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ON SYSTEMS, MAXIMAL Γ-HYPERIDEALS AND COMPLETE PRIME Γ-RADICALS IN Γ-SEMIHYPERGROUPS

KISHOR F. PAWAR AND SAFOORA J. ANSARI

Abstract. In this paper *c-system* , *n-system* and complete prime Γ-radical in a Γ-semihypergroup are introduced and studied while definition of *m-system* has been revised. Maximal Γ-hyperideals are studied and the conditions under which maximal Γ-hyperideals are prime and vice versa are investigated. It is proved that intersection of distinct maximal Γ-hyperideals in Γ-semihypergroup is non-empty. Relations between completely prime Γ-hyperideals, *c-system* and complete prime Γ-radicals are examined.

Key Words: Systems, Γ-semihypergroup, Γ-hyperideal, Γ-radical
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1. Introduction

Study of hyperstructures was initiated by a French mathematician Marty [21] in the year 1934 when he defined hypergroups based on the notion of hyperoperation at the 8^{th} Congress of Scandinavian Mathematicians. Since then a number of different algebraic hyperstructures are being studied. In a nonempty set equipped with binary operation, the composition of two elements is an element while in an algebraic hyperstructure the composition of two elements yields a nonempty set. There are many researchers in several countries who study hyperstructures and extend their contributions through research articles and books.

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∗Address correspondence to Safoora J. Ansari; E-mail:ansari.sjacs@snjb.org
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Corsini wrote many articles and books on different algebraic hyperstructures [6, 7, 8, 9]. He has given applications of hyperstructures in various subjects like cryptography, coding theory, automata, probability, lattice theory, graph theory and rough sets [8]. A book [11] on hyperrings is written by Davvaz and Leoreanu-Fotea in 2007 which gives detailed insight on fundamentals of hyperring theory. In 2016 B. Davvaz has published another book [10] on semihypergroups which is very useful for beginners and graduates to understand basics of semihypergroups. Pawar et. al [24] are studying regular Γ-semihyperring, Vougiouklis studied represntation of hyperstructures [26]. For more information on hyperstructures refer the work of Jafarpour and Cristea [20], Hoskova-Mayerova [15, 16, 17, 19], Novak [22, 23], The notion of Γ-semigroup was first defined by Sen and Saha [25] as a suitable generalization of semigroup and ternary semigroup and thereafter many mathematicians began to study Γ-semigroups, extended and generalized many concepts and notions of semigroups to Γ-semigroups [2].

The study of Γ-semihypergroup was initiated by Davvaz et al. [5, 12, 13] as a generalization of three algebraic structures semigroup, semihypergroup and Γ-semigroup. They have given many examples and studied Γ-semihypergroups considering several notions. In this paper systems, maximal Γ-hyperideals and complete prime Γ-radicals in Γ- semihypergroup are introduced and studied. Study of hyperstructures become more flexible as it is different from usual binary and ternary operations and produce generalized results in addition to those existing in classical algebraic theory. Apart from its applications in other fields Hoskova and A. Maturo have found some interesting applications of hyperstructures in management of teaching and relationship in schools [16, 17, 18]

2. Preliminaries

We begin with recalling some basic definitions and results from [12, 13] require for our purpose. For detailed study reader is requested to refer [5, 15, 16, 17, 19, 22, 23, 26].

Definition 2.1. [19] Let H be a non empty set and $\circ : H \times H \to \wp^*(H)$ be a hyperopertion, where $\wp^*(H)$ is the family of all non-empty subsets of H. The pair $(H, \circ)$ is called a *hypergroupoid*.

For any two non-empty subsets A and B of H and $x \in H$,

$$A \circ B = \bigcup_{a \in A, b \in B} a \circ b, A \circ \{x\} = A \circ x \text{ and } \{x\} \circ A = x \circ A.$$

Definition 2.2. [18]A hypergroupoid $(H, \circ)$ is called a *semihypergroup* if for all $a, b, c \in H$ we have, $(a \circ b) \circ c = a \circ (b \circ c)$. In addition if for every $a \in H, a \circ H = H = H \circ a$, then $(H, \circ)$ is called a *hypergroup*.

For more details of hypergroup, semihypergroup see [10] .

Definition 2.3. [13] Let S and Γ be two nonempty sets. Then S is called a Γ-*semihypergroup* if every $\gamma \in \Gamma$ is a hyperoperation on S that is, $x\gamma y \subseteq S$ for every $x, y \in S$, and for every $\alpha, \beta \in \Gamma$ and $x, y, z \in S$ we have the associative property

$$x\alpha(y\beta z) = (x\alpha y)\beta z.$$

Let A and B be two nonempty subsets of S and $\gamma \in \Gamma$, we denote the following:

$$A\gamma B = \bigcup_{a \in A, b \in B} a\gamma b$$

also,

$$A\Gamma B = \bigcup \{a\gamma b \mid a \in A, b \in B \text{ and } \gamma \in \Gamma\}.$$

A Γ-semihypergroup S is said to be commutative if for every $x, y \in S$ and $\gamma \in \Gamma$ we have $x\gamma y = y\gamma x$.
If (S, γ) is a hypergroup for every $\gamma \in \Gamma$ then S is called a Γ-hypergroup.

