ a$ ($a \in \Sigma_{i=1}^{n} e_i \circ a \circ f_i$ or $a \in \Sigma_{i=1}^{n} a \circ e_i \circ f_i$).

A finite subset $\epsilon = \{(e_i; f_i); i = 1, 2....n\}$ of $S\times S$, where S is a ternary hypersemiring, is called an identity set if it is a left, a lateral and a right identity set of S.

An element $e \in S$ is called a hyperidentity or unital element of S if $a \in (e\circ e\circ a) \cap (e\circ a\circ e) \cap (a\circ e\circ e)$ for all $a \in S$.

Definition 2.6. [16] Let $(S,+,\circ)$ be a ternary hypersemiring. An additive subsemigroup I of S is called
(i) a left hyperideal of S if $s_1 \circ s_2 \circ i \subseteq I$ for all $s_1; s_2 \in S$ and $i \in I$.
(ii) a right hyperideal of S if $i \circ s_1 \circ s_2 \subseteq I$ for all $s_1; s_2 \in S$ and $i \in I$.
(iii) a lateral hyperideal of S if $s_1 \circ i \circ s_2 \subseteq I$ for all $s_1; s_2 \in S$ and $i \in I$.
(iv) a two sided hyperideal of S if I is both a left and a right hyperideal of S.
(v) a hyperideal of S if I is a left, a right and a lateral ideal of S.

Definition 2.7. [16] Let $(S,+,\circ)$ be a ternary hypersemiring. If A, B and C are non empty subsets of S, then $A\circ B\circ C = \cup\{\sum_{finite} a_i \circ b_i \circ c_i : a_i \in A, b_i \in B, c_i \in C\}$.

Throughout this paper, we denote $A \circ B \circ C$ by ABC.

Theorem 2.8. [12] *If A, B and C are respectively right, lateral and left hyperideals of a ternary hypersemiring S, then $ABC \subseteq A \cap B \cap C$.*

3. Radical and Primary Hyperideals

Throughout the paper, unless otherwise stated S stands for a ternary hypersemiring $(S, +, \circ)$ with zero. Z_0^- and Z_0^+ denote set of all negative integers with zero and set of all positive integers with zero respectively.

Definition 3.1. Let $\mathcal{C} = \{\Pi_{i=1}^{2n+1} a_i : a_i \in S, n \in Z_0^+\}$ be the class of all finite ternary products of elements of a ternary hypersemiring $(S, +, \circ)$. A hyperideal I is called complete ternary hyperideal or $\mathcal{C}$-ternary hyperideal if for any $A \in \mathcal{C}$, $I \cap A \neq \phi$ implies $A \subseteq I$.

Example 3.2. Consider the ternary hypersemiring $(Z_0^-, +, \circ)$, where "+" is the standard addition of integers and hyperoperation '$\circ$' is defined by $a \circ b \circ c = \{abc + kn : n \in Z_0^-\}$, k being a fixed positive integer. Then every hyperideal of the form $mZ_0^+, m \in Z_0^-$ is a $\mathcal{C}$-ternary hyperideal.

Example 3.3. Consider the ternary hypersemiring $([0, 1], +, \circ)$, where binary operation '+' and ternary hyperoperation '$\circ$' on S are defined by $a + b = max\{a, b\}$ and $a \circ b \circ c = [0, x]$ respectively, where $x = min\{a, b, c\}$. In this ternary hypersemiring, the hyperideal $[0, \frac{1}{2}]$ is a $\mathcal{C}$-ternary hyperideal.

Example 3.4. Corresponding the set $X = \{2, 3\}$, $(Z_0^-, +, \circ)$ forms a ternary hypersemiring, where ternary hyperoperation '$\circ$' is defined by $a \circ b \circ c = \{x.a.b.c : x \in X\}$. In this ternary hypersemiring, the hyperideal $18Z_0^-$ is not a $\mathcal{C}$-ternary hyperideal. Because $-18 \in \{(-1) \circ (-1) \circ (-1) \circ (-1) \circ (-1) \circ (-1) \circ (-1)\}$, hence $\{(-1) \circ (-1) \circ (-1) \circ (-1) \circ (-1) \circ (-1) \circ (-1)\} \cap 18Z_0^- \neq \varnothing$. But $-27 \in \{(-1) \circ (-1) \circ (-1) \circ (-1) \circ (-1) \circ (-1) \circ (-1)\}$, so $\{(-1) \circ (-1) \circ (-1) \circ (-1) \circ (-1) \circ (-1) \circ (-1)\} \nsubseteq 18Z_0^-$.

Proposition 3.5. *Intersection of arbitrary collection of $\mathcal{C}$-ternary hyperideals $\{I_i : i \in \Lambda\}$ of ternary hypersemiring $(S, +, \circ)$ is also a $\mathcal{C}$-ternary hyperideal.*

Proof. Let $A \in \mathcal{C}$ such that $A \cap (\bigcap_{i \in \Lambda} I_i) \neq \phi$, so $A \cap I_i \neq \phi$ for all $i \in \Lambda$. Since $\{I_i : i \in \Lambda\}$ are $\mathcal{C}$-ternary hyperideals of S, $A \subseteq I_i$ for all $i \in \Lambda$. So $A \subseteq (\bigcap_{i \in \Lambda} I_i)$. Hence $(\bigcap_{i \in \Lambda} I_i)$ is a $\mathcal{C}$-ternary hyperideal. □

Definition 3.6. Let $(R, +, \circ)$ and $(S, +, \circ)$ be ternary hypersemirings. A mapping $f : R \to S$ is said to be a homomorphism if $f(a+b) = f(a)+f(b)$ and $f(a \circ b \circ c) \subseteq f(a) \circ f(b) \circ f(c)$. In particular, a homomorphism is called a good homomorphism if $f(a \circ b \circ c) = f(a) \circ f(b) \circ f(c)$.

