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ON QUASI-IDEALS OF Γ-SEMIHYPERRINGS

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ABSTRACT. The concept of quasi-ideals, minimal quasi-ideals for Γ-semihyperrings are introduced. Characterization of quasi-ideals and of minimal quasi-ideals of Γ-semihyperring with the help of ideals of Γ-semihyperrings have been taken into account. It is proved that for a Γ-semihyperring R with zero, one can find a quasi-ideal of R generated by X, where X is any non empty subset of R. Also the notion of quasi-simple Γ-semihyperring is introduced and proved some properties in this respect.

Key Words: Γ-semihyperring, Quasi-ideal, Minimal quasi-ideal, Quasi-simple, Γ-semihyperring.
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1. Introduction and preliminaries

In 1953, Steinfeld [4] introduced quasi-ideals for rings as a generalization of the notion of one sided ideal. Furthermore the concept of quasi-ideal was introduced by Steinfeld [5] for semigroups in 1956 and then quasi-ideal has been widely studied in various algebraic structures viz. regular rings, semirings, Γ-semirings etc. Also Iseki [2] studied quasi-ideal for semiring without zero in 1958 and proved some important results. In 2006, Chinram [6] introduced the concept of quasi-simple Γ-semigroups, quasi-gamma-ideals for Γ-semigroups and provided their properties. Then, both Iseki [2] and Chinram [6] studied minimal quasi-ideals of semirings and Γ-semigroups respectively. Jagatap and Pawar

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[7] introduced the concepts of quasi-ideals and minimal quasi-ideals for Γ-semirings.

The notion of hypergroup was introduced by Marty [1] in 1934. Later many authors studied algebraic hyperstructure which are generalization of classical algebraic structure. In classical algebraic structure the composition of two element is an element while in an algebraic hyperstructure composition of two element is a set. Let H be a non empty set then, the map $\circ : H \times H \to \wp^*(H)$ is called a hyperopertion where $\wp^*(H)$ is the family of all non-empty subsets of H and the couple $(H, \circ)$ is called a hypergroupoid. Moreover, the couple $(H, \circ)$ is called a semihypergroup if for every $a, b, c \in H$ we have $(a \circ b) \circ c = a \circ (b \circ c)$.

The notion of Γ-semihyperring as a generalization of semiring, semihyperring and Γ-semiring was introduced by Dehkordi and Davvaz [8]. Also Pawar et al.[3] introduced regular (strongly regular) Γ-semihyperrings and gave it's characterization with the help of ideals of Γ-semihyperrings. The present paper is divided into four different sections. In section 2, the concept of quasi-ideals for Γ-semihyperrings is introduced with examples and some properties are given. It is found that for any non-empty subset X of a Γ-semihyperrings R, one can find a quasi-ideal generated by X. In section 3, the concepts of quasi-simple Γ-semihyperring and 0-quasi-simple Γ-semihyperrings are introduced and some important theorems have been proved. In section 4, the concept of minimal quasi-ideal for Γ-semihyperrings is introduced and it's characterization is given with the help of minimal ideals in Γ-semihyperrings.

Here are some useful definitions and the readers are requested to refer [8], for more details.

Definition 1.1. Let R be a commutative semihypergroup and Γ be a commutative group. Then R is called a Γ-semihyperring if there is a map $R \times \Gamma \times R \to \wp^*(R)$ (images to be denoted by $a\alpha b$ for all $a, b \in R$ and $\alpha \in \Gamma$) and $\wp^*(R)$ is the set of all non-empty subsets of R satisfying the following conditions:

(1) $a\alpha(b + c) = a\alpha b + a\alpha c$
(2) $(a + b)\alpha c = a\alpha c + b\alpha c$
(3) $a(\alpha + \beta)c = a\alpha c + a\beta c$
(4) $a\alpha(b\beta c) = (a\alpha b)\beta c$, for all $a, b, c \in R$ and for all $\alpha, \beta \in \Gamma$.

Definition 1.2. A Γ-semihyperring R is said to be commutative if $a\alpha b = b\alpha a$ for all $a, b \in R$ and $\alpha \in \Gamma$.

Definition 1.3. A Γ-semihyperring R is said to be with zero if there exists $0 \in R$ such that $a \in a + 0$ and $0 \in 0\alpha a, 0 \in a\alpha 0$ for all $a \in R$ and $\alpha \in \Gamma$.