Example 2.4. Let $S = [0, 1]$ and $\Gamma = \mathbb{N}$. For every $x, y \in S$ and $\gamma \in \Gamma$ we define $\gamma : S \times S \longrightarrow \wp^*(S)$ by $x\gamma y = \left[0, \frac{xy}{\gamma}\right]$. Then γ is a hyperoperation on S and $x\alpha(y\beta z) = \left[0, \frac{xyz}{\alpha\beta}\right] = (x\alpha y)\beta z$. This means that S is a Γ-semihypergroup.

Definition 2.5. [13] A nonempty subset A of Γ-semihypergroup S is said to be a Γ-*subsemihypergroup* if $A\Gamma A \subseteq A$ i.e. $a\gamma b \subseteq A$ for every $a, b \in A$ and $\gamma \in \Gamma$.

Definition 2.6. [13] A nonempty subset A of a Γ-semihypergroup S is said to be a left (right) Γ-*hyperideal* if $S\Gamma A \subseteq A$ ($A\Gamma S \subseteq A$).

A is said to be a two sided Γ-hyperideal or simply a Γ-hyperideal if it is both left and right Γ-hyperideal.

S is called a left (right) simple Γ-semihypergroup if it has no proper left (right) Γ-hyperideal. S is said to be a simple Γ-semihypergroup if it has no proper Γ-hyperideal.

Example 2.7. In example 2.4 let $T = [0, t]$ where $t \in [0, 1]$ then T is left (right) Γ-hyperideal of S.

Lemma 2.8. [12] *Let S be a Γ-semihypergroup and Λ be a non-empty set such that for every $\lambda \in \Lambda$, I_λ is an ideal of S. Then the following assertions hold:*

(1) $\bigcup_{\lambda \in \Lambda} I_\lambda$ *is an ideal of S;*
(2) $\bigcap_{\lambda \in \Lambda} I_\lambda$ *is an ideal of S.*

Definition 2.9. [12] Let A be a non-empty subset of a Γ-semihypergroup S. Then intersection of all Γ-hyperideals of S containing A is a Γ-hyperideal of S generated by A, and denoted by $< A >$.

Definition 2.10. [14]A Γ-hyperideal A of a Γ-semihypergroup S is said to be a *principal* Γ*-hyperideal* if A is a Γ-hyperideal generated by single element a and is denoted by (a).

To study more examples on Γ-semihypergroup and the notions of fundamental relations on Γ-semihypergroup, quotient Γ- semihypergroup, right Noetherian Γ-semihypergroups etc. see [13].

3. Systems in Γ-Semihypergroups

In this section we revise the definition of *m-system* and prove the results established in [14]. Here we also define *n-system* of a Γ- semihypergroup and prove some results.

Definition 3.1. [13] A proper Γ-hyperideal P of a Γ-semihypergroup S is said to be a *prime* Γ-hyperideal if for every Γ-hyperideal I, J of S $I\Gamma J \subseteq P$ implies $I \subseteq P$ or $J \subseteq P$. If Γ-semihypergroup S is commutative then a proper Γ-hyperideal P is prime if and only if a$\Gamma b \subseteq P$ implies $a \in P$ or $b \in P$, for any $a, b \in S$.

Theorem 3.2. [12] *Let S be a Γ-semihypergroup and P be a left ideal of S. Then P is a prime ideal of S if and only if for all $x, y \in S, x\Gamma S\Gamma y \subseteq P$ implies that $x \in P$ or $y \in P$.*

In [14] Kostaq Hila et. al. defined *m-system* as follows.

Definition 3.3. Let M be a Γ-semihypergroup. A subset H of M is said to be an *m-system* of M if and only if $c \in H, d \in H$ imply that there exist element $p \in M$ and $\alpha_1, \alpha_2 \in \Gamma$ such that $c\alpha_1 p \alpha_2 d \subseteq H$. The empty set is said to be considered as an *m-system*.

This definition requires $c\alpha_1 p\alpha_2 d$ to be a subset of H. Here we revise this definition of *m-system* and instead of asking for a subset we only require non-empty intersection as follows:

Definition 3.4. A non-empty set M of a Γ-semihypergroup S is said to be an *m-system* of M if for given $x \in M, y \in M$ there exists an element $s \in S$ and $\alpha, \beta \in \Gamma$ such that $x\alpha s\beta y \cap M \neq \phi$.

We prove following propositions with the revised definition.