Definition 3.7. Let $(S, +, \circ)$ be a ternary hypersemiring. Then a hyperideal I of S, is said to be a k-hyperideal if $x+y \in I, x \in S$ and $y \in I$ implies $x \in I$.

Proposition 3.8. *Let f be a good homomorphism from a ternary hypersemiring S to a ternary hypersemiring T and I, J be k-hyperideals of S and T respectively. Then the following hold.*

(i) If I is a $\mathcal{C}$-ternary hyperideal of S containing the set $\{x \in S :$ there exist $a, b \in S_1$ such that $x = a + b$ and $f(a) = f(b)\}$ and f is an onto homomorphism, then $f(I)$ is a $\mathcal{C}$-ternary hyperideal of T.

(ii) If J is a $\mathcal{C}$-ternary hyperideal of T, then $f^{-1}(J)$ is a $\mathcal{C}$-ternary hyperideal of S.

Proof. (i) Let $\prod_{i=1}^{2n+1} a_i \cap f(I) \neq \phi$ for some $a_1, a_2, ..., a_{2n+1} \in T$. So there exist $s_i \in S$ such that $f(s_i) = a_i$, $1 \leq i \leq 2n+1$. Then $\Pi_{i=1}^{2n+1} f(s_i) \cap f(I) = f(\Pi_{i=1}^{2n+1} s_i) \cap f(I) \neq \phi$, because f is a good homomorphism. So there exists $r \in \Pi_{i=1}^{2n+1} s_i$ such that $f(r) \in f(I)$. Thus $f(r) = f(i)$ for some $i \in I$, that implies $r + i \in I$. Since I is a k-hyperideal, $r \in I$. So $\Pi_{i=1}^{2n+1} s_i \cap I \neq \phi$. Thus $\Pi_{i=1}^{2n+1} s_i \subseteq I$, since I is a $\mathcal{C}$-ternary hyperideal of S. Hence $\Pi_{i=1}^{2n+1} a_i = \Pi_{i=1}^{2n+1} f(s_i) = f(\Pi_{i=1}^{2n+1} s_i) \subseteq f(I)$.

(ii) Let $\Pi_{i=1}^{2n+1} s_i \cap f^{-1}(J) \neq \phi$ for some $s_1, s_2, ..., s_{2n+1} \in S$. Suppose $t \in \Pi_{i=1}^{2n+1} s_i \cap f^{-1}(J)$, then $f(t) \in f(\Pi_{i=1}^{2n+1} s_i) \cap J$. It follows that $\Pi_{i=1}^{2n+1} f(s_i) \cap J \neq \phi$. Since J is a $\mathcal{C}$-ternary hyperideal of T, $f(\Pi_{i=1}^{2n+1} s_i) = \Pi_{i=1}^{2n+1} f(s_i) \subseteq J$ which implies $\Pi_{i=1}^{2n+1} s_i \subseteq f^{-1}(J)$. So $f^{-1}(J)$ is a $\mathcal{C}$-ternary hyperideal of S. □

Definition 3.9. A proper hyperideal P of a ternary hypersemiring S is called a prime hyperideal of S if for any hyperideals A, B and C of S, $A \circ B \circ C \subseteq P$, then $A \subseteq P$ or $B \subseteq P$ or $C \subseteq P$.

Definition 3.10. A hyperideal P of a ternary hypersemiring S is called completely prime if for the elements a, b and c of S, $abc \subseteq P$, then either $a \in P$ or $b \in P$ or $c \in P$.

In a commutative ternary hypersemiring, the notions of prime hyperideal and completely prime hyperideal are the same.

Definition 3.11. A hyperideal M in a ternary hypersemiring S is called maximal if $M \neq S$ and for any hyperideal $N \supseteq M$, either $N = M$ or $N = S$.

Definition 3.12. A non-empty subset A of a ternary hypersemiring $(S, +, \circ)$ is called an m-system whenever for any $a, b, c \in A$, $aSbSc \cap A \neq \emptyset$ or $aSSbSSc \cap A \neq \emptyset$ or $aSSbScS \cap A \neq \emptyset$ or $SaSbSSc \cap A \neq \emptyset$.

Theorem 3.13. *Let I be an m-system of a ternary hypersemiring $(S, +, \circ)$ and N be a hyperideal of S such that $N \cap I = \emptyset$. Then there exists a maximal hyperideal M of S containing N such that $M \cap I = \emptyset$. Moreover, M is also a prime hyperideal of S.*

Proof. Consider the collection of hyperideals $\aleph = \{A : A \supseteq N$, A is a hyperideal of S such that $A \cap I = \emptyset\}$. Clearly $\aleph$ is non-empty, since $N \in \aleph$. Under set inclusion relation, $\aleph$ forms a partially order set and any chain of elements in $\aleph$ has an upper bound which is their union. So by Zorn's Lemma, $\aleph$ contains a maximal element M. Therefore from the consideration of $\aleph$, M is the required maximal hyperideal of S containing N such that $M \cap I = \emptyset$.

If possible, let M be not a prime hyperideal of S. So there exist hyperideals J, K, L of S such that $JKL \subseteq M$ but $J \nsubseteq M$, $K \nsubseteq M$ and $L \nsubseteq M$. Now $M \subsetneq M + J$, $M \subsetneq M + K$, $M \subsetneq M + L$. So by the given condition and maximality of M, $(M + J) \cap I \neq \emptyset$, $(M + K) \cap I \neq \emptyset$ and $(M + L) \cap I \neq \emptyset$. Then there exist $i_1, i_2, i_3 \in I$ such that $i_1 = n_1 + j$, $i_2 = n_2 + k$ $i_3 = n_3 + l$ for some $n_1, n_2, n_3 \in M$, $j \in J, k \in K, l \in L$. Now $i_1s_1s_2i_2s_3s_4i_3 = (n_1 + j)s_1s_2(n_2 + k)s_3s_4(n_3 + l) \subseteq n_1s_1s_2n_2s_3s_4n_3 + js_1s_2n_2s_3s_4n_3 + n_1s_1s_2ks_3s_4n_3 + js_1s_2ks_3s_4n_3 + n_1s_1s_2n_2s_3s_4l + js_1s_2n_2s_3s_4l + n_1s_1s_2ks_3s_4l + js_1s_2ks_3s_4l \subseteq M$ for all $s_1, s_2, s_3, s_4 \in S$. This implies $i_1SSi_2SSi_3 \cap I \subseteq M \cap I = \phi$, which is a contradiction. Thus in any case, we get a contradiction that $M \cap I = \phi$. Hence M is a prime hyperideal of S. □