In this paper we consider zero element 0 in a Γ-semihyperring has properties $a \in a + 0$, 0+0=0 and $a\alpha 0 = 0\alpha a = \{0\}$ for every $a \in R$ and $\alpha \in \Gamma$.

Let A and B be two non-empty subsets of a Γ-semihyperring R and $x \in R$, then

$$A + B = \{x \mid x \in a + b, a \in A, b \in B\}$$
$$A\Gamma B = \{x \mid x \in a\alpha b, a \in A, b \in B, \alpha \in \Gamma\}.$$

Definition 1.4. A non empty subset R_1 of Γ-semihyperring R is called a Γ-subsemihyperring if it is closed with respect to the multiplication and addition, that is $R_1 + R_1 \subseteq R_1$ and $R_1\Gamma R_1 \subseteq R_1$.

Definition 1.5. A right (left) ideal I of a Γ-semihyperring R is an additive subsemihypergroup of $(R, +)$ such that $I\Gamma R \subseteq I(R\Gamma I \subseteq I)$. If I is both right and left ideal of R then we say that I is a two sided ideal or simply an ideal of R.

As the concept of algebraic hyperstructure are used in various fields of science, it is useful to introduce and study different concepts of classical algebraic structure analogously in algebraic hyperstructure. So we tried to introduce the concepts of quasi-ideals and minimal quasi-ideal of classical algebraic structure in Γ-semihyperring and found that there is lot of scope for work in related topic.

2. Quasi-ideals in Γ-semihyperrings

Here the concept of quasi-ideals for Γ-semihyperrings is introduced and it is characterized with the help of ideals of Γ-semihyperrings.

Definition 2.1. A subsemihypergroup Q of $(R, +)$ is quasi-ideal of Γ-semihyperring R if $(R\Gamma Q) \cap (Q\Gamma R) \subseteq Q$.

Example 2.2. [8] Let $R = \{a, b, c, d\}, \Gamma = \mathbb{Z}_2$, $\alpha = \bar{0}$ and $\beta = \bar{1}$. Then R is Γ-semihyperring with the following hyperoperations

$+$	a	b	c	d
a	$\{a,b\}$	$\{a,b\}$	$\{c,d\}$	$\{c,d\}$
b	$\{a,b\}$	$\{a,b\}$	$\{c,d\}$	$\{c,d\}$
c	$\{c,d\}$	$\{c,d\}$	$\{a,b\}$	$\{a,b\}$
d	$\{c,d\}$	$\{c,d\}$	$\{c,d\}$	$\{a,b\}$

β	a	b	c	d
a	$\{a,b\}$	$\{a,b\}$	$\{a,b\}$	$\{a,b\}$
b	$\{a,b\}$	$\{a,b\}$	$\{a,b\}$	$\{a,b\}$
c	$\{a,b\}$	$\{a,b\}$	$\{c,d\}$	$\{c,d\}$
d	$\{a,b\}$	$\{a,b\}$	$\{c,d\}$	$\{c,d\}$

For any $x, y \in R$ we define $x\alpha y = \{a, b\}$. Then $Q = \{a, b\}$ is quasi-ideal of Γ-semihyperring R.

A Γ-subsemihyperring of R need not be a quasi-ideal of Γ-semihyperring R.

Example 2.3. [3] Let $R = \{a, b, c, d\}$. Then R is commutative semihypergroup with following hyperoperations

$+$	a	b	c	d
a	$\{a\}$	$\{a,b\}$	$\{a,c\}$	$\{a,d\}$
b	$\{a,b\}$	$\{b\}$	$\{b,c\}$	$\{b,d\}$
c	$\{a,c\}$	$\{b,c\}$	$\{c\}$	$\{c,d\}$
d	$\{a,d\}$	$\{b,d\}$	$\{c,d\}$	$\{d\}$

$\cdot$	a	b	c	d
a	$\{a\}$	$\{a,b\}$	$\{a,b,c\}$	$\{a,b,c,d\}$
b	$\{a,b\}$	$\{b\}$	$\{b,c\}$	$\{b,c,d\}$
c	$\{a,b,c\}$	$\{b,c\}$	$\{c\}$	$\{c,d\}$
d	$\{a,b,c,d\}$	$\{b,c,d\}$	$\{c,d\}$	$\{d\}$

Then R is a Γ-semihyperring with hyperoperation $x\alpha y \to x \cdot y$ for $x, y \in R$ and $\alpha \in \Gamma$ where Γ-is any commutative group. Here $\{a, b\}$ is Γ-subsemihyperring of R but not quasi-ideal of Γ-semihyperring R.

