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DERIVED METABELIAN GROUPS FROM H_v-GROUPS

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Abstract. In this paper first we introduce and analyze a new definition of left and right commutators in H_v-group. Secondly, using commutators we introduce a new strongly equivalence relation π^* on an H_v-group H such that the quotient H/π^*, the set of all equivalence classes, is a metabelian group. Then we introduce metabelian H_v-groups and investigate some of their properties. Finally, we investigate some properties of commutators for the class of weak polygroups.

Key Words: H_v-group, metabelian group, metabelian H_v-group, weak polygroup.
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1. Introduction

In [12] Vougioklis introduced the notion of H_v-groups as a generalization of the notion of hypergroups. In general, the motivation for H_v-group is the following: we know that the quotient of a group with respect to a normal subgroup is a group. In 1934 F. Marty states that, the quotient of a group with respect to any subgroup is a hypergroup (see [9]). Vougioklis states that the quotient of a group with respect to any partition is an H_v-group. Since then the study of H_v-structures has been continued in many directions by T. Vougiouklis, B. Davvaz, S. Spartalis, A. Dramalidis, S. Hoskova, and some other mathematicians. We invite the readers for more study about hyperstructures theory and

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its applications to [2, 3, 4, 5, 11]. The first fundamental relation defined on hypergroups is the β^* -relation, introduced by Koskas [8] in 1970, in connection with the heart of a hypergroup and studied mainly by Corsini, Davvaz, Freni, Leoreanu, Vougiouklis. Later on, Freni [7] introduced the γ-relation on a hypergroup, as a generalization of the relation β, proving that γ^* is the smallest regular relation on a semihypergroup such that the corrisponding quotient is a commutative semigroup. In the class of hyperrings, several fundamental relations have been defined till now with respect to both (hyper)operations (addition and multiplication) for example see [6, 10, 14]. H. Aghabozorgi et.al., introduced and analyzed the notions of left and right commutators in polygroups and they provided a detailed structure description of derived subpolygroups of polygroups. In this paper we generalize the notions of left and right commutators in H_v-groups. Then using commutators we introduce a new strongly equivalence relation π^* on an H_v-group H such that the quotient H/π^* is a metabelian group. Moreover, we introduce metabelian H_v-groups and investigate some of their properties. Finally, we investigate some properties of commutators for the class of weak polygroups. In the following we recall some basic notions of H_v-group theory.

Let H be a non-empty set and $P^*(H)$ be the set of all non-empty subsets of H. Let $\cdot$ be a *hyperoperation* (or *join operation*) on H, that is, $\cdot$ is a function from $H \times H$ into $P^*(H)$. If $(a, b) \in H \times H$, its image under $\cdot$ in $P^*(H)$ is denoted by $a \cdot b$. The join operation is extended to subsets of H in a natural way, that is, for non-empty subsets A, B of H, $A \cdot B = \cup\{a \cdot b \mid a \in A, b \in B\}$. The notation $a \cdot A$ is used for $\{a\} \cdot A$ and $A \cdot a$ for $A \cdot \{a\}$. Generally, the singleton $\{a\}$ is identified with its member a. The structure $(H, \cdot)$ is called an H_v-group if $a \cdot (b \cdot c) \cap (a \cdot b) \cdot c \neq \emptyset$, for all $a, b, c \in H$, which means that

$$(\bigcup_{u \in a \cdot b} u \cdot c) \cap (\bigcup_{v \in b \cdot c} a \cdot v) \neq \emptyset$$

and $a \cdot H = H \cdot a = H$ for all $a \in H$. A non-empty subset K of an H_v-group $(H, \cdot)$ is called a H_v-subgroup if it is an H_v-group. Suppose that $(H, \cdot)$ and $(K, \circ)$ are two H_v-group. A function $f : H \longrightarrow K$ is called a *homomorphism* if $f(a \cdot b) \subseteq f(a) \circ f(b)$ for all a and b in H. We say that f is a *good homomorphism* if for all a and b in H, $f(a \cdot b) = f(a) \circ f(b)$. If $(H, \cdot)$ is an H_v-group and $\rho \subseteq H \times H$ is an equivalence, we set

$$A \overline{\overline{\rho}} B \Leftrightarrow a \, \rho \, b, \quad \forall a \in A, \forall b \in B,$$

for all pairs (A, B) of non-empty subsets of H.

