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ON CERTAIN Γ-HYPERIDEALS IN Γ-SEMIHYPERGROUPS

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Abstract. In this paper pseudo symmetric Γ-hyperideal in Γ-semihypergroups is introduced and characterized. It is proved that the class of pseudo symmetric Γ-semihypergroups contains left (right) duo Γ-semihypergroups, quasi commutative Γ-semihypergroups with unity, left (right) pseudo commutative Γ-semihypergroups and idempotent Γ-semihypergroups. The notions of completely prime Γ-hyperideal, semiprime Γ-hyperideal, partially semiprime Γ- hyperideal are also defined and completely prime Γ-hyperideal of a Γ-semihypergroup has been characterized in terms of prime Γ- hyperideal and pseudo symmetric Γ-hyperideal. The characterization of completely semiprime Γ-hyperideals of Γ-semihypergroups is presented. In Γ-semihypergroup n-semiprime Γ-hyperideal and n-partially semiprime Γ-hyperideal are introduced as a generalization of semiprime Γ-hyperideal and partially semiprime Γ-hyperideal of Γ-semihypergroup respectively. The notion of semi-extension of Γ-hyperideal in Γ-semihypergroup has also been defined. Some related results are proved along with establishing the relationship between n-semiprime Γ-hyperideals and semi-extension of Γ-hyperideal in commutative Γ-semihypergroups.

Key Words: Γ-semihypergroup, Γ-hyperideal, Pseudo Symmetric Γ-hyperideal, Completely Prime Γ-hyperideal, n-semiprime Γ-htperideal
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1. Introduction

The notion of Γ-semigroup was first defined by Sen and Saha [12] as a generalization of semigroup and ternary semigroup, and thereafter many mathematicians across the globe started studying Γ-semigroups, extended and generalized many concepts and notions of semigroup theory to Γ-semigroups.

Marty, a French mathematician originated the study of hyperstructure theory [7] when he defined hypergroups based on the notion of hyperoperation, at the 8th Congress of Scandinavian Mathematicians in the year 1934. Since then a number of different hyperstructures are being studied. In a nonempty set equipped with binary operation, the composition of two elements is an element, while in an algebraic hyperstructure, the composition of two elements yields a nonempty set. Corsini contributed to the theory of hypergroups by writing a book [10] and soon extended his contribution to the theory with the publication of another book [11] in 2003 which gives the applications of hyperstructures in various subjects like cryptography, coding theory, automata, probability, lattice theory, graph theory and rough sets to name a few. In 2007 Davvaz and Leoreanu-Fotea wrote a book [3] which was dedicated to the study of hyperring theory. In this book they studied and analyzed several kinds of hyperrings and presented an outline of applications of hyperrings in Chemistry and Physics. Recently (in 2016) Davvaz has authored another book [2] on semihypergroups which is specially designed for beginners, nonspecialists and graduates to have some basic understanding and knowledge of semihypergroups. Recently Pawar et. al [9] have started studying regular Γ-semihyperring.

The study of Γ-semihypergroup was initiated by Davvaz et al. [13, 4, 5] as a generalization of three algebraic structures viz. semigroup, semihypergroup and Γ-semigroup. They have given many examples and characterized Γ-semihypergroups considering several notions. Anjaneyulu [1] in 1980, introduced pseudo symmetric ideals in semigroups which is further extended to Γ-semigroup [6].

In this paper concepts of Γ-semigroup have been extended to Γ-semihypergroup. The notions of pseudo symmetric Γ-hyperideal of Γ-semihypergroup is introduced and its characterization is presented in terms of left (right) extension of Γ-hyperideal. It is also proved that the class of Γ-semihypergroups contains left (right) duo Γ-semihypergroups, quasi commutative Γ-semihypergroups with unity, left (right) pseudo commutative Γ-semihypergroups and idempotent Γ- semihypergroups.

Completely prime Γ- hyperideal, semiprime Γ- hyperideal, partially semiprime Γ-hyperideal and completely semiprime Γ-hyperideal in Γ- semihypergroup have been defined and studied. Characterization of completely prime Γ-hyperideal and completely semiprime Γ-hyperideal of Γ- semihypergroup is proved. In the final section n-semiprime Γ-hyperideal and n-partially semiprime Γ-hyperideal are introduced as a generalization of semiprime Γ-hyperideal and partially semiprime Γ-hyperideal. The concept of semi-extension of Γ-hyperideal is introduced and some results have been proved.

2. **Preliminaries**

Here some preliminaries from [5] are given which are required to study the paper, for more detail and examples reader is requested to refer [5].

Definition 2.1. Let H be a non empty set and $\circ : H \times H \to \wp^*(H)$ be a hyperopertion, where $\wp^*(H)$ is the family of all non-empty subsets of H. The pair $(H, \circ)$ is called a hypergroupoid.
For any two non-empty subsets A and B of H and $x \in H$,

$$A \circ B = \bigcup_{a \in A, b \in B} a \circ b, A \circ \{x\} = A \circ x \text{ and } \{x\} \circ A = x \circ A.$$

Definition 2.2. A hypergroupoid $(H, \circ)$ is called a semihypergroup, if for all $a, b, c \in H$ we have, $(a \circ b) \circ c = a \circ (b \circ c)$. In addition, if for every $a \in H, a \circ H = H = H \circ a$, then $(H, \circ)$ is called a hypergroup.

For more details of hypergroup, semihypergroup see ,[2], [3][10].

Definition 2.3. Let S and Γ be two nonempty sets. Then S is called a Γ-*semihypergroup* if every $\gamma \in \Gamma$ is a hyperoperation on S that is, $x\gamma y \subseteq S$ for every $x, y \in S$, and for every $\alpha, \beta \in \Gamma$ and $x, y, z \in S$ we have the associative property

$$x\alpha(y\beta z) = (x\alpha y)\beta z.$$

Let A and B be two nonempty subsets of S and $\gamma \in \Gamma$, we denote the following:

$$A\gamma B = \bigcup_{a \in A, b \in B} a\gamma b$$

also,

$$A\Gamma B = \bigcup \{a\gamma b \mid a \in A, b \in B \text{ and } \gamma \in \Gamma\}.$$

A Γ-semihypergroup S is said to be commutative if for every $x, y \in S$ and $\gamma \in \Gamma$ we have $x\gamma y = y\gamma x$.
If (S, γ) is a hypergroup for every $\gamma \in \Gamma$ then S is called a Γ-hypergroup.