Proposition 3.5. *Let S be a Γ-semihypergroup then A Γ-hyperideal P of S is prime if and only if its compliment $S \setminus P$ is an m-system.*

Proof. Assume that P is a prime Γ-hyperideal of S and $x, y \in S \setminus P$. If for all $s \in S$ and $\alpha, \beta \in \Gamma$, $x\alpha s\beta y \cap (S \setminus P) = \phi$ then $x\alpha s\beta y \subseteq P$ that is $x\alpha S\beta y \subseteq P$. By theorem 3.2 either $x \in P$ or $y \in P$ a contradiction. Hence $S \setminus P$ is an *m-system* of S.
On the other hand if $S \setminus P$ is a an *m-system* and for Γ-hyperideals I and J of $S, I\Gamma J \subseteq P$ does not imply that $I \subseteq P$ or $J \subset P$ then $I \nsubseteq P$ and $J \nsubseteq P$ which gives elements x and y in I and J respectively such that $x, y \notin P$. This implies that $x, y \in S \setminus P$. Thus there exists $s \in S, \alpha, \beta \in \Gamma$ such that $x\alpha s\beta y \cap (S \setminus P) \neq \phi$. Now $x\alpha s\beta y \subseteq I\Gamma S\Gamma J \subseteq I\Gamma J \subseteq P$ implies $x\alpha s\beta y \cap (S \setminus P) = \phi$ a contradiction. Hence P is a prime Γ-hyperideal of S. □

Proposition 3.6. *Let I be a Γ-hyperideal disjoint from an m-system M of a Γ-semihypergroup S. Then there exists an m-system T of S containing M such that T is maximal in the class of m-systems disjoint from I.*

Proof. Let I be a Γ-hyperideal and M be an *m-system* of S respectively such that $I \cap M = \phi$.
Let $\mathscr{A} = \{K \mid M \subseteq K \subseteq S, K$ is *m-system* of S and $K \cap I = \phi\}$ be ordered by set inclusion. Clearly $\mathscr{A} \neq \phi$ as $M \in \mathscr{A}$. Let $\mathscr{C} = \{C_i \in \mathscr{A} \mid i \in \Lambda\}$ be a chain in $\mathscr{A}$ and set $H = \bigcup_{i\in\Lambda} C_i$. Thus $C_i \subseteq H$ for all $i \in \Lambda$, $M \subseteq H \subseteq S$ and $H \cap I = \phi$. Moreover H is an *m-system* because for $x, y \in H$, $x \in C_j$ and $y \in C_k$ for some $j, k \in \Lambda$. Without loss of generality assume that $C_j \subseteq C_k$, thus $x, y \in C_k$ and C_k is an *m-system*

implies that $x\alpha s\beta y \cap C_k \neq \phi$ for some $s \in S$ and $\alpha, \beta \in \Gamma$ and $C_k \subseteq H$ hence $x\alpha s\beta y \cap H \neq \phi$. This implies that $H \in \mathscr{A}$ and that every chain in $\mathscr{A}$ is bounded above. Therefore by Zorn's lemma there exists a maximal element say T in $\mathscr{A}$. □

Proposition 3.7. *Let I be a Γ-hyperideal of a Γ-semihypergroup S disjoint from an m-system M of S. Then there exists a Γ-hyperideal P containing I such that P is maximal in the class of Γ-hyperideals containing I and disjoint from M. Moreover P is prime Γ-hyperideal of S.*

Proof. Let S be a Γ-semihypergroup and I be a Γ-hyperideal of S disjoint from an *m-system* M of S. Let $\mathscr{B} = \{A \mid I \subseteq A, A$ is Γ-hyperideal of S and $A \cap M = \phi\}$ be ordered by set inclusion. Clearly $\mathscr{B} \neq \phi$ and every chain in $\mathscr{B}$ has an upper bound hence by Zorn's lemma $\mathscr{B}$ has a maximal element say P. From proposition 3.6 above there exists an *m-system* $T \supseteq M$ which is maximal in the class of *m-systems* disjoint from P. We have $T = S \setminus P$ therefore P is prime Γ-hyperideal of S. □

Definition 3.8. [3] A proper Γ-hyperideal Q of a Γ-semihypergroup S is said to be a *completely prime* Γ-hyperideal if for any non-empty subsets A and B of S, $A\Gamma B \subseteq Q$ implies that either $A \subseteq Q$ or $B \subseteq Q$.

Definition 3.9. [3] A proper Γ-hyperideal I of a Γ-semihypergroup S is said to be a *semiprime* Γ-hyperideal if for nonempty subsets A and B of S, $A\Gamma S\Gamma B \subseteq I$ implies $A \subseteq I$ or $B \subseteq I$.

Definition 3.10. [3] A proper Γ-hyperideal I of a Γ-semihypergroup S is said to be a *partially semiprime* Γ-hyperideal if for a nonempty subset A of S, $A\Gamma S\Gamma A \subseteq I$ implies that $A \subseteq I$.

Definition 3.11. [3] A proper Γ-hyperideal I of a Γ-semihypergroup S is said to be a *completely semiprime* Γ-hyperideal if for a nonempty subset A of S $A\Gamma A \subseteq I$ implies that $A \subseteq I$.

Proposition 3.12. [3] *In a Γ-semihypergroup S following results hold.*

(1) *Every completely prime Γ-hyperideal is a prime Γ-hyperideal.*
(2) *Every semiprime Γ- hyperideal is partially semiprime Γ-hyperideal.*
(3) *Every completely prime Γ-hyperideal is a completely semiprime Γ-hyperideal.*
(4) *Every prime Γ-hyperideal is partially semiprime.*
(5) *Every completely prime Γ-hyperideal is a pseudo symmetric Γ-hyperideal.*

Definition 3.13. A non-empty subset N of a Γ-semihypergroup S is said to be an *n-system* if for given $x \in N$ there exists $s \in S$ and $\alpha, \beta \in \Gamma$ such that $x\alpha s\beta x \cap N \neq \phi$.