Definition 3.14. Let A be a hyperideal of a ternary hypersemiring $(S, +, \circ)$. The intersection of all prime hyperideals of S containing A is called prime radical or simply radical of A, denoted by $Rad(A)$. If the ternary hypersemiring S does not have any prime hyperideal containing A, define $Rad(A) = S$.

Example 3.15. Consider the ternary hypersemiring $(Z_0^-, +, \circ)$, where '$\circ$' is defined by $a \circ b \circ c = (abc)Z_0^+$. The radicals of the hyperideals $7Z_0^-$ and $4Z_0^-$ are $7Z_0^-$ and $2Z_0^-$ respectively.

Example 3.16. For the set $X = \{10, 20\}$, the radicals of the hyperideals $5Z_0^-$ and $6Z_0^-$ in the ternary hypersemiring $(Z_0^-, +, \circ)$, where '$\circ$' is defined by $a \circ b \circ c = \{x.a.b.c : x \in X\}$, are Z_0^-, $3Z_0^-$ respectively.

Notation 3.17. *For any hyperideal A of S, $\Re(A) = \{a \in S : a^{2n+1} \subseteq A$, for some integer $n \geq 0\}$.*

Theorem 3.18. *Let A be a hyperideal of a commutative ternary hypersemiring $(S, +, \circ)$. Then $\Re(A)$ is a hyperideal of S containing A and $\Re(A) \subseteq Rad(A)$.*

Proof. Let $a, b \in \Re(A)$ be arbitrary. Then there exist $m, n \in Z_0^+$ such that $a^{2m+1} \subseteq A$ and $b^{2n+1} \subseteq A$. If $m = n = 0$, then $\{a + b\} \subseteq A$, so $a + b \in \Re(A)$. If either $m > 0$ or $n > 0$, then $2m + 2n + 1 \geq 3$. Now $(a+b)^{2m+2n+1} \subseteq \sum_{r=0}^{2m+2n+1} \binom{2m+2n+1}{r=0} a^{2m+2n+1-r} b^r$. If $2m+2n+1-r < 2m+1$, then $r \geq 2n+1$. Otherwise $2m+2n+1-r \geq 2m+1$. So in each case, either $a^{2m+2n+1-r} \subseteq A$ or $b^r \subseteq A$. Thus $(a+b)^{2m+2n+1} \subseteq A$. Consequently $a + b \in \Re(A)$. Again, for any $x, y \in S$ and $a \in \Re(A)$, there exists $n \in Z_0^+$ such that $a^{2n+1} \subseteq A$. Now for any $t \in x \circ y \circ a$, $t^{2n+1} \subseteq (x \circ y \circ a)^{2n+1} = x^{2n+1} \circ y^{2n+1} \circ a^{2n+1} \subseteq A$, which implies $t \in \Re(A)$. So $x \circ y \circ a \subseteq \Re(A)$. Therefore $\Re(A)$ is a hyperideal of S. Also for any a, $a \in A \Rightarrow a^1 = \{a\} \subseteq A \Rightarrow a \in \Re(A)$. Hence $A \subseteq \Re(A)$.

Let $a \in \Re(A)$, then $a^{2n+1} \subseteq A$ for some $n \in Z_0^+$. Therefore for any prime hyperideal P of S containing A, $a^{2n+1} \subseteq P$ implies $a \in P$. So $a \in Rad(A)$ and hence $\Re(A) \subseteq Rad(A)$. □

Theorem 3.19. *Let A be a complete ternary k-hyperideal of a commutative ternary hypersemiring $(S, +, \circ)$. Then $Rad(A) \subseteq \Re(A) = \{a \in S : a^{2n+1} \subseteq A$ for some integers $n \in Z_0^+\}$.*

Proof. Let $p \notin \Re(A)$. Then $p^{2n+1} \nsubseteq A$ for any $n \in Z_0^+$. Since A is complete ternary k-hyperideal, $p^{2n+1} \cap A = \phi$ for all $n \in Z_0^+$. Now consider $D = \cup\{p^{2n+1} + A$, for any $n \in Z_0^+\}$. Let $a, b, c \in D$ be arbitrary. Then $a \circ b \circ c \subseteq p^{2m_1+1} \circ p^{2m_2+1} \circ p^{2m_3+1} + A \subseteq p^{2(m_1+m_2+m_3+1)+1} + A \subseteq D$. Since S contains hyperidentity, D is an m-system. Here $D \cap A = \phi$. Let if possible $t \in D \cap A$, then $t = x + y$, where $x \in p^{2n+1}$ and $y \in A$. Thus $t \in A$ and $y \in A$ implies $x \in A$ (since A is a k-hyperideal), which contradicts the fact that $p^{2n+1} \cap A = \phi$ for any $n \in Z_0^+$. Hence by Theorem 3.13, there is a prime hyperideal P containing A and disjoint from D. So $p^{2n+1} \cap P = \phi$ for any $n \in Z_0^+$. Thus $p \notin P \Rightarrow p \notin Rad(A)$, consequently $Rad(A) \subseteq \Re(A)$. □

Proposition 3.20. *Let A be a $\mathcal{C}$-ternary hyperideal of a ternary hypersemiring $(S, +, \circ)$. Then $Rad(A)$ is a $\mathcal{C}$-ternary hyperideal of the ternary hypersemiring S.*