Definition 2.4. A nonempty subset A of Γ-semihypergroup S is said to be a Γ-subsemihypergroup if $A\Gamma A \subseteq A$ i.e. $a\gamma b \subseteq A$ for every $a, b \in A$ and $\gamma \in \Gamma$.

Definition 2.5. A nonempty subset A of a Γ-semihypergroup S is said to be a left (right) Γ-hyperideal if $S\Gamma A \subseteq A$ ($A\Gamma S \subseteq A$).

A is said to be a two sided Γ-hyperideal or simply a Γ-hyperideal if it is both left and right Γ-hyperideal.

S is called a left(right) simple Γ-semihypergroup if it has no proper left (right) Γ-hyperideal. S is said to be a simple Γ-semihypergroup if it has no proper Γ-hyperideal.

Definition 2.6. Let A be a non-empty subset of a Γ-semihypergroup S. Then intersection of all Γ-hyperideals of S containing A is a Γ-hyperideal of S generated by A, and denoted by $< A >$.

Definition 2.7. A Γ-hyperideal A of a Γ-semihypergroup S is said to be a principal Γ-hyperideal if A is a Γ-hyperideal generated by single element a and is denoted by $J[a] =< a >$.

Also to study more examples on Γ-semihypergroup and the notions of fundamental relations on Γ-semihypergroup, quotient Γ- semihypergroup, right Noetherian Γ-semihypergroups , see [5].

3. Pseudo Symmetric Γ-hyperideal in Γ- semihypergroup

The concept of Γ-hyperideal (breifly “ideal”) of Γ-semihypergroup was introduced very recently by Dariush Heidari et al. [5] where they defined the notions of prime ideal and extension of an ideal in commutative Γ-semihyergroup. In this section, the notion of pseudo symmetric Γ-hyperideal of Γ-semihypergroup is introduced with examples and has been characterized in terms of left(right) extension of Γ-hyperideal. It is also proved that the class of Γ-semihypergroups contains left (right) duo Γ-semihypergroups, quasi commutative Γ-semihypergroups with unity, left (right) pseudo commutative Γ-semihypergroups and idempotent Γ-semihypergroups along with proving some results in this respect.

Definition 3.1. An element a of a Γ-semihypergroup S is said to be a left (right) identity of S if $s \in a\alpha s$ ($s \in s\alpha a$) for all $s \in S$ and $\alpha \in \Gamma$.

An element a of a Γ-semihypergroup is said to be a two sided identity or simply an identity if a is both left and right identity, i.e. $s \in a\alpha s \cap s\alpha a$ for all $s \in S$ and $\alpha \in \Gamma$. We denote identity element of a Γ-semihypergroup as '1'.

Example 3.2. Let $S = \{a, b, c, d\}$ and $\Gamma = \{\alpha, \beta\}$. Define a hyperoperation $\circ$ on S as follows:

$\circ$	a	b	c	d
a	a	{a, b }	{a, c}	{a, d}
b	a	{a, b }	{a, c}	{a, d}
c	a	b	c	d
d	a	b	c	d

Define a mapping $S \times \Gamma \times S \to \wp^*(S)$ by $x\gamma y = x \circ y$ for every $x, y \in S$ and $\gamma \in \Gamma$. Then S is a Γ-semihypergroup. Observe that each element of S is left identity of S but S does not have a right identity.

Example 3.3. Let $S = \{x, y\}$ and $\Gamma = \{\alpha, \beta\}$ defined as follows:

α	x	y
x	{x, y}	{x, y}
y	{x, y}	{x, y}

β	x	y
x	{x}	{y}
y	{y}	{x}

Then S is a Γ-semihypergroup and x is a two sided identity of S.

Definition 3.4. An element a of Γ-semihypergroup S is said to be an α-idempotent if $a \in a\alpha a$. An element a of Γ- semihypergroup S is said to be a Γ-idempotent or simply idempotent if $a \in a\alpha a$ for all $\alpha \in \Gamma$ i.e. $a \in a\Gamma a$.

Definition 3.5. A Γ-semihypergroup S is said to be an idempotent Γ-semihypergroup if every element in S is a Γ-idempotent.

Example 3.6. In example 3.3, x is α-idempotent as well as β-idempotent hence x is Γ-idempotent whereas y is only α-idempotent. And in example 3.2 each element is Γ-idempotent, hence S there is a Γ-semihypergroup.

Example 3.7. Let $S = \{0, 1\}$ and $\Gamma = \{\alpha, \beta, \gamma\}$ be defined as follows:

α	0	1
0	{1}	{0}
1	{0}	{1}

β	0	1
0	{0, 1}	{0, 1}
1	{0, 1}	{0, 1}

γ	0	1
0	{0}	{1}
1	{1}	{0}

Here S has neither a left(right) identity nor a Γ-idempotent element.

Example 3.8. Let $S = \{0, 1\}$ and $\Gamma = \{\alpha, \beta\}$ be defined as follows:

α	0	1
0	{0}	{0, 1}
1	{0,1}	{0,1}

β	0	1
0	{0, 1}	{0, 1}
1	{0, 1}	{0, 1}

S has two two-sided identities and both of them are Γ-idempotent.

Definition 3.9. A Γ-hyperideal I of a Γ-semihypergroup S is said to be a pseudo symmetric Γ-hyperideal if, for nonempty subsets A and B of S, $A\Gamma B \subseteq I$ implies that $A\Gamma S\Gamma B \subseteq I$. A Γ-semihypergroup S is called pseudo symmetric Γ-semihypergroup if every Γ-hyperideal of S is a pseudo symmetric Γ-hyperideal.