Example 3.14. [1] Let $S = (0,1), \Gamma = \{\gamma_n | n \in \mathbb{N}\}$ and for every $n \in \mathbb{N}$ define hyperoperation γ_n as follows:

$$x\gamma_n y = \left\{\frac{xy}{2^k} | 0 \leq k \leq n\right\}$$

Then S is a Γ-semihypergroup and the set $M_i = (0, 2^{-i})$, $i \in \mathbb{N}$ is an m-hypersystem of S and $N_i = (0, 4^{-i})$ where $i \in \mathbb{N}$ is an n-hypersystem of S.

Example 3.15. [1] In example 2.4 the set $T = [0, t]$ where $t \in [0, 1]$ is both an m-hypersystem as well as an n-hypersystem of Γ-semihypergroup S.

Theorem 3.16. *In a Γ-semihypergroup S a Γ-hyperideal is partially semiprime if and only if its complement is an n-system.*

Proof. Let S be a Γ-semihypergroup , N be an *n-system* in S and I be a partially semiprime Γ-hyperideal of S hence $\phi \neq I \neq S$ this implies that $\phi \neq S \setminus I \neq S$ so choose $x \in S \setminus I$. If for all $s \in S$ and $\alpha, \beta \in \Gamma, x\alpha s\beta x \cap (S \setminus I) = \phi$ then $x\alpha s\beta x \subseteq I$ for all $s \in S$ and $\alpha, \beta \in \Gamma$ which implies that $x \in I$, a contradiction. Hence $S \setminus I$ is an *n-system*. Conversely assume that I is a Γ-hyperideal of S whose complement is an *n-system* and let $A\Gamma S\Gamma A \subseteq I$ for a non-empty subset A of S. If $A \nsubseteq I$ then there must exists an element $x \in A$ such that $x \notin I$. Now $x \in S \setminus I$ hence there exists $s \in S, \alpha, \beta \in \Gamma$ such that $x\alpha s\beta x \cap (S \setminus I) \neq \phi$. But $x\alpha s\beta x \subseteq A\Gamma S\Gamma A \subseteq I$ hence $x\alpha s\beta x \cap (S \setminus I) = \phi$. Therefore I is a partially semiprime Γ-hyperideal of S. □

Theorem 3.17. *For every n-system N of a Γ-semihypergroup S containing an element a there exists an m-system M of S such that $a \in M$ and $M \subseteq N$.*

Proof. Let $a \in N$ and set $a_1 = a$ we have $x_1 \in S, \alpha_1, \beta_1 \in \Gamma$ such that $a_1\alpha_1 x_1\beta_1 a_1 \cap N \neq \phi$. Let $a_2 \in a_1\alpha_1 x_1\beta_1 a_1 \cap N$ then we have $x_2 \in S, \alpha_2, \beta_2 \in \Gamma$ such that $a_2\alpha_2 x_2\beta_2 a_2 \cap N \neq \phi$ and so on. Let $M = \{a_i \in N \mid i = 1, 2, 3, ...\}$ then $a_1 = a$ and $M \subseteq N$. Now consider $x, y \in M$ that is $x = a_i$ and $y = a_j$ for some i, j. We can assume that $i \leq j$ then $a_{j+1} \in a_j\alpha_j x_j\beta_j a_j \subseteq a_j\Gamma S\Gamma a_j$ and $a_j \in M$ implies that $a_j \in a_{j-1}\Gamma S a_{j-1}$ and hence $a_{j+1} \in a_{j-1}\Gamma S\Gamma a_{j-1}\Gamma S\Gamma a_j \subseteq a_{j-1}\Gamma S\Gamma S\Gamma S\Gamma a_j \subseteq a_{j-1}\Gamma S\Gamma a_j \subseteq a_i\Gamma S\Gamma a_j$. Thus $a_{j+1} \in x\Gamma S\Gamma y$ implies that there exists some $s \in S, \alpha, \beta \in \Gamma$ such that $a_{j+1} \in x\alpha s\beta y$ and so $x\alpha s\beta y \cap M \neq \phi$ showing that M is an *m-system*. □

4. Maximal Γ-Hyperideal in Γ-Semihypergroup

This section deals with maximal Γ-hyperideals in Γ-semihypergroup and study the conditions under which maximal Γ-hyperideals are prime and vice versa.

Definition 4.1. An element a of Γ-semihypergroup S is said to be an *α-idempotent* if $a \in a\alpha a$. An element a of Γ- semihypergroup S is said to be a Γ-*idempotent* or simply *idempotent* if $a \in a\alpha a$ for all $\alpha \in \Gamma$ i.e. $a \in a\Gamma a$.

Definition 4.2. A Γ-semihypergroup S is said to be an *idempotent* Γ-*semihypergroup* if every element in S is a Γ-idempotent.

