Journal of Hyperstructures 9 *(3rd CSC2019)* (*2020*), *54-61.*
ISSN: 2322-1666 print/2251-8436 online

THE EXPONENTIATED DISCRETE INVERSE RAYLEIGH DISTRIBUTION

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Abstract. In this paper, a new distribution called the exponentiated discrete inverse Rayleigh distribution is introduced, which is an extension of the discrete inverse Rayleigh distribution. This new discrete distribution is a discrete analogue of the continuous exponentiated inverse Rayleigh distribution. In this paper, we discuss the shapes of probability mass and hazard rate functions, the moments of the new distribution and data generation. The maximum likelihood estimation of the parameters is also studied. Finally, an example is given to demonstrate an application of the new distribution.

Key Words: Maximum likelihood estimation, Hazard rate function, discrete inverse Rayleigh distribution, Moment.
2010 Mathematics Subject Classification: Primary: 60E05; Secondary: 62F15.

1. Introduction

Discrete distributions are often applied to modeling count data. Data of a lifetime phenomenon can be collected in terms of number of years, number of days, number of hours and similar time periods which possess a discrete nature. Sometimes the lifetime of a component can be measured in terms of a discrete random variable that is not a time period, for example, the lifetime of a tyre can be expressed in terms of the number of kilometers that it can last or the lifetime of a printer cartridge can be stated in terms of the number of paper pages that can be printed

Received: 19 November 2019, Accepted 01 August 2020. Communicated by Behrouz Fathi-Vajargah;
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apparently by that printer cartridge. Therefore, discrete distributions are of considerable importance in reliability problems. As science made progress and the variety of problems increased, the need was felt increasingly for new distributions with more flexibility. Therefore, scientists, especially statisticians, intended to extend classical distributions to new distributions with higher flexibility. In this paper, we wish to generalize the discrete inverse Rayleigh distribution and introduce a new distribution called the *exponentiated discrete inverse Rayleigh distribution*.

In what follows, we introduce the new distribution and discuss some of its distributional properties in Section 2. The maximum likelihood (ML) estimation of the parameters is explored in Section 3. In Section 4, an example is provided. Finally, Section 5 is devoted to some concluding remarks.

2. The new distribution and some of its distributional properties

The discrete inverse Rayleigh distribution was discussed in detail by Hussain and Ahmad [3]. This distribution is a special case of the discrete inverse Weibull distribution that was introduced by [1]. Let Y possess the discrete inverse Rayleigh distribution, then the cumulative distribution function (cdf) of Y is given by

$$F_Y(t) = Pr(Y \leq t) = q^{([t]+1)^{-2}}, \qquad t \geq 0.$$

In addition, the corresponding probability mass function (pmf) of Y is given by

$$p_Y(n) = Pr(Y = n) = q^{(n+1)^{-2}} - q^{n^{-2}}, \quad n = 0, 1, 2, ...,$$

where $[t]$ denotes the integer part of t, $0 < q < 1$ and $q^{\infty} = 0$.

Now, we extend the above distribution using the method of exponentiation in order to attain a more flexible distribution called the exponentiated discrete inverse Rayleigh distribution. The cdf and pmf of the exponentiated discrete inverse Rayleigh distribution are given by

$$F(x) = 1 - \left(1 - q^{([x]+1)^{-2}}\right)^{\gamma}, \qquad x \geq 0, \tag{2.1}$$

and

$$p(x) = \left(1 - q^{x^{-2}}\right)^{\gamma} - \left(1 - q^{(x+1)^{-2}}\right)^{\gamma}, \qquad x = 0, 1, 2, ...,$$

respectively, where $0 < q < 1$ and $\gamma > 0$ are the parameters of the model and $q^{\infty} = 0$.

If X possesses cdf (2.1) with parameters q and γ, then we write $X \sim EDIR(q,\gamma)$. It is clear that the discrete inverse Rayleigh distribution is a special case of the exponentiated discrete inverse Rayleigh distribution that is obtained by setting $\gamma = 1$ in (2.1).