Example 3.10. Let $S = \{x, y, z\}$ and $\Gamma = \{\alpha, \beta\}$. Define the hyperoperation ∘ on S as follows:

∘	x	y	z
x	x	x	x
y	x	x	x
z	x	{x,y}	{x,z}

Now define a map $S \times \Gamma \times S \to \wp^*(S)$ as $a\gamma b = a \circ b$ for every $a, b \in S$ and for every $\gamma \in \Gamma$. Then S is a Γ-semihypergroup whose ideals are $\{x\}, \{x, y\}$ and $\{x, y, z\}$, and each of these ideals is pseudo symmetric Γ-hyperideal hence S is a a pseudo symmetric Γ-semihypergroup.

Example 3.11. Consider $S = \{x, y, z\}$, $\Gamma = \{\alpha, \beta\}$ and the hyperoperations are defined as follows:

α	x	y	z
x	x	{x, y}	z
y	{x, y}	{x,y}	z
z	z	z	z

α	x	y	z
x	{x, y}	{x, y}	z
y	{x, y}	y	z
z	z	z	z

Then S is a Γ-semihypergroup and $\{z\}$ is a pseudo symmetric Γ-hyperideal of S.

Example 3.12. Let $S = [0, 1]$ and $\Gamma = \mathbb{N}$ for every $x, y \in S$ and $\gamma \in \Gamma$ define

$$\gamma\colon S \times S \longrightarrow P^*(S)$$

as

$$x\gamma y = \left[0, \frac{xy}{\gamma}\right]$$

then S is a Γ-semihypergroup and $I = [0, t]$, $t \in [0, 1]$ is a pseudo symmetric Γ-hyperideal of S.

Example 3.13. Let $G = (\mathbb{Z}_8, \cdot)$ be a semigroup and $H_1 = \{0\}, H_2 = \{0, 2, 4, 6\}, H_3 = \mathbb{Z}_8$ be subsemigroups of G. Let $\Gamma = \{\gamma_1, \gamma_2, \gamma_3\}$ and define

$$x\gamma_i y = xH_i y, \qquad \forall x, y \in G, \quad 1 \le i \le 3$$

Then $H = \{0, 4\}$ is a pseudo symmetric Γ-hyperideal of G.

Example 3.14. Let $S = M_{2\times 2}(\mathbb{Z})$ and $\Gamma = \{\gamma_1, \gamma_2\}$

$$H_1 = \left\{ \begin{pmatrix} p & 0 \\ 0 & 0 \end{pmatrix} \mid p \text{ is prime} \right\}, H_2 = \begin{pmatrix} 0 & 0 \\ 0 & 2 \end{pmatrix}$$

Define $A\gamma_i B = AH_i B \qquad \forall A, B \in S, \quad 1 \le i \le 2$ then S is a Γ-semihypergroup and $T = \left\{ \begin{pmatrix} x & y \\ z & w \end{pmatrix} \mid x, y, z, w \in 2\mathbb{Z} \right\}$ is a pseudo symmetric Γ-semihypergroup of S.

Proposition 3.15. *Every commutative Γ-semihypergroup is a pseudo symmetric Γ-semihypergroup.*

Converse need not be true. That is a pseudo symmetric Γ- semihypergroup S need not be commutative.

Example 3.16. In example 3.10, S is a pseudo symmetric Γ- semihypergroup but $y\gamma z = y \circ z = \{x\}$ and $z\gamma y = z \circ y = \{x, y\}$. Therefore $y\gamma z \neq z\gamma y$ hence S is not commutative.

Note 3.17. Let S be a Γ-semihypergroup, A and B be nonempty subsets of S and $\alpha \in \Gamma$. Then $A\alpha A\alpha B$ is written as $(A\alpha)^2 B$ and consequently $A\alpha A\alpha \cdots$(n-times)B is denoted by $(A\alpha)^n B$.

Definition 3.18. A Γ-semihypergroup S is said to be quasi commutative if for all $x, y \in S$, there exists a natural number n such that $x\Gamma y = (y\Gamma)^n x$.

An immediate consequence of the above definition is stated in the following result.

Proposition 3.19. *Every commutative Γ-semihypergroup is quasi commutative.*

Lemma 3.20. *If S is quasi commutative Γ-semihypergroup then $x\Gamma S = S\Gamma x$ for all $x \in S$.*

Definition 3.21. [5] Let I be an ideal of commutative Γ-semihypergroup and $\phi \neq A \subseteq S$. Then the extension of I by A is defined as follows:

$$\text{(A:I)} = \{\text{x} \in \ S \ \mid \ A\Gamma x \subseteq I\}$$

Lemma 3.22. [5] *Let S be a commutative Γ-semihypergroup. If I is an ideal of S and $\phi \neq A \subseteq S$ and $\gamma \in \Gamma$ then the following statements are true.*

(1) (A:I) is an ideal of S ;
(2) $A \subseteq (A : I) \subseteq (A\Gamma A : I) \subseteq (A\gamma A : I)$;
(3) $If A \subseteq I$ then $(A : I) = S$.

Lemma 3.23. [5] *Let S be a commutative Γ-semihypergroup. If I is an ideal of S and $\phi \neq A \subseteq S$ and $\gamma \in \Gamma$. Then*

$$(A : I) = \bigcap_{a \in A} (a : I) = (A \setminus I : I)$$

Now following definition of ideal extension is splitted to define left (right) extension of Γ-hyperideal I in a Γ-semihypergroup S which is not necessarily commutative.

Definition 3.24. Let S be a Γ-semihypergroup and I be a Γ-hyperideal of S. If $\phi \neq X \subseteq S$ then left (right) extension of I by X is denoted by $I_l(X)$ *or* $(I : X)$ ($I_r(X)$ *or* $(X : I)$) and is defined as follows:

$$I_l(X) = \{y \in S \mid \{y\}\Gamma X \subseteq I\}$$
$$I_r(X) = \{y \in S \mid X\Gamma\{y\} \subseteq I\}$$

Following is the characterization of pseudo symmetric Γ-hyperideal in Γ- semihypergroup.