The relation ρ is called *strongly regular on the left(on the right)* if $x \,\rho\, y \Rightarrow a \cdot x \,\overline{\overline{\rho}}\, a \cdot y$ ($x \,\rho\, y \Rightarrow x \cdot a \,\overline{\overline{\rho}}\, y \cdot a$, respectively), for all $(x, y, a) \in H^3$. Moreover, ρ is called *strongly regular* if it is strongly regular on the right and on the left.

Theorem 1.1. *If $(H, \cdot)$ is an H_v-group and ρ is a strongly regular relation on H, then the quotient H/ρ is a group under the operation:*

$$\rho(x) \otimes \rho(y) = \rho(z), \text{ for all } z \in x \cdot y.$$

We denote $\rho(x)$ by $\bar{x}$ and instead of $\bar{x} \otimes \bar{y}$ we write $\bar{x}\bar{y}$. Let $(H, \cdot)$ be an H_v-group and $\mathcal{U}$ be the set of all finite products of elements of H. For all $n > 1$, we define the relation β_n on H, as follows:

$$a \,\beta_n\, b \Leftrightarrow \exists u \in \mathcal{U} : \{a, b\} \subseteq u,$$

and $\beta = \bigcup_{i=1}^{n} \beta_n$, where $\beta_1 = \{(x, x) \mid x \in H\}$ is the diagonal relation on H. Note that, in general, for an H_v-group may be $\beta \neq \beta^*$, where β^* is the transitive closure of β. The relation β^* is the smallest equivalence relation on an H_v-group H, such that the quotient H/β^* is a group.

Definition 1.2. Let H be a weak H_v-group and X be a non-empty subset of H. We define $Ass(X) = \{(x_1, x_2) \in H^2 | (x_{\sigma(1)} \cdot x_{\sigma(2)}) \cdot x_{\sigma(3)} = x_{\sigma(1)} \cdot (x_{\sigma(2)} \cdot x_{\sigma(3)}), \forall x_3 \in X, \forall \sigma \in S_3\}$. Moreover, if $Ass(X) = H \times H$, we say that X is a full associative subset of H.

Remark 1.3. An H_v-group H is called a hypergroup if and only if $Ass(H) = H^2$.

2. On the strongly regular relation π^*

In this section, we introduce and analyze a new definition of left and right commutators in H_v-group. Using commutators we introduce a new strongly equivalence relation π^* on an H_v-group H such that the set of all equivalence classes; $\pi^*(x), x \in H$ is a metabelian group. Moreover, we introduce metabelian H_v-groups and invetigate some of their properties.

Definition 2.1. Let $(H, \cdot)$ be an H_v-group and $(x, y) \in H^2$. We define

(1) $[x, y]_r = \{h \in H \mid \ x \cdot y \cap (y \cdot x) \cdot h \cap y \cdot (x \cdot h) \neq \emptyset\}$;
(2) $[x, y]_l = \{h \in H \mid \ x \cdot y \cap h \cdot (y \cdot x) \cap (h \cdot y) \cdot x \neq \emptyset\}$;
(3) $[x, y] = [x, y]_r \cup [x, y]_l$.

From now on we call $[x,y]_r$, $[x,y]_l$ and $[x,y]$ *right commutator x and y, left commutator x and y* and *commutator x and y* in H, respectively. Also we will denote $[H,H]_r$, $[H,H]_l$ and $[H,H]$ the sets of all right commutators, left commutators and commutators in H, respectively.

Example 2.2. Suppose that $H = \{0,1,2\}$. Consider the H_v-group $(H,\cdot)$, where $\cdot$ is defined on H as follows:

$\cdot$	0	1	2
0	1, 2	0, 1	0
1	0, 2	1	2
2	0	1, 2	1, 2

We can see that $\{1\} = [1,1]_r \neq [1,1]_l = H$.