In addition, the hazard rate function (hrf) of the new distribution is

$$r(x) = \frac{p(x)}{P(X \geq x)} = \begin{cases} 1-(1-q)^{\gamma} & x=0, \\ 1-\left(\dfrac{1-q^{(x+1)^{-2}}}{1-q^{x^{-2}}}\right)^{\gamma} & x=1,2,3,\cdots. \end{cases}$$

Figure 1 includes the plots of pmfs and hrfs of the exponentiated discrete inverse Rayleigh distribution for selected combinations of the parameters. From Figure 1, it can be seen that the pmf can be decreasing or increasing-decreasing. Moreover, the hrf can be also decreasing or increasing-decreasing.

2.1. **Moments.** The k-th moment of $X \sim EDIR(q,\gamma)$, provided that it converges, is given by

$$E(X^k) = \sum_{x=0}^{\infty} x^k \left\{ \left(1-q^{x^{-2}}\right)^{\gamma} - \left(1-q^{(x+1)^{-2}}\right)^{\gamma} \right\}.$$

Consequently, the mean and variance of the new distribution, provided that they converge, are given by

$$E(X) = \sum_{x=0}^{\infty} x \left\{ \left(1-q^{x^{-2}}\right)^{\gamma} - \left(1-q^{(x+1)^{-2}}\right)^{\gamma} \right\},$$

and

$$Var(X) = \sum_{x=0}^{\infty} x^2 \left\{ \left(1-q^{x^{-2}}\right)^{\gamma} - \left(1-q^{(x+1)^{-2}}\right)^{\gamma} \right\} - (E(X))^2.$$

respectively.

2.2. **Data generation from the new distribution.** Here, we provide a way to generate data from the new distribution. Let W be a random variable that follows a continuous exponentiated inverse Rayleigh distribution with the following cdf

$$F_W(w) = 1 - \left(1-\mathrm{e}^{-\alpha w^{-2}}\right)^{\gamma}, \qquad w>0, \alpha>0, \gamma>0,$$

pmf-hrEDIR-eps-converted-to.pdf

FIGURE 1. The pmfs (left panel) and hrfs (right panel) of the exponentiated discrete inverse Rayleigh distribution for selected combinations of the parameters.

then upon setting $q = \mathrm{e}^{-\alpha}$, we have for non-negative integer values of x

$$\begin{aligned} Pr([W] = x) &= p(x \leq W < x + 1) = F_W(x + 1) - F_W(x) \\ &= \left(1 - q^{x^{-2}}\right)^{\gamma} - \left(1 - q^{(x+1)^{-2}}\right)^{\gamma}, \end{aligned}$$

which is the pmf of the exponentiated discrete inverse Rayleigh distribution with parameters q and γ.

Thus we conclude that if the random variable W possesses a continuous exponentiated inverse Rayleigh distribution with parameters α and γ, then $X = [W]$ follows the exponentiated discrete inverse Rayleigh distribution with parameters $q = \mathrm{e}^{-\alpha}$ and γ. In other words, the exponentiated discrete inverse Rayleigh distribution with parameters q and γ is the discrete analogue of the continuous exponentiated inverse Rayleigh distribution with parameters $\alpha = -\ln(q)$ and γ. Therefore, for generating data from the new distribution, it is sufficient, first, to generate data from the continuous exponentiated inverse Rayleigh distribution with parameters $\alpha = -\ln(q)$ and γ. Afterwards we obtain the integer values of the generated data.

3. ML estimation of the parameters

The ML method is one of the common methods of parameter estimation that is carried out by maximizing the likelihood function of the parameters. Let $x_1, \cdots, x_n$ be an observed random sample from the exponentiated discrete inverse Rayleigh distribution with parameters q and γ, then the likelihood function of the parameters, given $x_1, \cdots, x_n$, is given by

$$\mathcal{L}(q,\gamma) = \prod_{i=1}^{n} \left[\left(1 - q^{x_i^{-2}}\right)^{\gamma} - \left(1 - q^{(x_i+1)^{-2}}\right)^{\gamma} \right].$$