Example 4.3. Let $S = \{a, b, c, d\}$ and $\Gamma = \{\alpha, \beta\}$. Define a hyperoperation ∘ on S as follows:

∘	a	b	c	d
a	a	{a, b }	{a, c}	{a, d}
b	a	{a, b }	{a, c}	{a, d}
c	a	b	c	d
d	a	b	c	d

Define a mapping $S \times \Gamma \times S \to \wp^*(S)$ by $x\gamma y = x \circ y$ for every $x, y \in S$ and $\gamma \in \Gamma$. Then S is a Γ-semihypergroup. Observe that each element is Γ-idempotent, hence S is an idempotent Γ-semihypergroup

Example 4.4. Let $S = \{x, y\}$ and $\Gamma = \{\alpha, \beta\}$ defined as follows:

α	x	y
x	{x, y}	{x, y}
y	{x, y}	{x, y}

β	x	y
x	{x}	{y}
y	{y}	{x}

Then S is a Γ-semihypergroup and x is α-idempotent as well as β-idempotent hence x is Γ-idempotent whereas y is only α-idempotent..

Definition 4.5. A Γ-semihypergroup S is said to be *globally idempotent* if $S\Gamma S = S$.

Definition 4.6. A proper Γ-hyperideal I of a Γ-semihypergroup is said to be a *maximal* Γ-*hyperideal* if there does not exists any Γ-hyperideal between I and S. That is for a Γ-hyperideal J if $I \subseteq J \subseteq S$ then either $J = I$ or $J = S$.

Theorem 4.7. *In a globally idempotent* Γ*-semihypergroup every maximal* Γ*-hyperideal is prime* Γ*-hyperideal.*

Proof. Let S be a globally idempotent Γ-semihypergroup and I be a maximal Γ-hyperideal of S. For Γ-hyperideals A and B of S let $A\Gamma B \subseteq I$. If $A \nsubseteq I$ and $B \nsubseteq I$ then there exist $a \in A$, $b \in B$ and $a, b \notin I$. Therefore $I \cup A$ and $I \cup B$ are Γ-hyperideals of S such that $I \subsetneq I \cup A \subseteq S$ and $I \subsetneq I \cup B \subseteq S$. Since I is maximal $I \cup A = S$ and $I \cup B = S$. Now $S = S\Gamma S = (I \cup A)\Gamma(I \cup B) = I\Gamma I \cup I\Gamma B \cup A\Gamma I \cup A\Gamma B \subseteq I$. Thus $I \cup A \subseteq I$ which is a contradiction. Hence I is a prime Γ-hyperideal. □

Theorem 4.8. *If I is a maximal* Γ*-hyperideal of a* Γ*-semihypergroyp S and complement of I contains an idempotent element then I is prime* Γ*-hyperideal.*

Proof. Let I be a maximal Γ-hyperideal of a Γ-semihypergroup S such that $S \setminus I$ contains an idempotent element x. let A and B be two Γ-hyperideas of S such that $A\Gamma B \subseteq I$. If neither $A \subseteq I$ nor $B \subseteq I$ then $I \cup A = S$ and $I \cup B = S$ and $S\Gamma S \subseteq I$. This implies that $x \in I$ which is a contradiction. Hence I is a prime Γ-hyperideal of S. □

Theorem 4.9. *Let* $\{I_\alpha\}_{\alpha\in\Delta}$ *be the collection of all distinct maximal* Γ*-hyperideals of a* Γ*-semihypergroup* S*. Setting* $J_\alpha = S \setminus I_\alpha$ *and* $I^* = \bigcap_{\alpha\in\Delta} I_\alpha$ *with card* $\Delta \geq 2$ *we have the following:*

(1) $J_\alpha \cap J_\beta = \phi$ *whenever* $\alpha \neq \beta$

(2) $S = [\bigcup_{\alpha\in\Delta} J_\alpha] \cup I^*$

(3) $I_\alpha \supseteq J_\beta$ *whenever* $\alpha \neq \beta$

(4) If A is any Γ*-hyperideal of S such that* $A \cap J_\alpha \neq \phi$ *then* $J_\alpha \subseteq A$.

(5) For $\alpha \neq \beta, J_\alpha\Gamma J_\beta \subseteq I^*$ *showing that* I^* *is nonempty.*

Proof. (1) If $\alpha \neq \beta$ then I_α and I_β are two distinct maximal Γ-hyperideals of S. This implies that $I_\alpha \cup I_\beta$ is also a Γ-hyperideal of S and $I_\alpha \cup I_\beta = S$. Hence $J_\alpha \cap J_\beta = \phi$.

(2) As $I^* = \bigcap_{\alpha\in\Delta} I_\alpha = S \setminus (\bigcup_{\alpha\in\Delta} J_\alpha)$. Therefore $S = [\bigcup_{\alpha\in\Delta} J_\alpha] \cup I^*$.

(3) Let $\alpha \neq \beta$. As $J_\beta \subseteq S, J_\beta = S \cap J_\beta = I_\alpha \cap J_\beta$. Hence $I_\alpha \supseteq J_\beta$ whenever $\alpha \neq \beta$.

(4) Let A be a Γ-hyperideal of Γ-semiypergroup S and $A \cap J_\alpha \neq \phi$ then $A \nsubseteq I_\alpha$ hence $I_\alpha \cup A = S$. Since $I_\alpha \cap J_\alpha = \phi$ we have $J_\alpha \cap A = J_\alpha$ which implies that $J_\alpha \subseteq A$.