Example 2.3. Suppose that $H = \{0,1,2,3\}$. Consider the commutative H_v-group $(H,\cdot)$, where $\cdot$ is defined on H as follows:

$\cdot$	0	1	2	3
0	0	1	0, 2	3
1	1	2	3	0
2	0, 2	3	0	1
3	3	0	1	0, 2

In this case we have $[1,3]_r = \{0,2\}$ and $[1,3]_l = \{0\}$ and so $[1,3]_r \neq [1,3]_l$.

Notice that the above example is a commutative H_v-group while $[x,y]_r \neq [x,y]_l \neq [x,y]$, for some $(x,y) \in H^2$.

Proposition 2.4. *Let $(H,\cdot)$ be a commutative H_v-group such that $(x \cdot y) \cdot z \subseteq x \cdot (y \cdot z)$, for all $(x,y,z) \in H^3$. Then $[y,x]_r = [x,y]_r = [x,y]_l = [y,x]_l = [y,x] = [x,y]$, for all $(x,y) \in H^2$.*

Proof. The proof is strightforward. □

Corollary 2.5. *Let $(H,\cdot)$ be a commutative hypergroup. Then, $[y,x]_r = [x,y]_r = [x,y]_l = [y,x]_l = [y,x] = [x,y]$, for all $(x,y) \in H^2$.*

Let $(H,\cdot)$ be an H_v-group, $n \in \mathbb{N}$ and $(x_1,...,x_n) \in H^n$. We mean by $\mathcal{F}(x_1,...,x_n)$ the set of all finite possible products of $x_1,...,x_n$, respectively. For example $\mathcal{F}(x_1,x_2) = \{x_1 \cdot x_2\}$, $\mathcal{F}(x_1,x_2,x_3) = \{x_1 \cdot (x_2 \cdot x_3), (x_1 \cdot x_2) \cdot x_3\}$ and $\mathcal{F}(x_1,x_2,x_3,x_4) = \{(x_1 \cdot (x_2 \cdot x_3)) \cdot x_4, ((x_1 \cdot x_2) \cdot x_3) \cdot x_4, x_1 \cdot (x_2 \cdot (x_3 \cdot x_4)), x_1 \cdot ((x_2 \cdot x_3) \cdot x_4), (x_1 \cdot x_2) \cdot (x_3 \cdot x_4)\}$. Thus we have $\mathcal{U} = \bigcup_{(x_1,...,x_n) \in H^n} \mathcal{F}(x_1,...,x_n)$. Moreover, suppose that $u \in \mathcal{F}(x_1,...,x_n)$ if and only if $u_\sigma \in \mathcal{F}(x_{\sigma(1)},...,x_{\sigma(n)})$, for every $\sigma \in S_n$.

Definition 2.6. Let H be an H_v-group. Suppose that $\pi = \bigcup_{m\geq 1} \pi_m$, where π_1 is the diagonal relation and for every integer $m > 1$, π_m is the relation defined as follows:

$$x \; \pi_m \; y \Leftrightarrow \exists u \in \mathcal{F}(x_1, ..., x_m), \exists \sigma \in S_m : \sigma(i) = i \;\; \text{if } x_i \notin \cup[H, H] \text{ such that } x \in u \quad \text{and} \quad y \in u_\sigma.$$

Obviously, the relation π is reflexive and symmetric. Now, let π^* be the transitive closure of π.

Theorem 2.7. *Let $(H, \cdot)$ be an H_v-group. The relation π^* is a strongly regular relation.*

Proof. We know that π^* is an equivalence relation. In order to prove that it is strongly regular, first we have to show that:

$$x\pi y \Rightarrow x \cdot z \; \overset{=}{\pi^*} \; y \cdot z, \;\; z \cdot x \; \overset{=}{\pi^*} \; z \cdot y, \tag{2.1}$$

for every $z \in H$. Suppose that $x\pi y$. Then, there exists $m \in \mathbb{N}$ such that $x\pi_m y$. Hence, $\exists u \in \mathcal{F}(x_1, ..., x_m), \exists \sigma \in S_m : \sigma(i) = i$ if $x_i \notin \cup[H, H]$ such that $x \in u$ and $y \in u_\sigma$.