Definition 2.4. An element e of Γ-semihyperring R is said to be a left (right) identity of R if $r \in e\alpha r (r \in r\alpha e)$ for all $r \in R$ and $\alpha \in \Gamma$. An element e of Γ-semihyperring R is said to be a two sided identity or simply an identity if e is both left and right identity, that is $r \in e\alpha r \cap r\alpha e$ for all $r \in R$ and $\alpha \in \Gamma$.

In example 2.3 $'a'$ is an identity element but in example 2.2, there is no identity element.

Definition 2.5. An element $e \in R$ is said to be an idempotent element of a Γ-semihyperring if $e \in e\alpha e$, for any $\alpha \in \Gamma$.

Definition 2.6. A Γ-semihyperring R is said to be an idempotent Γ-semihyperring if every element of R is an idempotent element.

Theorem 2.7. *Intersection of any family of quasi-ideals of* Γ*-semihyperring* R *is quasi-ideal of* Γ*-semihyperring* R *provided it is non empty.*

Theorem 2.8. *Any one sided ideal or two sided ideal of* Γ*-semihyperring* R *is quasi-ideal of* Γ*-semihyperring* R*.*

Converse of the above theorem need not be true. We illustrate this by the following examples.

Example 2.9. [3] Consider the following:

$$R = \{\begin{pmatrix} x & y \\ z & w \end{pmatrix} | x, y, z, w \in \mathbb{R}\}$$
$$\Gamma = \{z | z \in \mathbb{Z}\}$$
$$A_\alpha = \{\begin{pmatrix} \alpha a & 0 \\ 0 & \alpha b \end{pmatrix} | a, b \in \mathbb{R}, \alpha \in \Gamma\}.$$

Then R is a Γ-semihyperring under the matrix addition with hyperoperation $M\alpha N \to MA_\alpha N$ for all $M, N \in R$ and $\alpha \in \Gamma$.

Example 2.10.
(a) In example 2.9, $Q = \{\begin{pmatrix} 0 & a \\ 0 & b \end{pmatrix} | a, b \in \mathbb{R}\}$ is a quasi-ideal but not right ideal of Γ-semihyperring R.
(b) In example 2.9, $Q = \{\begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix} | a, b \in \mathbb{R}\}$ is a quasi-ideal but not a left ideal of Γ-semihyperring R.
(c) In example 2.9, $Q = \{\begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} | a \in \mathbb{R}\}$ is a quasi-ideal but neither left nor right ideal of Γ-semihyperring R.

From above example quasi-ideals of Γ-semihyperring R need not be left and right ideal of Γ-semihyperring R.

Definition 2.11. A quasi-ideal of Γ-semihyperring R generated by a non-empty subset X of R is defined as the smallest quasi-ideal of R containing X and it is denoted by $(X)_q$

Thus $(X)_q = \cap_{i\in\Delta}\{Q_i \mid Q_i$ is a quasi-ideal and $X \subseteq Q_i\}$ where Δ denotes an indexing set.

$N_0X = \{t \in R \mid t \in \sum_{i=1}^{n} n_i x_i,\ x_i \in X, n \in N_0\}$

Now with the help of theorem given below one can find quasi-ideal generated by X where X is a any non empty subset of Γ-semihyperring with zero.

Theorem 2.12. *Let X be any non-empty subset of Γ-semihyperring R with zero. Then a quasi-ideal of R generated by X is given by $(X)_q = N_0X+(R\Gamma X)\cap(X\Gamma R)$ where N_0 denotes the set of non negative integers.*