(5) Let $\alpha \neq \beta$. If for some $x \in J_\alpha, y \in J_\beta$ and some $\gamma \in \Gamma$, $x\gamma y \nsubseteq I^*$ then there exists $a \in x\gamma y$ such that $a \notin I^*$. This implies that $a \in S \setminus I^* = \bigcup\limits_{\alpha \in \Delta} J_\alpha$ hence $a \in J_\delta$ for some $\delta \in \Gamma$. If $J_\alpha \neq J_\delta$ then $J_\alpha \cap J_\delta = \phi$ and $J_\alpha \subseteq (S \setminus J_\delta) = I_\delta$. Thus $a \in x\gamma y \subseteq J_\alpha \Gamma J_\beta \subseteq I_\delta \Gamma J_\beta \subseteq I_\delta = S \setminus J_\delta$ which is a contradiction. If $J_\alpha = J_\delta$ then for $\alpha \neq \beta, J_\alpha \cap J_\beta = \phi$ hence $J_\beta \subseteq S \setminus J_\alpha = I_\alpha$ and $a \in x\gamma y \subseteq J_\alpha \Gamma J_\beta \subseteq J_\alpha \Gamma I_\alpha \subseteq I_\alpha = S \setminus J_\alpha = S \setminus J_\delta$ which is again a contradiction. Hence for $\alpha \neq \beta, J_\alpha \Gamma J_\beta \subseteq I^*$ and therefore I^* is nonempty. □

Theorem 4.10. *Let S be a Γ-semihypergroup containing maximal Γ-hyperideals and let I^* be the intersection of all maximal Γ-hyperideals of S. Then every prime Γ-hyperideal of S containing I^* is a maximal Γ-hyperideal of S.*

Proof. Let $\{I_\alpha \mid I_\alpha \text{is a maximal } \Gamma\text{-hyperideal of } S \text{ for } \alpha \in \Delta\}, I^* = \bigcap\limits_{\alpha \in \Delta} I_\alpha$ and let P be a prime Γ-hyperideal of S containing I^* that is $P \supseteq I^* = \bigcap\limits_{\alpha \in \Delta} I_\alpha = S \setminus (\bigcup\limits_{\alpha \in \Delta} J_\alpha)$ where $J_\alpha = S \setminus I_\alpha$. This shows that $\bigcup\limits_{\alpha \in \Delta} J_\alpha \nsubseteq P$ and by (4) of theorem 4.9 we get a subset Π of Δ such that $P = S \setminus (\bigcup\limits_{\alpha \in \Pi} J_\alpha) = \bigcap\limits_{\alpha \in \Pi} I_\alpha$ and Π is non-empty. If $|\Pi| = 1$ then $P = I$ and this would imply that P is maximal Γ-hyperideal.
Next we show that $|\Pi| \ngeq 2$. If not then for $\gamma \in \Pi$ let $I' = \bigcap\limits_{\alpha \in \Pi, \alpha \neq \gamma} I_\alpha$ then

$$P = I' \cap I_\gamma \tag{2.1}$$

and $I' \Gamma I_\gamma \subseteq P$ implies that $I' \subseteq P$ or $I_\gamma \subseteq P$. If $I' \subseteq P$ then from (1) $P = I'$. Also $I' = P = I' \cap I_\gamma$ implies that $I' \subseteq I_\gamma$ by (3) of theorem 4.9 we have if $\gamma \neq \alpha$ then $J_\gamma \subseteq I_\alpha$ that is $J_\gamma \subseteq I'$. Thus $J_\gamma \subseteq I_\gamma$ which is a contradiction to $J_\gamma \cap I_\gamma = \phi$. Also if $I_\gamma \subseteq P$ then from (1) $I_\gamma \subseteq I'$. Since I_γ is a maximal Γ-hyperideal and $I' \neq S$ we have $I' = I_\gamma$. As $J_\gamma \subseteq I' = I_\gamma$ we arrive at a contradiction again as $J_\gamma \cap I_\gamma = \phi$. Hence $|\Pi| \ngeq 2$ and we conclude that P is maximal Γ-hyperideal. □

5. Complete Prime Γ-Radical in Γ-Semihypergroup

This section introduces *c-system* and complete prime Γ-radical in Γ-semihypergroup. Characterization of completely prime Γ-hyperideals is presented along with proving some nice results regarding complete prime Γ-radical in Γ-semihypergroup.

Proposition 5.1. *The non-empty intersection of any family of prime* Γ*-hyperideals of a* Γ*-semihypergroup is partially semiprime.*

Proof. Let $\{P_\alpha\}_{\alpha\in\Lambda}$ be a family of prime Γ-hyperideals of a Γ- semihypergroup S and $P = \bigcap_{\alpha\in\Lambda} P_\alpha \neq \phi$. Then P is a Γ-hyperideal. Let A be the nonempty subset of S and $A\Gamma S\Gamma A \subseteq P$. Therefore $A \subseteq P_\alpha$ for all $\alpha \in \Gamma$ where each P_α is prime and hence partially semiprime Γ-hyperideal. This implies that $A \subseteq P_\alpha$ for all $\alpha \in \Gamma$ hence $A \subseteq \bigcap_{\alpha\in\Lambda} P_\alpha$ that is $A \subseteq P$ and P is partially semiprime Γ-hyperideal of S. □