Thus, the log-likelihood function is

$$\ell(q,\gamma) = \sum_{i=1}^{n} \ln \left[\left(1 - q^{x_i^{-2}}\right)^{\gamma} - \left(1 - q^{(x_i+1)^{-2}}\right)^{\gamma} \right].$$

Upon differentiating the log-likelihood function with respect to the parameters and equating them with zero, we have

$$\begin{aligned}
\frac{\partial \ell(q,\gamma)}{\partial q} &= -\sum_{i=1}^{n} \frac{\gamma\left(1-q^{x_i^{-2}}\right)^{\gamma-1} x_i^{-2} q^{x_i^{-2}-1}}{\left(1-q^{x_i^{-2}}\right)^{\gamma}-\left(1-q^{(x_i+1)^{-2}}\right)^{\gamma}} \\
&\quad + \sum_{i=1}^{n} \frac{\gamma\left(1-q^{(x_i+1)^{-2}}\right)^{\gamma-1} (x_i+1)^{-2} q^{(x_i+1)^{-2}-1}}{\left(1-q^{x_i^{-2}}\right)^{\gamma}-\left(1-q^{(x_i+1)^{-2}}\right)^{\gamma}} = 0, \\
\frac{\partial \ell(q,\gamma)}{\partial \gamma} &= \sum_{i=1}^{n} \frac{\left(1-q^{x_i^{-2}}\right)^{\gamma} \ln\left(1-q^{x_i^{-2}}\right)}{\left(1-q^{x_i^{-2}}\right)^{\gamma}-\left(1-q^{(x_i+1)^{-2}}\right)^{\gamma}} \\
&\quad - \sum_{i=1}^{n} \frac{\left(1-q^{(x_i+1)^{-2}}\right)^{\gamma} \ln\left(1-q^{(x_i+1)^{-2}}\right)}{\left(1-q^{x_i^{-2}}\right)^{\gamma}-\left(1-q^{(x_i+1)^{-2}}\right)^{\gamma}} = 0.
\end{aligned}$$

In this paper, the ML estimates of the parameters are obtained with the help of the function *optim* in R [4].

4. Data application

In this section, we apply a data set taken from Student [5] which was also analyzed by [3]. The data are given in Table 1.

Table 1. The data set taken from Student [5].

Number of cells	0	1	2	3	4	5
Frequency	213	128	37	18	3	1

We fitted the discrete inverse Rayleigh distribution and the exponentiated discrete inverse Rayleigh distribution to the data of Table 1. The ML estimates, as well as the values of the Akaike information criterion (AIC) for both distributions are calculated and presented in Table 2. From Table 2 we see that the exponentiated discrete inverse Rayleigh distribution has a better fit than the discrete inverse Rayleigh distribution in the sense of AIC.

Table 2. The ML estimates of the parameters, as well as the calculated values of the AIC ($\widehat{q}$ and $\widehat{\gamma}$ denote the ML estimates of q and γ, respectively).

Distribution	$\widehat{q}$	$\widehat{\gamma}$	AIC
Discrete inverse Rayleigh	0.5335	—	912.3641
Exponentiated discrete inverse Rayleigh	0.46477	1.19712	910.7537

5. Concluding remarks

In this paper, we introduced a new distribution entitled "the exponentiated discrete inverse Rayleigh distribution" which is an extension of the discrete inverse Rayleigh distribution and discussed some of its distributional properties. Then, we compared the fit of the new distribution with that of the discrete inverse Rayleigh distribution with the help of a data set taken from Student [5]. The data set was previously analyzed by [3] as well. We know that AIC is increasing with respect to the number of parameters [2], however, though the number of the parameters of the new distribution is more than that of the discrete inverse Rayleigh distribution, the calculated value of the AIC for the exponentiated discrete inverse Rayleigh distribution is less than that of the discrete inverse Rayleigh distribution, that shows that the exponentiated discrete inverse Rayleigh distribution possesses a better fit than the discrete inverse Rayleigh distribution.

Acknowledgments

The authors wish to thank Dr. Zahra Danesh Kaftroudi for her help.

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