Proof. Let $a_1 \circ a_2 \circ a_3 \circ .... \circ a_{2n+1} \cap Rad(A) \neq \phi$ for some $a_1, a_2, a_3..., a_{2n+1} \in S$ and integers $n \in Z_0^+$. Then there exists $x \in a_1 \circ a_2 \circ a_3 \circ .... \circ a_{2n+1}$ such that $x^{2m+1} \subseteq A$, where $m \in Z_0^+$. Also $x^{2m+1} \subseteq (a_1 \circ a_2 \circ a_3 \circ .... \circ a_{2n+1})^{2m+1}$ implies $(a_1 \circ a_2 \circ a_3 \circ .... \circ a_{2n+1})^{2m+1} \cap A \neq \phi$. Since A is a $\mathcal{C}$-ternary hyperideal of S, $(a_1 \circ a_2 \circ a_3 \circ .... \circ a_{2n+1})^{2m+1} \subseteq A$. Now for any $y \in a_1 \circ a_2 \circ a_3 \circ .... \circ a_{2n+1}$, $y^{2m+1} \subseteq A$, whence $y \in Rad(A)$, i.e., $a_1 \circ a_2 \circ a_3 \circ .... \circ a_{2n+1} \subseteq Rad(A)$. Thus $Rad(A)$ is a $\mathcal{C}$-ternary hyperideal of S. □

Proposition 3.21. *Let A, B and C are hyperideals of a ternary hypersemiring S. Then*

(1) *$A \subseteq Rad(A)$.*
(2) *$A \subseteq B \Rightarrow Rad(A) \subseteq Rad(B)$.*
(3) *$Rad(Rad(A)) = Rad(A)$.*
(4) *$Rad(A) = Rad(A^{2n+1})$ for any $n \in Z_0^+$.*
(5) *$Rad(A + B) = Rad(Rad(A) + Rad(B))$.*
(6) *If S is commutative and A, B, C are complete ternary k-hyperideals of S, then $Rad(ABC) = Rad(A \cap B \cap C) = Rad(A) \cap Rad(B) \cap Rad(C)$.*

Proof. (1) Follows immediately from the Definition 3.14.

(2) Suppose $A \subseteq B$. Then any prime hyperideal P containing B also contains A. Therefore $Rad(A) \subseteq Rad(B)$.

(3) By (1) and (2), $A \subseteq Rad(A) \Rightarrow Rad(A) \subseteq Rad(Rad(A))$. Now let $x \in Rad(Rad(A))$ and $\{P_i\}_{i \in I}$ be the collection of all prime hyperideals containing A. Then $Rad(A) \subseteq P_i$ for all $i \in I$. So $x \in Rad(Rad(A)) \subseteq P_i$ for all $i \in I$. Hence $x \in Rad(A)$. Therefore $Rad(Rad(A)) = Rad(A)$.

(4) Since A is a hyperideal of S, $A^{2n+1} \subseteq A$ for all $n \in Z_0^+$. By (2), $Rad(A) \supseteq Rad(A^{2n+1})$. Let $x \in Rad(A)$. So x is in the set of all prime hyperideals containing A. If possible, let $x \notin Rad(A^{2n+1})$. Then there exists a prime hyperideal P containing A^{2n+1} and $x \notin P$. Here $A^{2n+1} \subseteq P$ implies $A \subseteq P$, because P is a prime hyperideal, which contradicts the fact that x in the set of all prime hyperideals containing A. Hence $Rad(A) = Rad(A^{2n+1})$ for any $n \in Z_0^+$.

(5) We have $A \subseteq Rad(A)$ and $B \subseteq Rad(B)$. So $A + B \subseteq Rad(A) + Rad(B)$ and thus by (2), $Rad(A+B) \subseteq Rad(Rad(A)+Rad(B))$. Again $A \subseteq A + B$ and $B \subseteq A + B$, which implies $Rad(A) \subseteq Rad(A + B)$ and $Rad(B) \subseteq Rad(A + B)$. Hence $Rad(A) +$

$Rad(B) \subseteq Rad(A+B)$. Thus by (2) and (3), $Rad(Rad(A) + Rad(B)) \subseteq Rad(Rad(A+B)) = Rad(A+B)$. Therefore $Rad(A+B) = Rad(Rad(A) + Rad(B))$.

(6) Clearly $ABC \subseteq A \cap B \cap C$. Then by (2), $Rad(ABC) \subseteq Rad(A \cap B \cap C)$. Let $x \in Rad(A \cap B \cap C)$. So there exists $m \in Z_0^+$ such that $x^{2m+1} \subseteq A \cap B \cap C$. Then $x^{6m+3} = x^{2m+1} \circ x^{2m+1} \circ x^{2m+1} \subseteq ABC$, which implies $x \in Rad(ABC)$. Hence $Rad(ABC) = Rad(A \cap B \cap C)$.

For the second equality, let $x \in Rad(A \cap B \cap C)$. Then there exists $n \in Z_0^+$ such that $x^{2m+1} \subseteq (A \cap B \cap C)$. Therefore $x^{2m+1} \subseteq A$, $x^{2m+1} \subseteq B$ and $x^{2m+1} \subseteq C$. This implies $x \in Rad(A)$, $x \in Rad(B)$ and $x \in Rad(C)$. So $x \in Rad(A) \cap Rad(B) \cap Rad(C)$. Conversely, let $x \in Rad(A) \cap Rad(B) \cap Rad(C)$. Then there exist $r, s, t \in Z_0^+$ such that $x^{2r+1} \subseteq A$, $x^{2s+1} \subseteq B$ $x^{2t+1} \subseteq C$. So $x^{(2r+1)(2s+1)(2t+1)} \subseteq A \cap B \cap C$, which implies $x \in Rad(A \cap B \cap C)$. Consequently, $Rad(A) \cap Rad(B) \cap Rad(C) \subseteq Rad(A \cap B \cap C)$. Hence $Rad(A \cap B \cap C) = Rad(A) \cap Rad(B) \cap Rad(C)$.

