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FUZZY N-FOLD OBSTINATE IDEALS IN MV-ALGEBRAS

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Abstract. In this paper, we introduce the notion of n-fold Boolean ideals of an MV-algebra and consider the quotient algebras induced by n-fold Boolean ideals. Also we prove that I is a n-fold Boolean ideal of an MV-algebra if and only if A/I is a n+1-bounded MV-algebra if and only if A/I is a subdirect product of algebras $Ł_k$, with $2 \leq k \leq n$.
Finally, we introduce the notion of fuzzy n-fold obstinate ideals in MV-algebras. We give some characterizations of fuzzy n-fold obstinate ideals.

Key Words: n-fold Boolean ideal, n-bounded MV-algebra, fuzzy n-fold obstinate.
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1. Introduction

Chang invented MV-algebras in order to give an algebraic proof of the completeness theorem of the infinite-valued logic of Łukasiewicz [2, 3]. Also, Mundici in the study of AFC^*-algebras, shows that category of MV-algebras is equivalent to category of lu-groups [17].

Ideal theory is a major tool in an MV-algebra, i.e., a certain type of ideals is useful to characterize an MV-algebra.

Then these classes of algebras have been intensively studied by many researcher. In particular, emphasis has been put an the filter theory of BL-algebras [12]. In [13] and [16] the authors defined the notion of n-fold Boolaen filters, n-fold fanstastic filters, n-fold normal filters in

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BL-algebras and studied the relations among many types n-fold filters in BL-algebras.
MV-algebras as well as BL-algebras are important logical algebras. This motivates us to study the notion of n-fold Boolean ideals in MV-algebras. Hence we introduce concept of n-fold Boolean ideals MV-algebra and we state and prove the extension theorem of this n-fold ideals and several characterizations of n-fold Boolean ideals of MV-algebras.

In addition, n-fold Boolean ideals are important, because we prove that the quotient algebras induced by n-fold Boolean ideals are n+1-bounded MV-algebras that are generated by all finite Łukasiewicz chains of n + 1 elements or less.

Also, several characterizations of this n-fold ideals are given. We prove that every semi-maximal ideal of an MV-algebra is a n-fold Boolean ideal but the inverse this theorem is not true in general.

The concept of fuzzy set was introduced by Zadeh (1965) [20]. This idea has been applied to other algebraic structures such as semi-group, group, ideals, modules and topologies. In 1991, Xi [19] applied the concept of fuzzy sets to BCK-algebras and proposed the notion of fuzzy implicative ideals. Afterwards, Hoo [11] proved that fuzzy implicative and fuzzy Boolean ideals are equivalent in MV-algebras.
In [9] results regarding fuzzy obstinate ideals of MV-algebras were obtained.

We introduce the notion of fuzzy n-fold obstinate ideals in MV-algebras. We give some characterizations of fuzzy n-fold obstinate ideals and establish the extension theorem of this class of ideals and study some properties of them.

Definition 1.1. [2] An MV-algebra is a structure $(A, \oplus, {}^*, 0)$ where $\oplus$ is a binary operation, *, is a unary operation, and 0 is a constant such that the following axioms are satisfied for any $a, b \in A$:
$(MV1)$ $(A, \oplus, 0)$ is an abelian monoid,
$(MV2)$ $(a^*)^* = a$,
$(MV3)$ $0^* \oplus a = 0^*$,
$(MV4)$ $(a^* \oplus b)^* \oplus b = (b^* \oplus a)^* \oplus a$.