Suppose that $z \in H$. We have $x \cdot z \subseteq (u) \cdot z$, $y \cdot z \subseteq (u_\sigma) \cdot z$ and $\sigma(i) = i$ if $x_i \notin \cup[H, H]$. Now, suppose that $x_{m+1} = z$ and we define the permutation $\sigma' \in S_{m+1}$ as follows:

$$\sigma'(i) = \sigma(i), \text{ for all } 1 \leq i \leq m \text{ and } \sigma'(m + 1) = m + 1.$$

Now let $u' = (u) \cdot z \in \mathcal{F}(x_1, ..., x_{m+1})$. Thus, $x \cdot z \subseteq u'$ and $y \cdot z \subseteq u'_{\sigma'}$ such that $\sigma'(i) = i$ if $x_i \notin \cup[H, H]$. Therefore, $x \cdot z \; \overset{=}{\pi^*} \; y \cdot z$. Similarly, we have $z \cdot x \; \overset{=}{\pi^*} \; z \cdot y$. Now, if $x\pi^* y$ then there exists $k \in \mathbb{N}$ and $(x = u_0, u_1, \ldots, u_k = y) \in H^{k+1}$ such that $x = u_0 \pi u_1 \pi \ldots \pi u_{k-1} \pi u_k = y$. Hence, by the above results, we obtain

$$x \cdot z = u_0 \cdot z \; \overset{=}{\pi^*} \; u_1 \cdot z \; \overset{=}{\pi^*} \; u_2 \cdot z \; \overset{=}{\pi^*} \; \ldots \; \overset{=}{\pi^*} \; u_{k-1} \cdot z \; \overset{=}{\pi^*} \; u_k \cdot z = y \cdot z$$

and so $x \cdot z \; \overset{=}{\pi^*} \; y \cdot z$.

Similarly, we can prove that $z \cdot x \; \overset{=}{\pi^*} \; z \cdot y$. Therefore, π^* is a strongly regular relation on H. $\square$

If H is an H_v-group we denote $Met(H) = \bigcup_{x\in\cup[H,H], y\in H} [x, y]$.

Definition 2.8. The H_v-group H is called metabelian if and only if $Met(H) = \omega_H$, where ω_H is the kernel of the canonical homomorphism $\varphi_H : H \longrightarrow H/\beta^*$; i.e. $\omega_H = \varphi^{-1}(1_{H/\beta^*})$.

Example 2.9. Suppose that $H = \{0, 1, 2, 3\}$. Consider the non-commutative H_v-group $(H, \cdot)$, where $\cdot$ is defined on H as follows:

$\cdot$	0	1	2	3
0	0	1, 2	2	3
1	1	1	H	3
2	2	0, 1, 2	2	2, 3
3	3	1, 3	3	H

In this case we can see that $\cup[H, H] = \omega_H = H$, and so $Met(H) = \omega_H$ which means that H is a metabelian H_v-group.

Remark 2.10. If G is a group then G is a metabelian group if and only if $[[x, y], z] = 1_G$, for every $(x, y, z) \in G^3$.

Theorem 2.11. *Let $(H, \cdot)$ be an H_v-group. Then,*

(i) *H/π^* is a metabelian group;*

(ii) *π is the smallest eqivalent relation such that H/π^* is a metabelian group.*

Proof. (i). According toTheorem 1.1 H/π^* is a group. Now let $(\bar{x}, \bar{y}, \bar{z}) \in (H/\pi^*)^3$. We shall prove that $[[\bar{x}, \bar{y}], \bar{z}] = 1_{H/\pi^*}$. To do this suppose that $\bar{a} = [\bar{x}, \bar{y}]$. Without lossing the generality we have $x \cdot y \cap (y \cdot x) \cdot a \neq \emptyset$ and so $a \in [H, H]$. Therefore $a \cdot z\pi^* z \cdot a$ so $\bar{a}\bar{z} = \bar{z}\bar{a}$. Consequently $[[\bar{x}, \bar{y}], \bar{z}] = 1_{H/\pi^*}$, which means that $Met(H/\pi^*) = \{1_{H/\pi^*}\} = \omega_{H/\pi^*}$.
(ii). Suppose that ρ is a strongly regular relation on H such that H/ρ is a metabelian group. Now let $a\pi b$ so there exists $u \in \mathcal{F}(x_1, ..., x_m)$, and there exist $\sigma \in S_m$ such that $\sigma(i) = i$ if $x_i \notin \cup[H, H]$ and $x \in u$ and $y \in u_\sigma$. If $x_i \in [H, H]$ then there exists $(s, t) \in H^2$ such that $s \cdot t \cap (t \cdot s) \cdot x_i \cap t \cdot (s \cdot x_i) \neq \emptyset$ or $s \cdot t \cap x_i \cdot (t \cdot s) \cap (x_i \cdot t) \cdot s \neq \emptyset$. Therefore $\rho(x_i) = [\rho(s), \rho(t)]$ or $\rho(x_i) = [\rho(s), \rho(t)]^{-1}$ and so $\rho(x_i)$ commutes with all elements of H/π^*. Hence $\rho(a) = \rho(b)$ and so $\pi \subseteq \rho$. Consequently $\pi^* \subseteq \rho$ holds. $\square$