Proof. Let $Q = N_0X + (R\Gamma X) \cap (X\Gamma R)$. For any $x \in X$, $x \in 1x + 0 \in N_0X + (R\Gamma X) \cap (X\Gamma R) = Q$. Therefore we get $X \subseteq Q$. Let $x, y \in Q = N_0X + (R\Gamma X) \cap (X\Gamma R)$. Then $x \in x_1 + x_2, y \in y_1 + y_2; x_1, y_1 \in N_0X$ and $x_2, y_2 \in (R\Gamma X)\cap(X\Gamma R)$. Hence $x_1+y_1 \in \sum_{i=1}^{m} n_i x_i+\sum_{i=1}^{k} m_i y_i \in N_0X$, where $x_i, y_i \in X$ and $m, k, n_i, m_i \in N_0$. As $R\Gamma X$ is a left ideal and $X\Gamma R$ is a right ideal we have $(R\Gamma X) \cap (X\Gamma R)$ is a Γ-subsemihyperring of R. Hence, $x_2+y_2 \in (R\Gamma X)\cap(X\Gamma R)$. Therefore $x+y \in (x_1+x_2)+(y_1+y_2) = (x_1 + y_1) + (x_2 + y_2) \in N_0X + (R\Gamma X) \cap (X\Gamma R) = Q$. This shows that Q is a subsemihypergroup of $(R, +)$. Now, we have

$$\begin{aligned}
(R\Gamma Q) \cap (Q\Gamma R) &\subseteq (R\Gamma Q)\\
&= R\Gamma(N_0X + (R\Gamma X) \cap (X\Gamma R))\\
&= N_0(R\Gamma X) + R\Gamma((R\Gamma X) \cap (X\Gamma R))\\
&\subseteq N_0(R\Gamma X) + R\Gamma(R\Gamma X)\\
&\subseteq R\Gamma X \qquad \text{(since } R\Gamma X \text{ is a left ideal).}
\end{aligned}$$

Similarly we can show that $(R\Gamma Q) \cap (Q\Gamma R) \subseteq X\Gamma R$. Therefore we get $(R\Gamma Q) \cap (Q\Gamma R) \subseteq (R\Gamma X) \cap (X\Gamma R) \subseteq Q$. This shows that Q is a quasi-ideal of R containing X. Let Q_1 be a quasi-ideal of R containing X. Then clearly $Q \subseteq Q_1$ since $(R\Gamma X) \cap (X\Gamma R) \subseteq (R\Gamma Q_1) \cap (Q_1\Gamma R) \subseteq Q_1$ and $N_0X \subseteq Q_1$. Therefore Q is the smallest quasi-ideal of R containing X. □

Corollary 2.13. *Let 'a' be an element of* Γ*-semihyperring* R *with zero. Then a quasi-ideal of* R *generated by 'a' is given by* $(a)_q = N_0 a + (R\Gamma a) \cap (a\Gamma R)$ *where* N_0 *denotes the set of non negative integers.*

Theorem 2.14. *Intersection of a right ideal and a left ideal of* Γ*-semihyperring* R *is a quasi-ideal of* R*.*

Any quasi-ideal of Γ-semihyperring can be written as an intersection of a left ideal and a right ideal of Γ-semihyperring provided Γ-semihyperring has an identity element, see the following theorem.

Theorem 2.15. *If* Γ*-semihyperring* R *has an identity element* e *then every quasi-ideal of* Γ*-semihyperring* R *can be expressed as an intersection of a left ideal and a right ideal of* Γ*-semihyperring* R*.*

Proof. Let R be a Γ-semihyperring with an identity element e and Q be a quasi-ideal of Γ-semihyperring R. As R has an identity element then clearly $Q \subseteq (R\Gamma Q)$ and $Q \subseteq (Q\Gamma R)$. It gives $Q \subseteq (R\Gamma Q) \cap (Q\Gamma R)$. But Q is quasi-ideal of Γ-semihyperring R implies that $(R\Gamma Q) \cap (Q\Gamma R) \subseteq Q$. Therefore we get $Q = (R\Gamma Q) \cap (Q\Gamma R)$ where $R\Gamma Q$ and $Q\Gamma R$ are respectively the left and right ideals of Γ-semihyperring R. □

Theorem 2.16. *If* Q *is a quasi-ideal and* T *is a* Γ*-subsemihyperring of* R *then* $Q \cap T$ *is a quasi-ideal of* T*.*

In following theorems 2.17 to 2.19 we characterized quasi-ideals of idempotent Γ-semihyperring with the help of ideals of Γ-semihyperring.