Note that we define $1 = 0^*$ and the auxiliary operation $\odot$ as follow:

$$x \odot y = (x^* \oplus y^*)^*.$$

We say that the element $x \in A$ has order n and we write $ord(x) = n$, if n is the smallest natural number such that $nx = 1$. We say that the element x has a finite order, and write $ord(x) < \infty$. An MV-algebra

A is locally finite if every non-zero element of A has finite order. We recall that the natural order determines a bounded distributive lattice structure such that

$$x \vee y = x \oplus (x^* \odot y) = y \oplus (x \odot y^*) \quad \text{and} \quad x \wedge y = x \odot (x^* \oplus y) = y \odot (y^* \oplus x).$$

Lemma 1.2. [4] *In each MV-algebra, the following relations hold for all $x, y, z \in A$:*
(1) $x \leq y$ *if and only if* $y^* \leq x^*$,
(2) *If* $x \leq y$, *then* $x \oplus z \leq y \oplus z$ *and* $x \odot z \leq y \odot z$,
(3) $x \leq y$ *if and only if* $x^* \oplus y = 1$ *if and if* $x \odot y^* = 0$,
(4) $x, y \leq x \oplus y$ *and* $x \odot y \leq x, y$, $x \leq nx = x \oplus x \oplus \cdots \oplus x$ *and* $x^n = x \odot x \odot \cdots \odot x \leq x$,
(5) $x \oplus x^* = 1$ *and* $x \odot x^* = 0$,
(6) *If* $x \in B(A)$, *then* $x \wedge y = x \odot y$, *for any* $y \in A$,
(7) $x \odot y \leq z \leftrightarrow x \leq y^* \oplus z$,
(8) *If* $x \leq y$ *and* $z \leq t$, *then* $x \oplus z \leq y \oplus t$.
An element $a \in A$ is called complemented if there is an element $b \in A$ such that $a \vee b = 1$ and $a \wedge b = 0$. We denote the set of complemented of A by $B(A)$.

Definition 1.3. [2] An ideal of an MV-algebra A is a nonempty subset I of A satisfying the following conditions:
$(I1)$ If $x \in I$, $y \in A$ and $y \leq x$ then $y \in I$,
$(I2)$ If $x, y \in I$, then $x \oplus y \in I$.
We denote by $Id(A)$ the set of ideals of an MV-algebra A.

Definition 1.4. [4] Let I be an ideal of an MV-algebra A. Then I is a proper if $I \neq A$. Proper ideal P is a prime if and only if for all $x, y \in A$, $x \odot y^* \in P$ or $y \odot x^* \in P$.
• [1] An ideal I of an MV-algebra A is called a Boolean ideal if $x \wedge x^* \in I$, for all $x \in A$.
• [1] P is a primary ideal of an MV-algebra A if it is a proper ideal such that for every $a, b \in A$ such that $a \odot b \in P$, there exists an integer $n > 0$ such that $a^n \in P$ or $b^n \in P$.

• [5] An ideal I is a quasi-implicative if for any $x \in A$ such that $x^n \in I$ for some $n \geq 1$, then $x \in I$.

Lemma 1.5. [4, 18] *M is a maximal ideal of an MV-algebra A if and only if for any $x \notin M$, $(nx)^* \in M$, for some integer $n \geq 1$.*

Remark 1.6. [4] In an MV-algebra M, the distance function is
$d: \quad M \times M \longrightarrow M, \qquad d(x,y) := (x \odot y^*) \oplus (y \odot x^*)$.
Suppose that I is an ideal of an MV-algebra A. Define $x \sim_I y$ if and only if $d(x,y) \in I$ if and only if $x \odot y^* \in I$ and $y \odot x^* \in I$. Then $\sim_I$ is a congruence relation on A. The set of all congruence classes is denoted by A/I then $A/I = \{[x] : x \in A\}$, where $[x] = \{y \in A : x \sim_I y\}$. We can easily to see that $x \in I$ if and only if $x/I = 0/I$. The MV-algebra operations on A/I given by $x/I \oplus y/I = (x \oplus y)/I$ and $(x/I)^* = x^*/I$, are well defined. Hence $(A/I, \oplus, *, [0])$ becomes an MV-algebra [4, 18].

Definition 1.7. [6] Let I be a proper ideal of A. The intersection of all maximal ideals of A which contain I is called the radical of I and it is denoted by $Rad(I)$. It is proved that

$$Rad(I) = \{a \in A : na \odot a \in I, \quad \text{for all} \quad n \in \mathbb{N}\}.$$

Definition 1.8. [6] A proper ideal I of A is said to be a semi-maximal ideal of A if $Rad(I) = I$. Hence I is semi-maximal ideal if and only if $na \odot a \in I$, implies $a \in I$, for all $a \in A$ and $n \in \mathbb{N}$.