Proposition 2.12. *The H_v-group H is metabelian if and only if $\beta^* = \pi^*$.*

Proof. Suppose that H is a metabelian H_v-group. We need to prove that H/β^* is a metabelian group. Let $a \in \cup[H, H]$ and $x \in H$. If $y \in [a, x]$ then $y \in \omega_H$. Also we have $a \cdot x \cap (x \cdot a) \cdot y \cap x \cdot (a \cdot y) \neq \emptyset$. or $a \cdot x \cap y \cdot (x \cdot a) \cap (y \cdot x) \cdot a \neq \emptyset$. Consequently $\beta^*(a)\beta^*(x) = \beta^*(x)\beta^*(a)$. Thus we have $\beta^* \supseteq \pi^*$ and so $\beta^* = \pi^*$. Conversely suppose that $\beta^* = \pi^*$ we shall prove that $Met(H) = \bigcup_{x \in \cup[H,H], y \in H} [x, y] \subseteq \omega_H$. To do this suppose that $a \in \cup[H,H]$ and $x \in H$. If $y \in [a,x]_r$ then $a\cdot x \cap (x\cdot a)\cdot y \cap x\cdot(a\cdot y) \neq \emptyset$. Because $\pi^*(a)\pi^*(x) = \pi^*(x)\pi^*(a)$ we conclude that $\pi^*(y) = \beta^*(y) = 1_{H/\beta^*}$. Hence $y \in \omega_H$. Similarly if $y \in [a,x]_r$ we have a similar result. □

Let $(H,\cdot)$ be an H_v-group. Then, we define $Z_v(H) = \{h | x\cdot h \cap h\cdot x \neq \emptyset, \forall x \in H\}$ and we call it the weak center of H.

Theorem 2.13. *Let $(H,\cdot)$ be an H_v-group and $\cup[H,H] \subseteq Z_v(H)$. Then, H is metabelian.*

Proof. Suppose that $(H,\cdot)$ be an H_v-group and $\cup[H,H] \subseteq Z_v(H)$, we need to prove that $\pi \subseteq \beta^*$. To do this suppose that $a\pi b$ so there exists $u \in \mathcal{F}(x_1,...,x_m)$, and there exists $\sigma \in S_m$ such that $\sigma(i) = i$ if $x_i \notin [H,H]$ and $x \in u$ and $y \in u_\sigma$. By induction on m we show that $u \cap u_\sigma \neq \emptyset$. Because $\cup[H,H] \subseteq Z_v(H)$, for $m = 2$ it is obvious. Now let it is true for all $k < m$. Because $u = u'\cdot u''$ and $u_\sigma = u'_{\sigma_1}\cdot u''_{\sigma_2}$, where $u' \in \mathcal{F}(x_1,...,x_k)$, and $u'' \in \mathcal{F}(x_{k+1},...,x_m)$, for some $k < m$ and $\sigma_1, \sigma_2 \in S_m$, we have $u' \cap u'_{\sigma_1} \neq \emptyset$ and $u'' \cap u''_{\sigma_2} \neq \emptyset$. Therefore $u \cap u_\sigma \neq \emptyset$ and so we have $a\beta^* b$. Consequently $\beta^* = \pi^*$. □

Let $(G,\cdot)$ be a group and R be an equivalence relation on G. In $\bar{G} = G/R$ consider the hyperoperation $\odot$ defined by $\bar{x} \odot \bar{y} = \{\bar{z} | z \in \bar{x}\cdot\bar{y}\}$, where $\bar{x}$ denotes the equivalence class of the element x. Then, $(\bar{G},\odot)$ is an H_v-group which is not necessary a hypergroup.