□

Proposition 3.22. *Let I be a hyperideal in a commutative ternary hypersemiring S. Then $Rad(I) = Rad(\Re(I))$.*

Proof. Since $I \subseteq \Re(I)$, Proposition 3.21(2) implies the inclusion $Rad(I) \subseteq Rad(\Re(I))$. Now for reverse inclusion, let P be any prime hyperideal containing I. Then it is sufficient to show that $\Re(I) \subseteq P$. Consider $x \in \Re(I)$. Then $x^{2n+1} \subseteq I \subseteq P$ for some integer $n \in Z_0^+$. So $x \in P$, that implies $\Re(I) \subseteq P$. Thus $Rad(I) = Rad(\Re(I))$. □

Theorem 3.23. *Let S_1 and S_2 be commutative ternary hypersemirings, $f : S_1 \to S_2$ be a good homomorphism and I be a k-hyperideal of S_2. Then $f^{-1}(Rad(I)) = Rad(f^{-1}(I))$.*

Proof. Let $x \in f^{-1}(Rad(I))$. Then $f(x) \in Rad(I)$. So there exists an integer $n \in Z_0^+$ such that $f^{2n+1}(x) = f(x^{2n+1}) \subseteq I$, which implies $x^{2n+1} \subseteq f^{-1}(I)$. Hence $x \in$ Rad $(f^{-1}(I))$.

Conversely, let $x \in Rad(f^{-1}(I))$. Then there exists an integer $n \in Z_0^+$ such that $x^{2n+1} \in (f^{-1}(I))$. Thus $f^{2n+1}(x) = f(x^{2n+1}) \subseteq I$. So $f(x) \in Rad(I)$, which implies $x \in f^{-1}$(Rad (I)). Thus $Rad(f^{-1}(I)) \subseteq f^{-1}(Rad(I))$. Therefore f^{-1}(Rad (I))=Rad $(f^{-1}(I))$. □

Theorem 3.24. *Let S_1 and S_2 be commutative ternary hypersemirings, $f : S_1 \to S_2$ be a good epimorphism and I be a k-hyperideal of S_1 such* *that* $\{x \in S_1 :$ *there exist* $a, b \in S_1$ *such that* $x = a + b$ *and* $f(a) = f(b)\} \subseteq I$. *Then* $f(Rad\ (I))$=$Rad\ (f(I))$.

Proof. Let $x \in f(Rad(I))$. Then there exists $a \in Rad(I)$ such that $f(a) = x$. So there exists $m \in Z_0^+$ such that $a^{2m+1} \subseteq I$. Now $x^{2m+1} = (f(a))^{2m+1} = f(a^{2m+1}) \subseteq f(I)$, since $a^{2m+1} \subseteq I$. Thus $x \in Rad(f(I))$. Hence $f(Rad(I)) \subseteq Rad(f(I))$.

For the converse part, let $x \in Rad(f(I))$. So $x^{2n+1} \subseteq f(I)$ for some $n \in Z_0^+$. Also there exists an element $a \in S$ such that $f(a) = x$. Now $f(a^{2n+1}) = (f(a))^{2n+1} = x^{2n+1} \subseteq f(I)$. Thus for any element $p \in a^{2n+1}$, there is an element $i \in I$ such that $f(p) = f(i)$. By the given condition, $p + i \in I$ and hence $p \in I$. So $a^{2n+1} \subseteq I$, which implies $a \in Rad(I)$. Thus $x = f(a) \in f(Rad(I))$. □

Definition 3.25. A hyperideal A of a ternary hypersemiring S is called primary hyperideal of S if for any $a, b, c \in S$, $abc \subseteq A$ and $a \notin A, b \notin A$, implies there exists an integer $n \in Z_0^+$ such that $c^{2n+1} \subseteq A$.

Theorem 3.26. *Let* A *be a primary* $\mathcal{C}$*-ternary hyperideal of a commutative ternary hypersemiring* $(S, +, \circ)$*, then* $Rad(A)$ *is a prime hyperideal of* S.

Proof. Let $a \circ b \circ c \subseteq Rad(A)$ and $a \notin Rad(A)$, $b \notin Rad(A)$. Now for any element $x \in a \circ b \circ c$, there exists an integer $n \in Z_0^+$ such that $x^{2n+1} \subseteq A$. This implies $x^{2n+1} \subseteq (a \circ b \circ c)^{2n+1} = a^{2n+1} \circ b^{2n+1} \circ c^{2n+1}$. So $a^{2n+1} \circ b^{2n+1} \circ c^{2n+1} \cap A \neq \phi$. Because A is $\mathcal{C}$-ternary hyperideal, $a^{2n+1} \circ b^{2n+1} \circ c^{2n+1} \subseteq A$. Now $a \notin Rad(A)$ and $b \notin Rad(A)$ implies $a^{2n+1} \cap A = \phi$ and $b^{2n+1} \cap A = \phi$ respectively. For any $p \in a^{2n+1}$, $q \in b^{2n+1}$, $r \in c^{2n+1}$, we have $p \notin A$ and $q \notin A$. Here $p \circ q \circ r \subseteq a^{2n+1} \circ b^{2n+1} \circ c^{2n+1} \subseteq A$. Since A is a primary hyperideal, there exists an integer $m \in Z_0^+$ such that $r^{2m+1} \subseteq A$. Also $r^{2m+1} \subseteq (c^{2n+1})^{2m+1}$. Hence $(c^{2n+1})^{2m+1} \cap A \neq \phi$, which implies $(c^{2n+1})^{2m+1} \subseteq A$ and hence $c \in Rad(A)$. So $Rad(A)$ is a prime hyperideal of S. □

Theorem 3.27. *Let* I *be a proper hyperideal of a ternary hypersemiring* $(S, +, \circ)$. *Then* $Rad(I) = \{s \in S$*: every m-system in* S *which contains* s *has a non-empty intersection with* $I\}$

Proof. Consider $\Omega = \{s \in S$: every m-system in S which contains s has a non-empty intersection with I$\}$. Let $x \in Rad(I)$ and $\{P_\lambda : \lambda \in \Lambda\}$ be the collection of all prime hyperideals of S containing I. Then $x \in P_\lambda$ for all $\lambda \in \Lambda$. If possible, let there exists an m-system A which contains x and has empty intersection with I. Then by Theorem 3.13, there exists a prime hyperideal P_λ such that $A \cap P_\lambda = \phi$. Since $x \in P_\lambda$, we arrive at a contradiction. So $Rad(I) \subseteq \Omega$.