Lemma 1.9. *I is a maximal ideal of A if and only if A/I is a locally finite MV-algebra.*

Definition 1.10. [4] Let $n \geq 2$ be an integer. By an n-bounded MV-algebra we shall mean an algebra satisfying the equation

$$(n-1)x = nx$$

The variety of n-bounded MV-algebras will be denoted by U_n.

Theorem 1.11. [4] *Let A be an MV-algebra and $n \geq 2$ an integer. Then $A \in U_n$ if and only if A is a subdirect product of algebra $Ł_k$, with $2 \leq k \leq n$, where $Ł_k$ i.e, the n element Łukasiewicz chains ($Ł_k = \{0, 1/(k-1), 2/(k-1), \cdots, (k-2)/(k-1), 1\}$).*

Definition 1.12. [20] A fuzzy set in A is a mapping $\mu : A \to [0,1]$. Let μ be a fuzzy set in A. For $t \in [0,1]$, the set $\mu^t = \{x \in A : \mu(x) \geq t\}$ is called a level subset of μ.
For any fuzzy sets μ, ν in A, the binary relation $\subseteq$ is defined as

$$\mu \subseteq \nu \text{ if and only if } \mu(x) \leq \nu(x) \text{ for all } x \in A.$$

Definition 1.13. [20] Let X, Y be two sets, μ be a fuzzy subset of X, μ' be a fuzzy subset of Y and $f : X \to Y$ be a homomorphism. The image of μ under f denoted by $f(\mu)$ is a fuzzy set of Y defined by:
for all $y \in Y$, $f(\mu)(y) = sup_{x \in f^{-1}(y)} \mu(x)$, if $f^{-1}(y) \neq \emptyset$ and $f(\mu)(y) = 0$

if $f^{-1}(y) = \emptyset$.
The preimage of μ' under f denoted by $f^{-1}(\mu')$ is a fuzzy set of X defined by: for all $x \in X$, $f^{-1}(\mu')(x) = \mu'(f(x))$.

Definition 1.14. [10] Let A be an MV-algebra. Then a fuzzy set μ in A is a fuzzy ideal of A, if it satisfies
$(MV1)$ $\mu(0) \geq \mu(x)$, for all $x \in A$,
$(MV2)$ $\mu(y) \geq \mu(x) \wedge \mu(y \odot x^*)$, for all $x, y \in A$.

Proposition 1.15. [10, Proposition 2.1] *Let A be an MV-algebra and $\mu : A \to [0,1]$ be a fuzzy set on A. Then μ is called a fuzzy ideal on A, if and only if*
(1) $\mu(x) \leq \mu(0)$, *for all $x \in A$ and*
(2) $\mu(x \oplus y) \geq \mu(x) \wedge \mu(y)$, *for all $x, y \in A$,*
(3) *If $x \leq y$, then $\mu(x) \geq \mu(y)$.*

Definition 1.16. [7] μ is called a fuzzy Boolean ideal, if $\mu(x \wedge (nx)^*) = \mu(0)$, for all $x \in A$.

Theorem 1.17. [10] *Let μ be a fuzzy ideal in A. For any $x, y \in A$, the following hold:*
(1) $\mu(x \oplus y) = \mu(x) \wedge \mu(y)$,
(2) $\mu(x \vee y) = \mu(x) \wedge \mu(y)$.

2. n-fold Boolean ideals in MV-algebras

Form now on $(A, \oplus, *, 0, 1)$ or simply A is an MV-algebra.

Definition 2.1. Let I be an ideal of A. I is called n-fold Boolean ideal of A, if it satisfies: $x \wedge (nx)^* \in I$.

In particular, 1-fold Boolean ideals are Boolean ideals.
The following example shows that n-fold Boolean ideals exist and that an ideal is not n-fold Boolean ideal of A, in general.