Theorem 3.25. *Let I be a Γ-hyperideal of a Γ-semihypergroup S. Then I is pseudo symmetric Γ-hyperideal of S if and only if for every nonempty subset X of $S, I_r(X)$ is a Γ-hyperideal of S.*

Theorem 3.26. *Let I be a Γ-hyperideal of a Γ-semihypergroup S. Then I is pseudo symmetric Γ-hyperideal of S if and only if for every nonempty subset X of $S, I_l(X)$ is a Γ-hyperideal of S.*

Definition 3.27. A Γ-semihypergroup S is said to be a left (right) duo Γ-semihypergroup if every left (right) Γ-hyperideal of S is a two sided Γ-hyperideal. A Γ-semihypergroup S is said to be a duo Γ-semihypergroup if it is both left duo Γ-semihypergroup and right duo Γ-semihypergroup.

Notation 3.28. Let S be a Γ-semihypergroup. If S has an identity, then we denote $S^1 = S$ and if S does not have an identity, let S^1 be the

Γ-semihypergroup S with an identity adjoined (usually denoted by the symbol '1').

Lemma 3.29. *A* Γ*-smihypergroup* S *is a duo* Γ*-semihypergroup if and only if* $x\Gamma S^1 = S^1\Gamma x$ *for all* $x \in S$.

Theorem 3.30. *Every left duo* Γ*-semihypergroup is a pseudo symmetric* Γ*-semihypergroup.*

Proof. Let S be a left duo Γ-semihypergroup, I be a Γ-hyperideal and X be a nonempty subset of S. Consider $I_l(X)$ the left extension of I by X and let $y \in I_l(X)$ then we have $S\Gamma\{y\} \subseteq I_l(X)$. This implies that $S\Gamma I_l(X) \subseteq I_l(X)$. Therefore $I_l(X)$ is a left Γ-hyperideal of S and hence $I_l(X)$ is two sided Γ-hyperideal of S. Therefore by theorem (3.26), I is a pseudo symmetric Γ-hyperideal of S. Hence S is a pseudo symmetric Γ-semihypergroup. □

On the similar lines one can prove following theorem.

Theorem 3.31. *Every right duo* Γ*-semihypergroup is pseudo symmetric* Γ*-semihypergroup.*

Corollary 3.32. *Every duo* Γ*-semihypergroup* S *is pseudo symmetric* Γ*- semihypergroup.*

Proof. Let S be a duo Γ-semihypergroup, that is S is left duo Γ- semihypergroup as well as right duo Γ- semihypergroup. Then by theorem 3.30 and theorem 3.31, S is a pseudo symmetric Γ-semihypergroup. □

Corollary 3.33. *If* S *is quasi commutative* Γ*-semihypergroup with unity then* S *is pseudo symmetric* Γ*-semihypergroup.*

Proof. Let S be a quasi commutative Γ-semihypergroup, by lemma 3.20, $x\Gamma S = S\Gamma x$ for all $x \in S$. Since S is with unity 1, $S^1 = S$. Therefore $x\Gamma S^1 = S^1\Gamma x$ for all $x \in S$ and by lemma 3.29, it follows that S is a duo Γ-semihypergroup. Finally by corollary 3.32, we see that S is a pseudo symmetric Γ-semihypergroup. □

Definition 3.34. A Γ-semihypergroup is said to be left pseudo commutative if for any nonempty subsets X, Y and Z of S, $X\Gamma Y\Gamma Z = Y\Gamma X\Gamma Z$. A Γ-semihypergroup is said to be right pseudo commutative if for any nonempty subsets X, Y and Z of S, $X\Gamma Y\Gamma Z = X\Gamma Z\Gamma Y$.
Obviously if S is a commutative Γ-semihypergroup then it is both left pseudo commutative and right pseudo commutative Γ- semihypergroup. But converse need not be true.

Example 3.35. In example (3.10), S is both left pseudo commutative and right pseudo commutative but S is not commutative for $y\gamma z \neq z\gamma y$.

Theorem 3.36. *Every left (right) pseudo commutative* Γ- *semihypergroup is pseudo symmetric.*

Theorem 3.37. *Every idempotent* Γ*-semihypergroup is pseudo symmetric* Γ*-semihypergroup.*

4. Prime, Completely Prime, Semiprime and Completely Semprime Γ-Hyperideals:

In this section the notions of completely prime Γ-hyperideal, semiprime Γ-hyperideal, partially semiprime Γ-hyperideal and completely semiprime Γ-hyperideal are defined and several relations amongst them are explored. Characterizations of completely prime Γ-hyperideal and completely semiprime Γ-hyperideals of a Γ-semihypergroup are also presented.

Definition 4.1. [5] A proper Γ-hyperideal P of a Γ-semihypergroup S is said to be a prime Γ-hyperideal if for every Γ-hyperideal I, J of S, $I\Gamma J \subseteq P$ implies $I \subseteq P$ or $J \subseteq P$. If Γ-semihypergroup S is commutative, then a proper Γ-hyperideal P is prime if and only if a$\Gamma b \subseteq P$ implies $a \in P$ or $b \in P$, for any $a, b \in S$.

Definition 4.2. A proper Γ-hyperideal P of a Γ-semihypergroup S is said to be a completely prime Γ-hyperideal if for any two nonempty subsets A and B of S, $A\Gamma B \subseteq P$ implies $A \subseteq P$ or $B \subseteq P$.

From above two definitions, it is clear that in a commutative Γ-semihypergroup, every prime Γ-hyperideal is completely prime.

Definition 4.3. A proper Γ-hyperideal I of a Γ-semihypergroup S is said to be a semiprime Γ-hyperideal if for nonempty subsets A and B of S, $A\Gamma S\Gamma B \subseteq I$ implies $A \subseteq I$ or $B \subseteq I$.