Proposition 2.14. *Let $(G,\cdot)$ be a group and R be an equivalence relation on G and $(\bar{G},\odot)$ be the associated H_v-group. Then,*

$$\overline{[G,G]} = \{\bar{z} | \bar{z} \cap [G,G] \neq \emptyset\} \subseteq \cup[\bar{G},\bar{G}].$$

Proof. Let $\bar{a} \in \overline{[G,G]}$. Then, $\bar{a} \cap [G,G] \neq \emptyset$, thus there exist $g \in \bar{a}$ and $x, y \in G$ such that $g = x^{-1}\cdot y^{-1}\cdot x\cdot y$. We show that

$$\bar{x} \odot \bar{y} \cap (\bar{y} \odot \bar{x}) \odot \bar{a} \cap \bar{y} \odot (\bar{x} \odot \bar{a}) \neq \emptyset.$$

We have $y\cdot x \in \bar{y}\cdot\bar{x}$, then $\overline{y\cdot x} \in \bar{y} \odot \bar{x}$ and so we have

$$x\cdot y = (y\cdot x)\cdot x^{-1}\cdot y^{-1}\cdot x\cdot y = (y\cdot x)\cdot g \in \overline{y\cdot x}\cdot\bar{a}.$$

Therefore $\overline{x\cdot y} \in (\bar{y} \odot \bar{x}) \odot \bar{a}$. Also, $x \in \bar{x}, y \in \bar{y}$, then $x\cdot y \in \bar{x}\cdot\bar{y}$, so $\overline{x\cdot y} \in \bar{x} \odot \bar{y}$. Thus $\overline{x\cdot y} \in \bar{x} \odot \bar{y} \cap (\bar{y} \odot \bar{x}) \odot \bar{a}$. On the other hand $y^{-1}\cdot x\cdot y = x\cdot x^{-1}\cdot y^{-1}\cdot x\cdot y = x\cdot g \in \bar{x}\cdot\bar{a}$, then $\overline{y^{-1}\cdot x\cdot y} \in \bar{x} \odot \bar{a}$. So $x\cdot y = y\cdot y^{-1}\cdot x\cdot y \in \bar{y}\cdot\overline{y^{-1}\cdot x\cdot y} \subseteq \bar{y}\cdot(\bar{x} \odot \bar{a})$. Thus $\overline{x\cdot y} \in \bar{y} \odot (\bar{x} \odot \bar{a})$. Therefore

$$\overline{x\cdot y} \in \bar{x} \odot \bar{y} \cap (\bar{y} \odot \bar{x}) \odot \bar{a} \cap \bar{y} \odot (\bar{x} \odot \bar{a}) \neq \emptyset.$$

Then $\bar{a} \in [x, y]_r$, so $\bar{a} \in \cup[\bar{G}, \bar{G}]$. $\square$

In the following example we show that in Proposition 2.14 the equality does not necessarily hold.

Example 2.15. Consider the group $G = \mathbb{Z}_6 = \{0, 1, 2, 3, 4, 5\}$. Let the partion $\{\{0, 1, 2\}, \{3, 4, 5\}\}$ of $\mathbb{Z}_6$. In this case we have $\bar{G} = G/R = \{\bar{0}, \bar{3}\}$, because

$$\bar{0} \odot \bar{3} = \{\bar{z} | z \in \{0, 1, 2\} \cdot \{3, 4, 5\}\} = \{\bar{0}, \bar{3}\}.$$

For all $a \in G$ we have

$$\bar{0} \odot \bar{3} \cap (\bar{0} \odot \bar{3}) \odot \bar{a} \cap \bar{3} \odot (\bar{0} \odot \bar{a}) \neq \emptyset.$$

Therefore, $\cup[\bar{G}, \bar{G}] = \bar{G} = \{\bar{0}, \bar{3}\}$, but $\bar{3} \cap [G, G] = \bar{3} \cap \{0\} = \emptyset$.