Theorem 2.17. *Let* L *be a left ideal of idempotent* Γ*-semihyperring* R*. Then for any element* e *of* R*,* $e\Gamma L$ *is a quasi-ideal of* R*.*

Proof. First we show that $e\Gamma L = L \cap (e\Gamma R)$. As $e \in R$ and L is a left ideal of R we have $e\Gamma L \subseteq L$ and $e\Gamma L \subseteq e\Gamma R$ implies $e\Gamma L \subseteq L \cap (e\Gamma R)$. Now let x be an element of $L \cap (e\Gamma R)$. Hence we have $x \in L$ and $x \in e\alpha r$ for some $\alpha \in \Gamma$ and $r \in R$. As x is an idempotent element and $x \in e\alpha r$ gives $x \in e\alpha e\alpha r\beta x \subseteq e\alpha L$ since L is a left ideal and $x \in L$. This shows that $L \cap (e\Gamma R) \subseteq e\Gamma L$. Hence combining both the inclusions we get $e\Gamma L = L \cap (e\Gamma R)$. As L is a left ideal and $e\Gamma R$ is a right ideal of R, we get $L \cap (e\Gamma R)$ is a quasi-ideal of R by Theorem 2.14. Therefore $e\Gamma L$ is a quasi-ideal of R. □

Theorem 2.18. *Let* R *be a right ideal of idempotent* Γ*-semihyperring* R*. Then for any element* e *of* R*,* $R\Gamma e$ *is a quasi-ideal of* R*.*

Theorem 2.19. *Let R be idempotent Γ-semihyperring. Then for any elements $e, f \in R$, $e\Gamma R\Gamma f$ is a quasi-ideal of R.*

Proof. First we show that $e\Gamma R\Gamma f = (e\Gamma R) \cap (R\Gamma f)$. As $e\Gamma R$ is a right ideal $e\Gamma R\Gamma f = (e\Gamma R)\Gamma f \subseteq e\Gamma R$ and $e\Gamma R\Gamma f = e\Gamma(R\Gamma f) \subseteq R\Gamma f$ since $R\Gamma f$ is a left ideal. Therefore $e\Gamma R\Gamma f \subseteq (e\Gamma R) \cap (R\Gamma f)$. For reverse inclusion let $x \in (e\Gamma R) \cap (R\Gamma f)$. Then $x \in e\alpha a$ and $x \in b\beta f$ for some $a, b \in R, \alpha, \beta \in \Gamma$. Then clearly for $\gamma \in \Gamma, x \in e\alpha a\gamma b\beta f \subseteq e\Gamma R\Gamma f$. This shows that $(e\Gamma R) \cap (R\Gamma f) \subseteq e\Gamma R\Gamma f$. By combining both the inclusion we get $e\Gamma R\Gamma f = (e\Gamma R) \cap (R\Gamma f)$. As $R\Gamma f$ is a left ideal and $e\Gamma R$ is a right ideal of R we have $(e\Gamma R) \cap (R\Gamma f)$ is a quasi-ideal of R by theorem 2.14. Therefore, $e\Gamma R\Gamma f$ is a quasi-ideal of R. □

3. Quasi-Simple Γ-semihyperrings

In this section analogous to quasi-simple Γ-semigroup defined by Chinram in [6] the notion of a quasi-simple Γ-semihyperring is introduced and attempts have been made to prove some results. Further, quasi-simple Γ-semihyperring is characterized with the help of ideals and elements of a Γ-semihyperring.

Definition 3.1. A Γ-semihyperring R without zero is said to be a quasi-simple Γ-semihyperring if R is the only quasi-ideal of R.

A Γ-semihyperring R defined in example 2.3 is a quasi-simple Γ-semihyperring.

Definition 3.2. A Γ-semihyperring R with zero is said to be a 0-quasi-simple Γ-semihyperring if R has no proper nonzero quasi-ideal.

Example 3.3. Let $R = \mathbb{Q}, \Gamma = \{\gamma_\alpha | \alpha \in \mathbb{N}\}$ and $A_\alpha = \alpha\mathbb{Z}^+$, we define $x\alpha y \to xA_\alpha y, \alpha \in \Gamma$ and $x, y \in R$. Then R is a Γ-semihyperring under ordinary addition and multiplication. Here R is a 0-quasi-simple Γ-semihyperring.