Conversely, let $x \in \Omega$ and $\{P_\lambda : \lambda \in \Lambda\}$ be the collection of all prime hyperideals of S containing I. If possible, let $x \notin Rad(I)$. Then there exists $\lambda \in \Lambda$ such that $x \notin P_\lambda$. By Theorem 3.13, P_λ^c is an m-system of S, which contains x and has empty intersection with I, which is a contradiction. Therefore $\Omega \subseteq Rad(I)$. □

Definition 3.28. Let A be a primary complete ternary k-hyperideal. A is called P-primary complete ternary k-hyperideal, whenever $Rad(A) = P$ is a prime hyperideal of a commutative ternary hypersemiring S.

Example 3.29. In the ternary hypersemiring $(Z_0^-, +, \circ)$, where hyperoperation '$\circ$' is defined by $a \circ b \circ c = \{n(abc) : n \in Z_0^-\}$, $P = 2Z_0^-$ is a prime hyperideal. Here the primary complete ternary k-hyperideal $8Z_0^-$ is a P-primary complete ternary k-hyperideal, because $Rad(8Z_0^-) = P$.

Proposition 3.30. *If A is a complete ternary k-hyperideal and P be a hyperideal of a commutative ternary hypersemiring $(S, +, \circ)$, then A is a P-primary complete ternary k-hyperideal of S if and only if*

(1) $A \subseteq P \subseteq Rad(A)$ *and*
(2) $a \circ b \circ c \subseteq A$ *and* $a, b \notin A$ *implies* $c \in P$.

Proof. If A is a P-primary complete ternary k-hyperideal, then the conditions (1), (2) are clearly true. For the converse part, let $a \circ b \circ c \subseteq A$ and $a, b \notin A$. Then by the given conditions, $c \in P \subseteq Rad(A)$, which implies $c^{2n+1} \subseteq A$ for some integer $n \in Z_0^+$. So A is a primary hyperideal. To show that $Rad(A) = P$, let $x \in Rad(A)$. Then there exists a least positive integer m such that $x^{2m+1} \subseteq A$. If $m = 0$, then by (1), $x \in P$. If $m \geq 1$, then $x^{2m-1} \nsubseteq A$. Since A is a $\mathcal{C}$-ternary hyperideal, $x^{2m-1} \cap A = \phi$. Now let $y, z \in x^{2m-1}$. Then $y \circ z \circ x \subseteq x^{2m-1} \circ x^{2m-1} \circ x \subseteq A$. So by (2), $x \in P$. Hence by (1), $P = Rad(A)$, thus A is a P-primary complete ternary k-hyperideal of S. □

Proposition 3.31. *Let A be a proper hyperideal of ternary hypersemiring S. Then A is a primary hyperideal of S if and only if for any hyperideals I, J, K of S, if $IJK \subseteq A$, $I \nsubseteq A$ and $J \nsubseteq A$, then $K \subseteq \Re(A)$.*

Proof. Let A be a primary hyperideal such that $IJK \subseteq A$, $I \nsubseteq A, J \nsubseteq A$. Then there exist $i \in I, j \in J$ such that $i \notin A$ and $j \notin A$. Take $k \in K$. Since $ijk \subseteq IJK \subseteq A$, there exists an integer $n \in Z_0^+$ such that $k^{2n+1} \subseteq A$ i.e., $k \in \Re(A)$. Therefore $K \subseteq \Re(A)$.

Conversely, let $a \circ b \circ c \subseteq A, a \notin A, b \notin A$. Since $\langle a \rangle \circ \langle b \rangle \circ \langle c \rangle \subseteq \langle a \circ b \circ c \rangle \subseteq A$, $\langle a \rangle \nsubseteq A$, and $\langle b \rangle \nsubseteq A$, we have $\langle c \rangle \subseteq \Re(A)$. Thus $c^{2n+1} \subseteq A$. So A is primary. □

Proposition 3.32. *Let f be a good homomorphism from a ternary hypersemiring S to a ternary hypersemiring T and I, J be k-hyperideals of S and T respectively. Then*

(i) If I is a primary hyperideal of S such that $\{x \in S_1$: there exist $a, b \in S_1$ such that $x = a + b$ and $f(a) = f(b)\} \subseteq I$ and f is an epimorphism, then $f(I)$ is a primary hyperideal of T.

(ii) If J is a primary hyperideal of T, then $f^{-1}(J)$ is a primary hyperideal of S.

Proof. (i) Let $a \circ b \circ c \subseteq f(I)$, where $a, b, c \in T$ and $a \notin f(I), b \notin f(I)$. As f is an onto homomorphism, $f(a_1) = a, f(b_1) = b, f(c_1) = c$ for some $a_1, b_1, c_1 \in S$, where $a_1 \notin I, b_1 \notin I$. Here $f(a_1 \circ b_1 \circ c_1) = f(a_1)f(b_1)f(c_1) \subseteq f(I)$. So for any $x \in a_1 \circ b_1 \circ c_1$, there exists $i \in I$ such that $f(x) = f(i)$. Thus $x + i \in I$ and hence $x \in I$. Therefore $a_1 \circ b_1 \circ c_1 \subseteq I$ and $a_1 \notin I, b_1 \notin I$, which implies $c_1^{2n+1} \subseteq I$ for some $n \in Z_0^+$. So $c^{2n+1} = f(c_1^{2n+1}) \subseteq f(I)$. Hence $f(I)$ is a primary hyperideal of T.