Definition 2.16. Let R be an equivalence relation on G. R is called good if and only if $\overline{[G, G]} \supseteq \cup[\bar{G}, \bar{G}]$.

Remark 2.17. If R is a good relation on $(G, \cdot)$, then by Proposition 2.14 we have $\overline{[G, G]} = \cup[\bar{G}, \bar{G}]$.

Proposition 2.18. *Let R be an equivalence relation on $(G, \cdot)$ such that for every $x \in G$ there exists $y \in [G, G]$ that $(x, y) \in R$. Then, R is a good relation.*

Proof. Suppose that $\bar{b} \in [\bar{G}, \bar{G}]$, then $\bar{b} = \bar{g}$, where $g \in [G, G]$. So $\bar{b} \cap [G, G] \neq \emptyset$. Therefore $b \in \overline{[G, G]}$. $\square$

Theorem 2.19. *Let $(G, \cdot)$ be a metabelian group and R be a good relation on G. Then $\bar{G}$ is a metabelian H_v-group.*

Proof. Let $\bar{a} \in \cup[\bar{G}, \bar{G}]$ and $\bar{x} \in \bar{G}$. we prove that $\bar{a} \odot \bar{x} \cap \bar{x} \odot \bar{a} \neq \emptyset$. To do this end, because R is a good relation we have $\bar{a} \in \overline{[G, G]}$. So there exists $g \in [G, G]$ such that $\bar{a} = \bar{g}$. Because G is a metabelian group we have $g \cdot x = x \cdot g$. Consequently $g \cdot x \in \bar{g} \cdot \bar{x} = \bar{a} \cdot \bar{x}$ and $x \cdot g \in \bar{x} \cdot \bar{g} = \bar{x} \cdot \bar{a}$, and so $\overline{g \cdot x} \in \bar{a} \odot \bar{x} \cap \bar{x} \odot \bar{a}$ which means $\bar{a} \odot \bar{x} \cap \bar{x} \odot \bar{a} \neq \emptyset$. So $\cup[\bar{G}, \bar{G}] \subseteq Z_v(\bar{G})$. Using Theorem 2.13 we conclude that $\bar{G}$ is a metabelian H_v-group. $\square$

3. On metableian weak polygroups

In this section we study the notion of commutators for the class of weak polygroups introduced in [5].

Definition 3.1. ([5]) The H_v-group P is called a *weak polygroup* and denoted by $\langle P, \cdot, e, ^{-1} \rangle$, where $^{-1} : P \to P$, $x \rightsquigarrow x^{-1}$ is a map, if the following conditions hold:

(1) P has a scalar identity e; (i.e., $e \cdot x = x \cdot e = x$, for every $x \in P$);
(2) $x \in y \cdot z$ implies $y \in x \cdot z^{-1}$ and $z \in y^{-1} \cdot x$.

Remark 3.2. A weak polygroup P is called a polygroup if and only if $Ass(P) = P^2$.

Definition 3.3. A non-empty subset K of a weak polygroup $\langle P, \cdot, e, ^{-1} \rangle$ is a weak subpolygroup of P if

(1) $x, y \in K$ implies $x \cdot y \subseteq K$;
(2) $x \in K$ implies $x^{-1} \in K$.

Definition 3.4. Let X be a non-empty subset of a weak polygroup $\langle P, \cdot, e, ^{-1} \rangle$. Let $\{A_i| \ \ i \in J\}$ be the family of all weak subpolygroups of P which contain X. Then $\cap_{i \in J} A_i$ is called the weak subpolygroup generated by X. This weak subpolygroup is denoted by $< X >$. If $X = \{x_1, x_2, ..., x_n\}$, then the weak subpolygroup $< X >$ is denoted by $< x_1, x_2, ..., x_n >$. In a special case $< \cup[P,P]_r >$, $< \cup[P,P]_l >$ and $< \cup[P,P] >$ are denoted by P'_r, P'_l, P', respectively.