Definition 4.4. A proper Γ-hyperideal I of a Γ-semihypergroup S is said to be a partially semiprime Γ-hyperideal if for a nonempty subset A of S, $A\Gamma S\Gamma A \subseteq I$ implies that $A \subseteq I$.

Definition 4.5. A proper Γ-hyperideal I of a Γ-semihypergroup S is said to be a completely semiprime Γ-hyperideal if for a nonempty subset A of S $A\Gamma A \subseteq I$ implies that $A \subseteq I$.

Proposition 4.6. *In a* Γ*-semihypergroup* S *following results hold.*

(1) Every completely prime Γ*-hyperideal is a prime* Γ*-hyperideal.*
(2) Every semiprime Γ*- hyperideal is partially semiprime* Γ*-hyperideal.*
(3) Every completely prime Γ*-hyperideal is a completely semiprime* Γ*-hyperideal.*
(4) Every prime Γ*-hyperideal is partially semiprime.*
(5) Every completely prime Γ*-hyperideal is a pseudo symmetric* Γ*-hyperideal.*

Proposition 4.7. *In a* Γ*-semihypergroup* S *with unity 1*

(1) every completely semiprime Γ*-hyperideal is a partially semiprime* Γ*-hyperideal.*
(2) every completely prime Γ*-hyperideal is a partially semiprime* Γ*-hyperideal.*

Proposition 4.8. *Every completely semiprime* Γ*-hyperideal of a* Γ*-semihypergroup is pseudo symmetric* Γ*-hyperideal.*

Example 4.9. In example (3.10), $\{x\}$ is a pseudo symmetric Γ- hyperideal but $\{x\}$ is not completely semiprime. Because $\{x, y\}\Gamma\{x, y\} \subseteq \{x\}$ but $\{x, y\} \nsubseteq \{x\}$.

Theorem 4.10. [4] *Let* S *be a* Γ*-semihypergroup and* P *be a left ideal of* S*. Then* P *is a prime ideal of* S *if and only if for all* $x, y \in S, x\Gamma S\Gamma y \subseteq P$ *implies that* $x \in P$ *or* $y \in P$*.*

Corollary 4.11. *Let* S *be a* Γ*-semihypergroup and* P *be a* Γ*-hyperideal of* S*. Then* P *is a prime* Γ*-hyperideal of* S *if and only if for all* $x, y \in S, x\Gamma S\Gamma y \subseteq P$ *implies that* $x \in P$ *or* $y \in P$*.*

Theorem 4.12. *In a* Γ*-semihypergroup* S*, a* Γ*-hyperideal of* S *is a completely prime* Γ*-hyperideal if and only if it is a prime* Γ*-hyperideal and pseudo symmetric* Γ*-hyperideal.*

Proof. Proof follows from proposition 4.6$(1), (5)$ and by corollary 4.11. □

Theorem 4.13. *In a* Γ*-semihypergroup* S *with unity 1, a* Γ*-hyperideal* I *is completely semiprime if and only if it is partially semiprime and pseudo symmetric* Γ*-hyperideal.*

Proof. Follow from proposition 4.7 (1) and proposition 4.8. □

5. n-Semiprime Γ-Hyperideals of Γ-Semihypergroups

In the recent past Kostaq Hila et al.[8] studied n-prime Γ-hyperideals in Γ-semihypergroups. In this section the notions of n-semiprime Γ-hyperideal and n-partially semiprime Γ-hyperideal of a Γ-semihypergroup are introduced as a generalization of semiprime Γ-hyperideal and partially semiprime Γ-hyperideal of Γ-semihypergroup and semi-extension of Γ-hyperideal in Γ-semihypergroup is defined. Some results have been proved along with establishing the relationship between n-semiprime Γ-hyperideal and semi-extension of Γ-hyperideal in commutative Γ- semihypergroup.

Let $n \geq 2$ be an integer and $A_1, A_2, ..., A_{n-1}, A_n$ be any n subsets of a Γ-semihypergroup S. Let j be any integer $(2 \leq j \leq n-1)$, we then define

$$\begin{aligned}
\widetilde{A}_{(1;n)} &= A_2\Gamma S\Gamma A_3\Gamma S\Gamma ... A_{n-1}\Gamma S\Gamma A_n \\
\widetilde{A}_{(2;n)} &= A_1\Gamma S\Gamma A_3\Gamma S\Gamma ... A_{n-1}\Gamma S\Gamma A_n \\
\widetilde{A}_{(j;n)} &= A_1\Gamma S\Gamma A_2\Gamma S\Gamma ... A_{j-1}\Gamma S A_{j+1}\Gamma S\Gamma ... A_{n-1}\Gamma S\Gamma A_n \\
\widetilde{A}_{(n;n)} &= A_1\Gamma S\Gamma A_2\Gamma S\Gamma ... A_{n-2}\Gamma S\Gamma A_{n-1}
\end{aligned}$$

Definition 5.1. Let S be a Γ- semihypergroup. A Γ- hyperideal I of S is said to be an n- semiprime Γ- hyperideal if for any n subsets $A_1, A_2, ..., A_{n-1}, A_n$ of S, $A_1\Gamma S\Gamma A_2\Gamma S\Gamma ... A_{n-1}\Gamma S\Gamma A_n \subseteq I$ implies that there exists an integer $j(1 \leq j \leq n)$ such that

$$\widetilde{A}_{(1;n)}, \widetilde{A}_{(2;n)}, ..., \widetilde{A}_{(j-1;n)}, \widetilde{A}_{(j+1;n)}, ..., \widetilde{A}_{(n-1;n)}, \widetilde{A}_{(n;n)} \subseteq I$$

Definition 5.2. A Γ-hyperideal I of a Γ-semihypergroup S is said to be n-partially semiprime if for any n subsets $A_1, A_2, ..., A_{n-1}, A_n$ of S with $A_1 = A_2 = ... = A_{n-1} = A_n = A$, $A_1\Gamma S\Gamma A_2\Gamma S\Gamma ... A_{n-1}\Gamma S\Gamma A_n \subseteq I$ implies that $\widetilde{A}_{(n;n)}$ is a subset of I.