(ii) Suppose J is a primary hyperideal of T. Let $a \circ b \circ c \subseteq f^{-1}(J)$ for some $a, b, c \in S$ and $a \notin f^{-1}(J), b \notin f^{-1}(J)$. Now $f(a) \circ f(b) \circ f(c) = f(a \circ b \circ c) \subseteq J$ and $f(a) \notin J, f(b) \notin J$. As J is a primary hyperideal of T, $f(c^{2n+1}) = f(c)^{2n+1} \subseteq J$ for some $n \in Z_0^+$. So $c^{2n+1} \subseteq f^{-1}(J)$. Consequently $f^{-1}(J)$ is a primary hyperideal of S. □

Theorem 3.33 (The Prime Avoidance Theorem)**.** *Let I be an arbitrary hyperideal in a ternary hypersemiring $(S, +, \circ)$ and $P_1, P_2, ..., P_n$ be k-hyperideals of S such that at least $n-2$ of which are $\mathcal{C}$-ternary hyperideals as well as completely prime hyperideals. If $I \subseteq P_1 \cup P_2 \cup ... \cup P_n$, then $I \subseteq P_i$, for some i.*

Proof. The proof is by induction on $n \geq 2$. For $n = 2$ suppose $I \subseteq P_1 \cup P_2$. If $I \nsubseteq P_1$, then there exists $x \in I$ such that $x \notin P_1$. Since $I \subseteq P_1 \cup P_2$, so $x \in P_2$. Take $y \in I \cap P_1$. Then $x + y \in I \subseteq P_1 \cup P_2$. If $x + y \in P_1$, then $x \in P_1$(since P_1 is a k-hyperideal),which is a contradiction. Thus $x + y \in P_2$, which implies $y \in P_2$. So $I \cap P_1 \subseteq P_2$. Now $I = (I \cap P_1) \cup (I \cap P_2) \subseteq P_2$. So either $I \subseteq P_1$ or $I \subseteq P_2$.

Assume the result is true for $n - 1$, $n \geq 3$. Let $I \subseteq P_1 \cup P_2 \cup ... \cup P_n$, where at least $n - 2$ of the P_i are completely prime. Suppose that

$I \nsubseteq P_1 \cup P_2 \cup ..P_{i-1} \cup P_{i+1}.. \cup P_n$ for all i. Then there exists $x_i \in I$ such that $x_i \notin P_j$ for all $i \neq j$. So we must have $x_i \in P_i$. Since $n \geq 3$, at least one of the P_i is completely prime hyperideal. Without loss of generality, let us assume that P_1 is a completely prime hyperideal. Consider the set $X = \{x_1\} + x_2^{n+1} \circ x_3 \circ ... \circ x_n \subseteq I \subseteq P_1 \cup P_2 \cup ... \cup P_n$. Here $x_2^{n+1} \circ x_3 \circ ... \circ x_n \subseteq P_i$, where $i \neq 1$(since P_i is a hyperideal and $x_i \in P_i$). Now for each $y \in x_2^{n+1} \circ x_3 \circ ... \circ x_n$, $x_1 + y \in P_i$ for some i. If for $i \geq 2$, $x_1 + y \in P_i$, then $x_1 \in P_i$, which is a contradiction. Thus $x_1 + y \in P_1$ and hence $y \in P_1$. So $(x_2^{n+1} \circ x_3 \circ ... \circ x_n) \cap P_1 \neq \phi$, which implies $(x_2^{n+1} \circ x_3 \circ ... \circ x_n) \subseteq P_1$. Hence $x_k \in P_1$ for some $k = 2, 3, .., n$, which is also a contradiction. Therefore $I \subseteq P_1 \cup P_2 \cup ..P_{i-1} \cup P_{i+1}.. \cup P_n$ for some i. By induction assumption, $I \subseteq P_i$ for some i. □

Definition 3.34. Let $I, I_1, I_2, ..., I_n$ be hyperideals of a ternary hypersemiring S. The collection $\{I_1, I_2, ..., I_n\}$ is said to be a cover of I if $I \subseteq I_1 \cup I_2 \cup ... \cup I_n$. We call such a cover of I efficient, if I is not contained in the union of any $n-1$ of the hyperideals $I_1, I_2, ..., I_n$.

Proposition 3.35. *Let $(S, +, \circ)$ be a commutative ternary hypersemiring and let $\{Q_1, Q_2, ..., Q_n\}$ be an efficient covering of the hyperideal I, where $Q_1, Q_2, ..., Q_n$ are k-hyperideals of S. If $Rad(Q_i) \nsubseteq Rad(Q_j)$ for each $i \neq j$, then no Q_k is a primary hyperideal of S.*

Proof. We first prove that for efficient covering $\{Q_1, Q_2, ..., Q_n\}$ of I, $\cap_{i\neq k} Q_i = \cap_{i=1}^n Q_i$ for all k. Let $x \in \cap_{i\neq k} Q_i$. Since the cover is efficient, there exists $x_k \in Q_k \cap I$ such that $x_k \notin \cup_{i\neq k} Q_i$. Now consider the element $x + x_k$ in I. If $x + x_k \in Q_i$ for $i \neq k$, then $x_k \in Q_i$ for all $i \neq k$, which is a contradiction. Thus $x + x_k \in Q_k$ and hence $x \in Q_k$. So $\cap_{i\neq k} Q_i = \cap_{i=1}^n Q_i$. Now if possible, let Q_k be a primary hyperideal of S. Here $I \circ I \circ Q_1^{n+1} \circ Q_2 \circ .. \circ Q_{k-1} \circ Q_{k+1} \circ .. \circ Q_n \subseteq Q_i$ for all $i \neq k$. Since $I \cap (\cap_{i=1}^n Q_i) = I \cap (\cap_{i\neq k} Q_i) \subseteq I \cap Q_k \subseteq Q_k$, we get $I \circ I \circ Q_1^{n+1} \circ Q_2 \circ .. \circ Q_{k-1} \circ Q_{k+1} \circ .. \circ Q_n \subseteq Q_k$. As $I \nsubseteq Q_k$, by Proposition 3.31, $Q_i \subseteq \Re(Q_k)$. Therefore by Proposition 3.22, $Rad(Q_i) \subseteq Rad(\Re(Q_k)) = Rad(Q_k)$, which contradicts the hypothesis. Hence the result. □

Using Proposition 3.35, we obtain the following Theorem.