Theorem 3.4. *Let R be a Γ-semihyperring without zero then R is a quasi-simple Γ-semihyperring if and only if $(R\Gamma a) \cap (a\Gamma R) = R$ for $a \in R$.*

Proof. Let R is a quasi-simple Γ-semihyperring and $a \in R$. As $R\Gamma a$ is a left ideal and $a\Gamma R$ is a right ideal of Γ-semihyperring R it gives $(R\Gamma a) \cap (a\Gamma R)$ is a quasi- ideal of R by theorem 2.14. Since R is a quasi-simple Γ-semihyperring we get $(R\Gamma a) \cap (a\Gamma R) = R$. Conversely suppose that $(R\Gamma a) \cap (a\Gamma R) = R$, for $a \in R$ and Q is a any quasi-ideal of Γ-semihyperring R. For any $q \in Q$ we have $R = (R\Gamma q) \cap (q\Gamma R)$ by assumption. Hence $R = (R\Gamma q) \cap (q\Gamma R) \subseteq (R\Gamma Q) \cap (Q\Gamma R) \subseteq Q$ since $q \in Q$ and Q is a quasi-ideal of a Γ-semihyperring R. Therefore $R \subseteq Q$. Thus $R = Q$ as $Q \subseteq R$. Hence R is a quasi-simple Γ-semihyperring. □

Theorem 3.5. *Let R be a Γ-semihyperring without zero then R is a quasi-simple Γ-semihyperring if and only if $(a)_q = R$, for $a \in R$.*

Proof. Let R be a Γ-semihyperring without zero. If we consider R as a quasi-simple Γ-semihyperring then clearly $(a)_q = R$, for $a \in R$. Conversely suppose that $(a)_q = R$, for $a \in R$ and let Q be a quasi-ideal of R. Then for any $a \in Q$ we have $(a)_q = R$ by assumption. $R = (a)_q \subseteq Q$. Therefore $Q = R$, since $Q \subseteq R$ and $R \subseteq Q$. □

In the following theorem it is proved that if R is a Γ-semihyperring without zero and intersection of a quasi-simple Γ-semihyperring $R_1 \subseteq R$ with a quasi-ideal Q of R is non empty, then $R_1 \subseteq Q$.

Theorem 3.6. *Let R be a Γ-semihyperring without zero, Q be a quasi-ideal of R and R_1 be a Γ-subsemihyperring of R. If R_1 is quasi-simple with $R_1 \cap Q \neq \phi$ then $R_1 \subseteq Q$.*

Proof. Let R_1 be a quasi-simple Γ-subsemihyperring with $R_1 \cap Q \neq \phi$ and $a \in R_1 \cap Q$. Now $a \in R_1$ and R_1 is quasi-simple gives $(R_1\Gamma a) \cap (a\Gamma R_1) = R_1$ by theorem 3.4. Therefore we have $R_1 = (R_1\Gamma a) \cap (a\Gamma R_1) \subseteq (R_1\Gamma Q) \cap (Q\Gamma R_1) \subseteq (R\Gamma Q) \cap (Q\Gamma R) \subseteq Q$ since Q is a quasi-ideal. Thus we get $R_1 \subseteq Q$. □

Theorem 3.7. *Let R be a Γ-semihyperring with zero. If R is 0-quasi-simple Γ- semihyperring with identity element e then $(R\Gamma a) \cap (a\Gamma R) = R$, for $a \in R - \{0\}$.*

Theorem 3.8. *Let Q be a quasi-ideal and R_1 be a Γ-subsemihyperring of a Γ-semihyperring R with zero and containing an identity element e. If R_1 is 0-quasi-simple with $R_1 - \{0\} \cap Q \neq \phi$ then $R_1 \subseteq Q$.*

4. Minimal Quasi-Ideals in Γ-semihyperrings

Here minimal quasi-ideals for Γ-semihyperrings is introduced and it is characterized with the help of minimal ideals of Γ-semihyperring. Also relationship between quasi-simple Γ-semihyperring and minimal quasi-ideal of Γ-semihyperring is established.

Definition 4.1. A quasi-ideal Q of Γ-semihyperring R is said to be a minimal quasi-ideal of Γ-semihyperring R if Q does not contain any other proper quasi-ideal of Γ-semihyperring R.

In example 2.2, $Q = \{a, b\}$ is quasi-ideal of Γ-semihyperring R and it does not contain any proper quasi-ideal. Therefore $\{a, b\}$ is a minimal quasi-ideal of Γ-semihyperring R.