Proposition 5.3. *In a Γ-semihypergroup S following results can be proved.*

1. *Every n-semiprime Γ-hyperideal is an n-partially semiprime Γ-hyperideal.*
2. *A Γ-hyperideal is semiprime if and only if it is 2-semiprime Γ-hyperideal.*
3. *I is partially semiprime Γ-hyperideal of S if and only if I is 2-partially semiprime.*

Proof. (2) Let I be a semiprime Γ-hyperideal of S and for any subsets A_1 and A_2 of S let $A_1\Gamma S\Gamma A_2 \subseteq I$. Then we have $\tilde{A}_{(1;2)} \subseteq I$ or $\tilde{A}_{(2;2)} \subseteq I$, therefore I is 2-prime Γ-hyperideal of S.
Conversely assume that I is a 2-prime Γ-hyperideal of S and for any two subsets A_1 and A_2 of S, let $A_1\Gamma S\Gamma A_2 \subseteq I$, this implies that is $A_1 \subseteq I$ or $A_2 \subseteq I$, and so I is semiprime Γ-hyperideal of S. □

Definition 5.4. Let I be a Γ-hyperideal of a Γ-semihypergroup S and A be any nonempty subset of S. The left (right) semi-extension of I by A is denoted by $[I : A)$ ($(A : I]$) and is defined as follows:

$$\begin{aligned} [I : A) &= \{x \in S \mid x\Gamma S\Gamma A \subseteq I\} \quad \textit{and} \\ (A : I] &= \{x \in S \mid A\Gamma S\Gamma x \subseteq I\} \end{aligned}$$

If $A = \{a\}$ then we write $[I : a)$ and $(a : I]$ respectively for left (and right) semi-extension of I by an element a.
In a commutative Γ-semihypergroup S, left semi- extension of a Γ-hyperideal I is same as its right semi-extension. To see this consider,

$$\begin{aligned} [I : A) &= \{x \in S \mid x\Gamma S\Gamma A \subseteq I\} \\ &= \{x \in S \mid A\Gamma S\Gamma x \subseteq I\} \\ &= (A : I] \end{aligned}$$

Lemma 5.5. *Let I be a Γ-hyperideal of a Γ-semihypergroup S. If A and B are nonempty subset of S and $\gamma \in \Gamma$ then following statements hold in S.*

(1) If $A \subseteq B$ then $(B : I] \subseteq (A : I]$ and $[I : B) \subseteq [I : A)$;
(2) (A:I] and [I:A) are Γ-hyperideals of S;
(3) $I \subseteq (A : I] \subseteq (A\Gamma A : I] \subseteq (A\gamma A : I]$ and $I \subseteq [I : A) \subseteq [I : A\Gamma A) \subseteq [I : A\gamma A)$;
(4) If $A \subseteq I$ then $(A : I] = S$ and $[I : A) = S$.

Lemma 5.6. *Let S be a Γ-semihypergroup and I be a Γ-hyperideal of S. If $\phi \neq A \subseteq S$ then*

$$(A : I] = \bigcap_{a \in A} (a : I] \ \textit{and} \ [A : I) = \bigcap_{a \in A} [I : a)$$

Theorem 5.7. *Let I be the partially semiprime Γ-hyperideal of a Γ-semihypergroup S then $I = (S : I]$ and $I = [I : S)$.*

Theorem 5.8. *In a Γ-semihypergroup S every $(n-1)$-semiprime Γ-hyprideal is an n-semiprime Γ-hyperideal for all integers $n \geq 3$.*

Proof. Let I be an $(n-1)$-semiprime Γ-hyperideal of S and $A_1, A_2, ...$
$.., A_{n-1}, A_n$ be any n subsets of S such that

$$A_1 \Gamma S \Gamma A_2 \Gamma S ... \Gamma A_{n-1} \Gamma S \Gamma A_n \subseteq I.$$

Let $X_1 = A_1 \Gamma S \Gamma A_2$ and $X_i = A_{i+1}$, $(i = 2, 3, ..., n-1)$ then $X_1 \Gamma S \Gamma X_2 \Gamma S ...$ $... \Gamma X_{n-2} \Gamma S \Gamma X_{n-1} \subseteq I$. Since I is $(n-1)$-semiprime Γ-hyperideal , there exists an integer $j, (1 \leq j \leq n-1)$ such that

$$\tilde{X}_{(1;n-1)}, \tilde{X}_{(2;n-1)}, ..., \tilde{X}_{(j-1;n-1)}, \tilde{X}_{(j+1;n-1)}, ..., \tilde{X}_{(n-1;n-1)} \subseteq I.$$

$Case\, 1$: Assume that $\tilde{X}_{(1;n-1)} \nsubseteq I$.

Then we have $\tilde{X}_{(2;n-1)}$, $\tilde{X}_{(3;n-1)}$,..., $\tilde{X}_{(n-1;n-1)} \subseteq I$ and $\tilde{X}_{(i;n-1)} = \tilde{A}_{(i+1;n)}$ for $2 \leq i \leq n-1$, therefore, $\tilde{A}_{(3;n)}, \tilde{A}_{(4;n)}, ..., \tilde{A}_{(n;n)} \subseteq I$. As $\tilde{A}_{(n;n)} \subseteq I$, and $\tilde{A}_{(n;n)} = A_1 \Gamma S \Gamma A_2 \Gamma S ... \Gamma A_{n-2} \Gamma S \Gamma A_{n-1} \subseteq I$, by hypothesis there exists an integer k $(1 \leq k \leq n-1)$ such that

$$\tilde{A}_{(1;n-1)}, \tilde{A}_{(2;n-1)}, ..., \tilde{A}_{(k-1;n-1)}, \tilde{A}_{(k+1;n-1)}, ..., \tilde{A}_{(n-1;n-1)} \subseteq I, \text{ then}$$

$$A_2 \Gamma S \Gamma A_3 \Gamma S ... \Gamma A_{n-2} \Gamma S \Gamma A_{n-1} = \tilde{A}_{(1;n-1)} \subseteq I$$

or

$$A_1 \Gamma S \Gamma A_3 \Gamma S ... \Gamma A_{n-2} \Gamma S \Gamma A_{n-1} = \tilde{A}_{(2;n-1)} \subseteq I$$