Example 2.2. Let $A = \{0, a, b, 1\}$, where $0 < a, b < 1$. Define $\odot$, $\oplus$ and $*$ as follows:

$\odot$	0	a	b	1
0	0	0	0	0
a	0	a	0	a
b	0	0	b	b
1	0	a	b	1

$\oplus$	0	a	b	1
0	0	a	b	1
a	a	a	1	1
b	b	1	b	1
1	1	1	1	1

$*$	0	a	b	1
	1	b	a	0

Then $(A, \oplus, \odot, *, 0, 1)$ is an MV-algebra [14], it is clear that $I_1 = \{0, a\}$ and $I_2 = \{0, b\}$ are n-fold Boolean ideals of A.

Example 2.3. Let $A = \{0, a, b, c, d, 1\}$, where $0 < a, b < c < 1$ and $0 < b < d < 1$. Define $\oplus$, $\odot$ and $*$ as follows:

$\odot$	0	a	b	c	d	1
0	0	0	0	0	0	0
a	0	a	0	a	0	a
b	0	0	0	0	b	b
c	0	a	0	a	b	c
d	0	0	b	b	d	d
1	0	a	b	c	d	1

$\oplus$	0	a	b	c	d	1
0	0	a	b	c	d	1
a	a	a	c	c	1	1
b	b	c	d	1	d	1
c	c	c	1	1	1	1
d	d	1	d	1	d	1
1	1	1	1	1	1	1

$*$	0	a	b	c	d	1
	1	d	c	b	a	0

Then $(A, \oplus, \odot, *, 0, 1)$ is an MV-algebra [14] and it is clear $I = \{0, a\}$ is an ideal of A but since $c \wedge c^* = c \wedge b = b \notin I$, it is not a 1-fold Boolean ideal of A.

Theorem 2.4. *Every n-fold Boolean ideal is a* $(n+1)$*-fold Boolean ideal of* A*.*

Proof. Let I be n-fold Boolean ideal of A and $x \wedge (nx)^* \in I$, for all $x \in A$. We must show that $x \wedge ((n+1)x)^* \in I$. We have $nx \leq (n+1)x$. Hence by Lemma 1.2 (1), $((n+1)x)^* \leq (nx)^*$. We imply that $((n+1)x)^* \wedge x \leq (nx)^* \wedge x \in I$. Thus $((n+1)x)^* \wedge x \in I$. □

The following example shows that the converse of Theorem 2.4, is not true in general.

Example 2.5. Let A be an MV-algebra from Example 2.3, $I = \{0, a\}$ is 2-fold Boolean ideal of A, while is not 1-fold Boolean ideal of A, since $c \wedge c^* = c \wedge b = b \notin I$.

Theorem 2.6. *Let $n \geq 1$, I_1 and I_2 two ideals of A such that $I_1 \subseteq I_2$. If I_1 is a n-fold Boolean ideal, then so is I_2.*

Proof. If I_1 is a n-fold Boolean ideal, then $x \wedge (nx)^* \in I_1$, for all $x \in A$. Since $I_1 \subseteq I_2$, we have $x \wedge (nx)^* \in I_2$, for all $x \in A$. Thus I_2 is a n-fold Boolean ideal of A. □

Remark 2.7. Let I and J be ideals of A. We have

$$I \vee J = (I \cup J] = \{a \in A : a \leq b \oplus c, \quad \text{for some } b \in I \text{ and } c \in J\}.$$

It is an ideal of A, [4, 18]. If I or J is a n-fold Boolean ideal, then by Theorem 2.6, we get that $I \vee J$ is a n-fold Boolean ideal.