Theorem 3.36 (The Primary Avoidance Theorem)**.** *Let I be an arbitrary hyperideal in a commutative ternary hypersemiring $(S, +, \circ)$ and $Q_1, Q_2, ..., Q_n$ be k-hyperideals of S such that at least $n-2$ of which are* *primary hyperideals. If $I \subseteq Q_1 \cup Q_2 \cup ... \cup Q_n$ and $Rad(Q_i) \nsubseteq Rad(Q_j)$ for each $i \neq j$, then $I \subseteq Q_i$ for some i.*

Proof. Without loss of generality, assume that the cover is efficient. By Proposition 3.35, $n \leqslant 2$. For $n = 2$, $I \subseteq Q_1 \cup Q_2$ implies either $I \subseteq Q_1$ or $I \subseteq Q_2$, which contradicts the fact that the cover is efficient. Hence $n = 1$. □

In the next Theorem, we extend the Primary Avoidance Theorem for class of complete ternary hyperideals in a ternary hypersemiring S.

Theorem 3.37 (Extended Version of Primary Avoidance Theorem)**.** *Let S be a commutative ternary hypersemiring and P_1, P_2,....,P_n be $\mathcal{C}$-ternary primary k-hyperideals of S, such that $Rad(P_i) \nsubseteq Rad(P_j)$ for all $i \neq j$. Let T be a hyperideal of S such that $aSS + T \nsubseteq \cup_{i=1}^{n} P_i$, for some $a \in S$. Then there exists a subset T_1 of T such that $a + T_1 \nsubseteq \cup_{i=1}^{n} P_i$.*

Proof. Assume that a lies in all of P_1, P_2,....,P_k but none of P_{k+1},....,P_n. If $k = 0$, then $a + 0 \notin \cup_{i=1}^{n} P_i$. So consider $k \geq 1$. Now $T \nsubseteq \cup_{i=1}^{k} P_i$. If $T \subseteq \cup_{i=1}^{k} P_i$, by Theorem 3.36, $T \subseteq P_i$ for some $1 \leq i \leq k$. Thus $aSS + T \subseteq P_i \subseteq \cup_{i=1}^{n} P_i$, which is a contradiction. So there exists an element $p \in T$ such that $p \notin \cup_{i=1}^{k} P_i$. Also $P_{k+1} \cap .... \cap P_n \nsubseteq P_1 \cup P_2 \cup ... \cup P_k$. If $P_{k+1} \cap .... \cap P_n \subseteq P_1 \cup P_2 \cup ... \cup P_k$, then by Theorem 3.36, $P_{k+1} \cap .... \cap P_n \subseteq P_j$ for some $1 \leqslant j \leqslant k$. Thus $Rad(P_{k+1}) \cap ... \cap Rad(P_n) = Rad(P_{k+1} \cap .... \cap P_n) \subseteq Rad(P_j)$ by Proposition 3.21. Since $(Rad(P_{k+1}))^{n-k} Rad(P_{k+2}) ... Rad(P_n) \subseteq Rad(P_{k+1} \cap .... \cap P_n) \subseteq Rad(P_j)$ and $Rad(P_j)$ is prime hyperideal, by Theorem 3.26, $Rad(P_l) \subseteq Rad(P_j)$ for $k+1 \leqslant l \leqslant n$, which contradicts the hypothesis. Thus there exists $c \in P_{k+1} \cap .... \cap P_n$ such that $c \notin P_1 \cup P_2 \cup ... \cup P_k$. Now $p \circ c \circ c \subseteq T$ and $p \circ c \circ c \subseteq P_{k+1} \cap .... \cap P_n$ but $p \circ c \circ c \nsubseteq P_1 \cup P_2 \cup ... \cup P_n$. If $p \circ c \circ c \subseteq P_1 \cup P_2 \cup ... \cup P_k$, then $p \circ c \circ c \subseteq P_i$ for some $1 \leqslant i \leqslant k$. This implies either $p \in Rad(P_i)$ or $c \in P_i$, which is also a contradiction. Consider $T_1 = p \circ c \circ c$, then $a + T_1 \nsubseteq \cup_{i=1}^{n} P_i$. Since each P_i is a $\mathcal{C}$-ternary primary k-hyperideal of S and $a \in \cup_{i=1}^{k} P_i - \cup_{j=k+1}^{n} P_j$, we have $T_1 \subseteq \cup_{j=k+1}^{n} P_j - \cup_{i=1}^{k} P_i$. □

4. Conclusion

In this paper, radical of hyperideals and primary hyperideals of a ternary hypersemiring have been introduced and studied. The prime and primary avoidance theorems for $\mathcal{C}$-hyperideals in ternary hypersemiring, have been generalized. There is a huge scope of further study on ternary hypersemirings, in terms of prime and primary hyperideals. Moreover, the results obtained in this article, can be extended to some other algebraic systems like gamma-semirings, partially ordered ternary semirings etc. and also to fuzzy and intuitionistic fuzzy settings.

Acknowledgments

The authors are thankful to the referees for their valuable suggestions towards the improvement of the paper.

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