Since I is Γ-hyperideal of S, we see that either

$$\tilde{A}_{(1;n)} \subseteq I \text{ or } \tilde{A}_{(2;n)} \subseteq I.$$

Hence $\tilde{A}_{(1;n)}, \tilde{A}_{(3;n)}, ..., \tilde{A}_{(n-1;n)}$ $\tilde{A}_{(n;n)} \subseteq I$ or $\tilde{A}_{(2;n)}, \tilde{A}_{(3;n)}, ..., \tilde{A}_{(n-1;n)}$ $\tilde{A}_{(n;n)} \subseteq I$.

$Case\, 2$: Assume that $\tilde{X}_{(1;n-1)} \subseteq I$.

Then there exists an integer $r (2 \leq r \leq n-1)$ such that

$$\tilde{X}_{(2;n-1)}, \tilde{X}_{(3;n-1)}, ..., \tilde{X}_{(r-1;n-1)}, \tilde{X}_{(r+1;n-1)}, ..., \tilde{X}_{(n-1;n-1)} \subseteq I$$

That is we have,

$$\tilde{A}_{(3;n)}, \tilde{A}_{(4;n)}, ..., \tilde{A}_{(r;n)}, \tilde{A}_{(r+2;n)}, ..., \tilde{A}_{(n;n)} \subseteq I.$$

Since $A_3 \Gamma S \Gamma A_4 \Gamma S ... \Gamma A_{n-1} \Gamma S \Gamma A_n = \tilde{X}_{(1;n-1)} \subseteq I$ and I is Γ-hyperideal, we have

$$\tilde{A}_{(1;n)} = A_2 \Gamma S \Gamma A_3 \Gamma S ... \Gamma A_{n-1} \Gamma S \Gamma A_n \subseteq I$$

and

$$\tilde{A}_{(2;n)} = A_1\Gamma S\Gamma A_3\Gamma S...\Gamma A_{n-1}\Gamma S\Gamma A_n \subseteq I.$$

Therefore $\tilde{A}_{(1;n)}, \tilde{A}_{(2;n)},...,\tilde{A}_{(r;n)}, \tilde{A}_{(r+2;n)},...,\tilde{A}_{(n-1;n)}$ $\tilde{A}_{(n;n)} \subseteq I$, which proves that I is an n-semiprime Γ-hyperideal of S. □

Following corollary says that for all integers $n \geq 2$, n-semiprime Γ-hyperideals are a generalization of semiprime Γ-hyperideals.

Corollary 5.9. *In a Γ-semihypergroup S every semiprime Γ- hyperideal of S is an n-semiprime Γ-hyperideal of S for all integers ≥ 2.*

Proof. Let I be a semiprime Γ-hyperideal of S, by proposition 5.3 (2) I is 2-semiprime Γ-hyperideal of S and by theorem 5.8, I is 3-semiprime Γ-hyperideal of S. Now applying theorem 5.8 repeatedly we see that I is n-semiprime Γ-hyperideal of S for all $n \geq 2$. □

Following is the theorem that establishes the relationship between n-semiprime Γ-hyperideals and semi-extension of Γ-hyperideals in commutative Γ- semihypergroups.

Theorem 5.10. *In a commutative Γ-semihypergroup S a Γ-hyperideal I is n-semiprime if and only if every semi-extension of I is an $(n-1)$-semiprime Γ-hyperideal of S for all integers $n \geq 3$.*

Proof. Let I be an n-semiprime Γ-hyperideal of a commutative Γ- semihypergroup S and X be any subset of S. Also let $A_1, A_2, ..., A_{n-2}, A_{n-1}$ be any $n-1$ subsets of S such that

$$A_1\Gamma S\Gamma A_2\Gamma S...\Gamma A_{n-2}\Gamma S\Gamma A_{n-1} \subseteq (X : I],$$

that is $X\Gamma S\Gamma A_1\Gamma S\Gamma A_2\Gamma S...\Gamma A_{n-2}\Gamma S\Gamma A_{n-1} \subseteq I$. Denote $X_1 = X$ and $X_i = A_{i-1}$ for all $i = 2, 3, ..., n$. Then $X_1\Gamma S\Gamma X_2\Gamma S...\Gamma X_{n-1}\Gamma S\Gamma X_n \subseteq I$, since I is n-semiprime Γ-hyperideal of S, there exists an integer $j(1 \leq j \leq n)$ such that

$$\tilde{X}_{(1;n)}, \tilde{X}_{(2;n)},...,\tilde{X}_{(j-1;n)}, \tilde{X}_{(j+1;n)},...,\tilde{X}_{(n-1;n)}, \tilde{X}_{(n;n)} \subseteq I.$$

It follows that there exists an integer $k = j-1(1 \leq k \leq n-1)$ such that $X\Gamma S\Gamma\tilde{A}_{(1;n-1)},...,X\Gamma S\Gamma\tilde{A}_{(k-1;n-1)}, X\Gamma S\Gamma\tilde{A}_{(k+1;n-1)},...,X\Gamma S\Gamma\tilde{A}_{(n-1;n-1)}$ are all subsets of I. Hence $\tilde{A}_{(1;n-1)}, \tilde{A}_{(2;n-1)},...,\tilde{A}_{(k-1;n-1)}, \tilde{A}_{(k+1;n-1)},....$ $...\tilde{A}_{(n-1;n-1)} \subseteq (X : I]$. Therefore $(X : I]$ is an $(n-1)$-semiprime Γ-hyperideal of S.