Theorem 4.2. *The intersection of a minimal right ideal and a minimal left ideal of Γ-semihyperring R is a minimal quasi-ideal of a Γ-semihyperring R.*

Proof. Let A be a minimal left ideal and B be a minimal right ideal of a Γ-semihyperring R. Then by theorem 2.14, $Q = A \cap B$ is a quasi-ideal of a Γ-semihyperring R. To show that Q is minimal quasi-ideal assume $Q_1 \subseteq Q$, where Q_1 is a quasi-ideal of a Γ-semihyperring R. Since $Q_1 \subseteq Q \subseteq A$ and A is left ideal we get $R\Gamma Q_1 \subseteq R\Gamma Q \subseteq R\Gamma A \subseteq A$. Similarly, we can show $Q_1\Gamma R \subseteq B$. But $R\Gamma Q_1$ is a left ideal and $Q_1\Gamma R$ is a right ideal of Γ-semihyperrring gives $R\Gamma Q_1 = A$ and $Q_1\Gamma R = B$ since A is a minimal left ideal and B is a minimal right ideal of Γ-semihyperrring R. Also, $Q = B \cap A = (R\Gamma Q_1) \cap (Q_1\Gamma R) \subseteq Q_1$ as Q_1 is a quasi-ideal. Therefore we get $Q = Q_1$ since $Q \subseteq Q_1$ and $Q_1 \subseteq Q$. This give that Q is a minimal quasi-ideal of a Γ-semihyperrring R. □

Theorem 4.3. *Every minimal quasi-ideal Q of a Γ-semihyperrring R is represented as $Q = (R\Gamma a) \cap (a\Gamma R)$ where a is any element of Q. Also $R\Gamma a$ is a minimal left ideal and $a\Gamma R$ is a minimal right ideal of a Γ-semihyperrring R.*

Proof. Let Q be a minimal quasi-ideal of a Γ-semihyperrring R and $a \in Q$. As $R\Gamma a$ is a left ideal and $a\Gamma R$ is a right ideal of a Γ-semihyperrring R, by theorem 2.14, $(R\Gamma a)\cap(a\Gamma R)$ is a quasi-ideal of a Γ-semihyperrring R. Also, $(R\Gamma a) \cap (a\Gamma R) \subseteq (R\Gamma Q) \cap (Q\Gamma R) \subseteq Q$ since $a \in Q$ and Q is a quasi-ideal of a Γ-semihyperrring R. As Q is minimal quasi-ideal of a Γ-semihyperrring R we get $Q = (R\Gamma a) \cap (a\Gamma R)$

Now to show $R\Gamma a$ is minimal left ideal of a Γ-semihyperrring R, suppose A is a left ideal of a Γ-semihyperrring R such that $A \subseteq R\Gamma a$. Therefore $(R\Gamma A) \subseteq R\Gamma(R\Gamma a) = (R\Gamma R)\Gamma a \subseteq R\Gamma a$. Hence $(R\Gamma A) \cap (a\Gamma R) \subseteq (R\Gamma a) \cap (a\Gamma R) = Q$. But $R\Gamma A$ is a left ideal and $a\Gamma R$ is a right ideal of a Γ-semihyperrring R. Therefore by theorem 2.14, we have $(R\Gamma A) \cap (a\Gamma R)$ is quasi-ideal of a Γ-semihyperrring R. So we get $Q = (R\Gamma A) \cap (a\Gamma R) \subseteq (R\Gamma A)$ since Q is a minimal quasi ideal of a Γ-semihyperrring R.

Now,

$$\begin{aligned} R\Gamma a &\subseteq (R\Gamma Q) && \text{(since } a \in Q) \\ &\subseteq R\Gamma(R\Gamma A) && \text{(since } Q \subseteq R\Gamma A) \\ &= (R\Gamma R)\Gamma A \\ &\subseteq R\Gamma A \end{aligned}$$

Therefore $R\Gamma a \subseteq R\Gamma A \subseteq A$ since A is a left ideal of R. As $R\Gamma a \subseteq A$ and $A \subseteq R\Gamma a$ implies $R\Gamma a = A$. This shows that $R\Gamma a$ is a minimal left ideal of R. On the similar lines we can show that $a\Gamma R$ is a minimal right ideal of R. □

Theorem 4.4. *Let Q be a quasi-ideal of a Γ-semihyperring R. If Q itself is a quasi-simple Γ-semihyperring then Q is a minimal quasi-ideal of R.*

Theorem 4.5. *A proper quasi-ideal of a Γ-semihyperring R is minimal if and only if the intersection of any distinct proper quasi-ideals is empty.*

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