Proposition 5.2. *In a* Γ*-semihypergroup* S *if* I *is a* Γ*-hyperideal and* Q *is a competely prime* Γ*-hyperideal then* $I \cap Q$ *is a completely prime* Γ*-hyperideal of* I *considering* I *as a* Γ*-semihypergroup.*

Proof. $I \cap Q$ is a Γ-hyperideal of I because I is a Γ-hyperideal. Let $A, B \subseteq I$ with $A\Gamma B \subseteq I \cap Q$ then $A\Gamma B \subseteq Q$ and Q is completely prime Γ-hyperideal hence either $A \subseteq Q$ or $B \subseteq Q$. If $A \subseteq Q$ then $I \cap A \subseteq I \cap Q$ that is $A \subseteq I \cap Q$. Similarly if $B \subseteq Q$ then $B \subseteq I \cap Q$. Hence $I \cap Q$ is a complteley prime Γ-hyperideal of S. □

Definition 5.3. A non-empty subset C of a Γ-semihypergroup S is said to be a *c-system* if for given $a, b \in C$ there exists $\alpha \in \Gamma$ such that $a\alpha b \cap C \neq \phi$.

Proposition 5.4. *Every* Γ*-subsemihypergroup of a* Γ*-semihypergroup is a c-system.*

Proof. Straight forward. □

Theorem 5.5. *A* Γ*-hyperideal* Q *of a* Γ*-semihypergroup* S *is completely prime if and only if its complement* $S \setminus Q$ *is a c-system.*

Proof. Let Q be a completely prime Γ-hyperideal of a Γ-semihypergroup S and $a, b \in S \setminus Q$. If $a\alpha b \cap (S \setminus Q) = \phi, \forall\alpha \in \Gamma$ then $a\alpha b \subseteq Q, \forall\alpha \in \Gamma$. By definition of completely prime Γ-hyperideal we have either $a \in Q$ or $b \in Q$ which is a contradiction. Hence $S \setminus Q$ is a *c-system.*
Conversely assume that $S \setminus Q$ is *c-system* and for nonempty subsets X

and Y of Γ-semihypergroup S such that $X\Gamma Y \subseteq Q$. If neither $X \subseteq Q$ nor $Y \subseteq Q$ then there exist $x \in X$ and $y \in Y$ such that $x \notin Q$ and $y \notin Q$. Hence $x, y \in S \setminus Q$. There exists $\alpha \in \Gamma$ such that $x\alpha y \cap (S \setminus Q) \neq \phi$. As $x\alpha y \cap (S \setminus Q) \subseteq X\Gamma Y \cap (S \setminus Q)$ we have $X\Gamma Y \cap (S \setminus Q) \neq \phi$ which implies that $X\Gamma Y \nsubseteq Q$. Hence Q is a completely prime Γ-hyperideal of S. □

Proposition 5.6. *In a Γ-semihypergoup S if I is a Γ-hyperideal disjoint from a c-system C of S then threre exists a c-system $U \supseteq C$ which is maximal in the class of c-systems disjoint from I.*

Proof. Apply Zorn's lemma to the family of c-systems containing C and disjoint from I. □

Proposition 5.7. *In a Γ-semihypergroup S if I is a Γ-hyperideal disjoint from a c-system C of S then there exists a completely prime Γ-hyperideal Q of S which is maximal in the class of Γ-hyperideals containing I and disjoint from C.*

Proof. Let I be a Γ-hyperideal of a Γ-semihypergroup S disjoint from a *c-system* C of S and $\mathscr{G} = \{J \mid J \text{ is a } \Gamma\text{-hyperideal of } S,\ I \subseteq J \subseteq S,\ J \cap C = \phi\}$ be ordered by set inclusion. $\mathscr{G} \neq \phi$ since $I \in \mathscr{G}$. We see that every chain in $\mathscr{G}$ has an upper bound hence by Zorn's lemma, $\mathscr{G}$ has a maximal element say Q. From proposition 5.6 there exists a *c-system* $U \supseteq C$ which is maximal in the class of *c-systems* disjoint from Q and we find $U = S \setminus Q$. Hence Q is completely prime Γ-hyperideal of S. □

Definition 5.8. [14] Let S be a Γ-semihypergroup. The intersection K_S of all ideals of S if it is nonempty is called the *Kernel* of S.

Definition 5.9. [14] Let S be a Γ-semihypergroup. The *prime radical* $P(S)$ (or $rad.(S)$) of S is the intersection of all prime ideals of S.
If I is a Γ-hyperideal of S , the *prime radical* $P(I)$ (or $rad.(I)$) of I is the intersection of all prime ideals containing I.

Definition 5.10. Let S be a Γ-semihypergroup. The intersection of all completely prime Γ-hyperideals of S is called *complete prime* Γ-radical of S and is denoted by $c.rad(S)$.
If I is a Γ-hyperideal of S then the intersection of all completely prime Γ-hyperideals of S containing I is called a *complete prime* Γ*-radical* of I and is denoted by $c.rad(I)$.