Conversely assume that every semi-extension of I is an $(n-1)$- semiprime Γ-hyperideal of S and let $A_1, A_2, ..., A_{n-1}, A_n$ be any n subsets of S such that $A_1\Gamma S\Gamma A_2\Gamma S...\Gamma A_{n-1}\Gamma S\Gamma A_n \subseteq I$. Then we have $A_1\Gamma S\Gamma A_2\Gamma S.....\Gamma A_{n-1} \subseteq [I : A_n)$, that is $A_1\Gamma S\Gamma A_2\Gamma S...\Gamma A_{n-1} \subseteq (A_n : I]$ because S is commutative. Since $(A_n : I]$ is $(n-1)$-semiprime Γ-hyperideal of S, it follows that there exists an integer $j(1 \leq j \leq n-1)$ such that

$$\tilde{A}_{(1;n-1)}, \tilde{A}_{(2;n-1)},...,\tilde{A}_{(j-1;n-1)},\ \tilde{A}_{(j+1;n-1)},...,\tilde{A}_{(n-1;n-1)} \subseteq (A_n : I].$$

Now suppose that $\tilde{A}_{(j;n-1)} \nsubseteq (A_n : I]$. This yields

$$\tilde{A}_{(1;n)}, \tilde{A}_{(2;n)},...,\tilde{A}_{(j-1;n)},\ \tilde{A}_{(j+1;n)},...,\tilde{A}_{(n-1;n)} \subseteq I.$$

Observe that either $\tilde{A}_{(j;n)} \subseteq I$ or $\tilde{A}_{(n;n)} \subseteq I$.
Because for every integer $k(1 \leq k \leq n)$ and $k \neq j$, we have

$$A_1\Gamma S\Gamma A_2\Gamma S...\Gamma A_{k-1}\Gamma S\Gamma A_{k+1}\Gamma S...\Gamma A_{n-1}\Gamma S\Gamma A_n \subseteq (A_k : I]$$

For all $l = 1, 2, ..., k-1$ denote $X_l = A_l$ and for all $l = k, k+1, ..., n-1$ denote $X_l = A_{l+1}$. We obtain

$$X_1\Gamma S\Gamma X_2\Gamma S...\Gamma X_{n-2}\Gamma S\Gamma X_{n-1} \subseteq (A_k : I].$$

Hence there exists an integer $r(1 \leq r \leq n-1)$ such that

$$\tilde{X}_{(1;n-1)}, \tilde{X}_{(2;n-1)},...,\tilde{X}_{(r-1;n-1)},\ \tilde{X}_{(r+1;n-1)},...,\tilde{X}_{(n-1;n-1)} \subseteq (A_k : I].$$

This implies that there exists an integer $i, (1 \leq i \leq n)$ and $i \neq k$ we can assume that $(i < k)$ such that

$$\tilde{A}_{(1;n)}, \tilde{A}_{(2;n)},...,\tilde{A}_{(i-1;n)},\ \tilde{A}_{(i+1;n)},...,\ \tilde{A}_{(k-1;n)},\\ \tilde{A}_{(k+1;n)},...,\tilde{A}_{(n-1;n)}, \tilde{A}_{(n;n)} \subseteq I.$$

Since $k \neq j$ and $i \neq k$ implies that $i \neq j$ (but $k = n$ is possible), we have $\tilde{A}_{(j;n)} \subseteq I$ or $\tilde{A}_{(n;n)} \subseteq I$. Hence

$$\tilde{A}_{(1;n)}, \tilde{A}_{(2;n)},....,\tilde{A}_{(n-2;n)}, \tilde{A}_{(n-1;n)} \subseteq I$$

or

$$\tilde{A}_{(1;n)}, \tilde{A}_{(2;n)},..,\tilde{A}_{(j-1;n)},\ \tilde{A}_{(j+1;n)},...,\tilde{A}_{(n-1;n)}, \tilde{A}_{(n;n)} \subseteq I.$$

This shows that I is an n-semiprime Γ-hyperideal of S. □

Theorem 5.11. *Let I be a Γ-hyperideal of a commutative Γ- semihypergroup S. Then n-semiprime Γ-hyperideals and $(n-1)$-semiprime Γ-hyperideals of S coincide for all integers $n \geq 3$ provided $x \in S\Gamma x$ for all $x \in S$.*

Theorem 5.12. *In a commutative Γ-semihypergroup S if I is a partially semiprime Γ-hyperideal as well as an n-semiprime Γ-hyperideal of S and if*

$\mathscr{A} = \{J : J$ *is an* $(n-1)$*-semiprime* Γ*-hyperideal of* S *and* $I \subseteq J\}$

then

$$I = \bigcap_{J \in \mathscr{A}} J.$$

Proof. As $I \subseteq J$, for all $J \in \mathscr{A}$ implies that $I \subseteq \bigcap_{J \in \mathscr{A}} J$. By lemma 5.6 we have $(S : I] = \bigcap_{x \in S} (x : I]$ and by theorem 5.7 $I = (S : I]$, we get $I = \bigcap_{x \in S} (x : I]$. By lemma 5.5 (3) $I \subseteq (x : I]$ for all $x \in S$ and I is an n-semiprime Γ-hyperideal of S, by theorem 5.10 $(x : I]$ is an $(n-1)$-semiprime Γ-hyperideal of S for all $x \in S$. Thus we get $(x : I] \in \mathscr{A}$ for all $x \in S$. Hence $\bigcap_{J \in \mathscr{A}} J \subseteq \bigcap_{x \in S} (x : I] = I$. Therefore $I = \bigcap_{J \in \mathscr{A}} J$. □

6. Conclusion

In this paper pseudo symmetric Γ-hyperideals,completely prime Γ-hyperideals, completely semiprime Γ-hyperideals are introduced and investigated. The notions of n-semiprime Γ-hyperideals and n-partially semiprime Γ-hyperideals are described as an extension of semiprime Γ-hyperideals and partially semiprime Γ-hyperideals in Γ-semihypergroups together with proving some results in this respect.

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