Proposition 3.5. *Let $\langle P, \cdot, e, ^{-1} \rangle$ be a weak polygroup. Then, for all $(x, y) \in P^2$ we have*

(1) $[x,y]_r \subseteq (x^{-1} \cdot y^{-1}) \cdot (x \cdot y) \cap x^{-1} \cdot (y^{-1} \cdot (x \cdot y))$;
(2) $[x,y]_l \subseteq (x \cdot y) \cdot (x^{-1} \cdot y^{-1}) \cap ((x \cdot y) \cdot x^{-1}) \cdot y^{-1}$.

Proposition 3.6. *Let $\langle P, \cdot, e, ^{-1} \rangle$ be a weak polygroup. Then, P is weak commutative if and only if $e \in [x,y]_r$ (resp. $e \in [x,y]_l$), for all $(x,y) \in P^2$.*

Proof. Let $e \in [x,y]_r$, for all $(x,y) \in P^2$. Then, $e \in (x^{-1} \cdot y^{-1}) \cdot (x \cdot y)$, for all $(x,y) \in P^2$ and so $x \cdot y \cap y \cdot x \neq \emptyset$ for all $(x,y) \in P^2$, that means P is weak commutative. The converse is obvious. $\square$

Proposition 3.7. *Let $\langle P, \cdot, e, ^{-1} \rangle$ be a weak polygroup that $x \cdot (y \cdot z) \subseteq (x \cdot y) \cdot z$, for all $(x,y,z) \in P^3$. Then, for all $(x,y) \in P^2$ we have*

(1) $[x,y]_r = [x^{-1}, y^{-1}]_l$;
(2) $P' = P'_r = P'_l$.

Proof. 1) Let $(x,y) \in P^2$ and $u \in [x,y]_r$. Then, $x \cdot y \cap (y \cdot x) \cdot u \cap y \cdot (x \cdot u) \neq \emptyset$ so there exists $t \in P$ such that $t \in x \cdot y \cap (y \cdot x) \cdot u \cap y \cdot (x \cdot u)$ thus $t \in x \cdot y \cap v \cdot u$ for some $v \in y \cdot x$. Since P is a weak polygroup then we have $v^{-1} \in x^{-1} \cdot y^{-1} \cap u \cdot (y^{-1} \cdot x^{-1}) \subseteq x^{-1} \cdot y^{-1} \cap u \cdot (y^{-1} \cdot x^{-1}) \cap (u \cdot y^{-1}) \cdot x^{-1}$ therefore $u \in [x,y]_l$ hence $[x,y]_r \subseteq [x^{-1}, y^{-1}]_l$. Similarly we have $[x^{-1}, y^{-1}]_l \subseteq [x,y]_r$.
2) Follows from (1). $\square$

Proposition 3.8. *Let $\langle P, \cdot, e, ^{-1} \rangle$ be a weak polygroup that $\cup[P, P]$ is a full associative subset of P. Then, for all $(x, y) \in P^2$ we have*

(1) $[x, y]_r = [x^{-1}, y^{-1}]_l$;
(2) $P' = P'_r = P'_l$.

Proof. 1) Let $(x, y) \in P^2$ and $u \in [x, y]_r$. Then, $x \cdot y \cap (y \cdot x) \cdot u \neq \emptyset$ so there exists $t \in P$ such that $t \in x \cdot y \cap (y \cdot x) \cdot u$ thus $t \in x \cdot y \cap v \cdot u$ for some $v \in y \cdot x$. Since P is a weak polygroup then we have $v^{-1} \in x^{-1} \cdot y^{-1} \cap u \cdot (y^{-1} \cdot x^{-1})$. Therefore $u \in [x, y]_l$ hence $[x, y]_r \subseteq [x^{-1}, y^{-1}]_l$. Similarly we have $[x^{-1}, y^{-1}]_l \subseteq [x, y]_r$.
2) Follows from (1). □

Corollary 3.9. *If P is a polygroup, then for all $(x, y) \in P^2$ we have*

(1) $[x, y]_r = [x^{-1}, y^{-1}]_l$;
(2) $P' = P'_r = P'_l$.

4. CONCLUSION

In this paper we introduce and analyze a generalization of the notion of commutators in H_v-groups. Several properties are investigated such as introduceing a new strongly equivalence relation π^* on an H_v-group H such that the quotient H/π^*, the set of all equivalence classes, is a metabelian group. This research can be continuated, for instance in study of some particular classes of H_v-groups.

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