Theorem 5.11. [14] *Let S be a Γ-semihypergroup and I an ideal of S. Then the Kernel of I exists and is equal to K_S.*

Theorem 5.12. *Let S be a Γ-semihypergroup and I a Γ-hyperideal of S. Then $rad.(I) = \{x \in S \mid$ every m-system of S containing x intersects $I\}$.*

Proof. Let $\Delta = \{x \in S \mid$ every m-*system* of S containing x intersects $I\}$, $x \in rad.(I)$ and M be an m-*system* of S containing x. Then $x \notin S \setminus M$ which is a prime Γ-hyperideal of S. If $M \cap I = \phi$ then $rad.(I) \subseteq S \setminus M$. Therefore $x \in S \setminus M$, a contradiction. Hence $M \cap I \neq \phi$ and $x \in \Delta$. Thus $rad.(I) \subseteq \Delta$. Now suppose that $x \in \Delta$. If $x \notin rad.(I)$ then there exists a prime Γ-hyperideal P of S containing I such that $x \notin P$. Then $x \in S \setminus P$ which is m-*system* of S and $I \cap (S \setminus P) = \phi$ hence $x \notin \Delta$ a contradiction. Hence $x \in rad.(I)$ and $\Delta \subseteq rad.(I)$. □

Theorem 5.13. *A Γ-hyperideal Q of a Γ-semihypergroup S is partially semiprime if and only if $rad.(Q) = Q$.*

Lemma 5.14. *Let S be a Γ-semihypergroup then*

$$c.rad.(S) = \{x \in S \mid \textit{every } c\textit{-system of } S \textit{ containing } x \textit{ intersects } K_S\}.$$

Proof. Let $\Delta = \{x \in S \mid$ every c-*system* of S containing x intersects $K_S\}$ and $x \notin c.rad.(S)$ implies that there exists a completely prime Γ- hyperideal Q of S such that $x \notin Q$, hence $x \in S \setminus Q$ which is c-*system* of S containing x and $(S \setminus Q) \cap K_S = \phi$. Therefore $x \notin \Delta$ and $\Delta \subseteq c.rad.(S)$. Now suppose that $x \notin \Delta$ then there exists a c-*system* C of S containing x and $C \cap K_S = \phi$. By proposition 5.7 there exists a completely prime Γ-hyperideal $Q^{'}$ of S which is maximal in the class of Γ-hyperideals of S containing K_S and disjoint from C. Now $x \notin Q^{'}$ and hence $x \notin c.rad.(S)$ therefore $c.rad.(S) \subseteq \Delta$ which implies that $c.rad.(S) = \{x \in S \mid$ every c-*system* of S containing x intersects $K_S\}$. □

Theorem 5.15. *Let I be a Γ-hyperideal of a Γ-semihypergroup S then $c.rad.(I) = I \cap c.rad.(S)$*

Proof. Let $\mathscr{E}$ be the collection of all completely prime Γ-hyperideals of Γ-semihypergroup S and $\mathscr{F}$ be the collection of all completely prime Γ-hyperideals of I, by proposition 5.2 it follows that if $Q \in \mathscr{E}$ then $I \cap Q \in \mathscr{F}$. So $c.rad.(I) = \bigcap\limits_{Q \in \mathscr{F}} Q \subseteq \bigcap\limits_{Q^{'} \in \mathscr{E}} (Q^{'} \cap I) = c.rad.(S) \cap I$. Suppose that $x \notin c.rad.(I)$ then there exists a c-*system* C of I containing x and $C \cap K_I = \phi$. Now C can be viewed as a c-*system* of S containing x and $C \cap K_S = \phi$ implies that $x \notin c.rad.(S)$. Hence $c.rad.(S) \subseteq c.rad.(I)$ and $c.rad.(I) \subseteq c.rad.(S) \cap I \subseteq c.rad.(S) \subseteq c.rad.(I)$ implies that $c.rad.(I) = I \cap c.rad.(S)$. □

6. Conclusion

In this paper *m-system*, *n-system*, *c-system* and maximal Γ- hyperideals in Γ-semihypergroups have been studied and it is proved that in a Γ-semihypergroup a Γ-hyperideal is prime (partially semiprime)if and only if its compliment is an *m-system* (*n-system*). It is also shown that if compliment of a maximal Γ-hyperideal contains an idempotent element then it becomes a prime Γ-hyperideal. The notion of complete prime Γ-radical have been introduced and its relation with *c-system* is explored. It is proved that the intersection of a Γ-hyperideal and completely prime Γ-hyperideal is completely prime Γ-hyperideal and a characterization of completely prime Γ-hyperideal in terms of *c-system* is established.

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Kishor Fakira Pawar
Department of Mathematics, School of Mathematical Sciences, Kavayitri Bahinabai Chaudhari North Maharashtra University, Jalgaon, 425 001, M.S., India
Email: `kfpawar@nmu.ac.in`

Ansari Safoora Samreen Jamal Abdul Nasir
Department of Mathematics, S.N.J.B.'s K.K.H.A. Arts, S.M.G.L. Commerce & S.P.H.J. Science College, Chandwad, District Nashik, 423 101, M.S., India
Email: `ansari.sjacs@snjb.org`