Lemma 2.8. *$\{0\}$ is a n-fold Boolean ideal of A if and only if every ideal I of A is a n-fold Boolean ideal.*

Theorem 2.9. *Let I be an ideal of A. Then I is a n-fold Boolean ideal of A if and only if every ideal of A/I is a n-fold Boolean ideal.*

Proof. Assume that I is a n-fold Boolean ideal of A. From Lemma 2.8, we have

$$\begin{aligned} x \wedge (nx)^* \in I \quad &\Leftrightarrow \quad (x \wedge (nx)^*)/I = 0/I, \\ &\Leftrightarrow \quad x/I \wedge (nx)^*/I = 0/I, \\ &\Leftrightarrow \quad x/I \wedge ((nx)/I)^* = 0/I, \\ &\Leftrightarrow \quad x/I \wedge (n(x/I))^* = 0/I \in \{[0]\}. \end{aligned}$$

Hence $\{[0]\}$ is a n-fold Boolean ideal of A/I, thus by Lemma 2.8, we conclude that every ideal of A/I is a n-fold Boolean ideal. □

The following example shows that the MV-homomorphic image of an n-fold Boolean ideal is not even an ideal.

Example 2.10. In Example 2.2, consider MV-homomorphism $f : A \to A$ such that $f(0) = 0$, $f(a) = 1$, $f(b) = 0$ and $f(1) = 1$. It is clear $I = \{0, a\}$ is a 1-fold Boolean ideal of A, while $f(I) = \{0, 1\}$ is not an ideal of A.

In the following theorem, we study inverse image of a n-fold Boolean ideal under a MV-homomorphism.

Theorem 2.11. *Let $f : A \to B$ be an onto MV-homomorphism and I be a n-fold Boolean ideal of B. Then inverse image of I is a n-fold Boolean ideal of A.*

Proof. Let I be a n-fold Boolean ideal of B. We show that for $x \in A$, $x \wedge (nx)^* \in f^{-1}(I)$. Since $f(x) \in B$ and I is a n-fold Boolean ideal of B, $f(x) \wedge (nf(x))^* \in I$ if and only if $f(x) \wedge (f(nx))^* \in I$ if and only if $f(x \wedge (nx)^*) \in I$ if and if $x \wedge (nx)^* \in f^{-1}(I)$. Hence $f^{-1}(I)$ is a n-fold Boolean ideal of A. □

Theorem 2.12. *Let I be a primary ideal and quasi-implicative ideal of A. Then I is a n-fold Boolean ideal.*

Proof. We have $0 = x \odot (x^*)^n = x \odot (nx)^* \in I$, for any $x \in A$. Since I is a primary ideal, so $x^m \in I$ or $((nx)^*)^m \in I$, for some integer $m \geq 1$. Since I is a quasi-implicative ideal, then $x \in I$ or $(nx)^* \in I$. Hence $x \wedge (nx)^* \leq x, (nx)^* \in I$, thus $x \wedge (nx)^* \in I$. Therefore I is a n-fold Boolean ideal. □

Proposition 2.13. *The following conditions are equivalent for any ideal I and any $n \geq 1$:*
(i) For all $x, y \in A$, $x \odot (y^* \oplus nx) \in I$ implies $x \in I$,
(ii) If $nx \odot x \in I$, for all $x \in A$, implies $x \in I$.

Proof. $(i) \rightarrow (ii)$ We obtain the result by setting $y = 1$ in the equation (i).
$(ii) \rightarrow (i)$ Suppose that $x \odot (y^* \oplus nx) \in I$, for all $x, y \in A$. Hence $x \odot nx \leq x \odot (y^* \oplus nx) \in I$. It follows that $x \odot nx \in I$, and by hypothesis we obtain $x \in I$. □

Theorem 2.14. *A proper ideal I is a n-fold Boolean ideal of A if and only if $nx \odot x \in I$, then $x \in I$, for all $n \in \mathbb{N}$.*

Proof. Let I be a n-fold Boolean ideal of A. We have $x \wedge (nx)^* \in I$. Suppose that $nx \odot x \in I$. We prove that $x \in I$. Since I is an ideal, $x \wedge (nx)^* \oplus (x \odot nx) = (x \odot (x^* \oplus (nx)^*) \oplus (x \odot (nx)) \in I$. On the other hand, $x \leq (x \odot nx) \vee x \in I$, thus $x \in I$.
Conversely, let $x \in A$. Setting $t = x \wedge (nx)^*$, we show that $t \in I$. Since $t \leq x$, we have $nt \leq nx$ and then $(nx)^* \wedge x \leq (nx)^* \leq (nt)^*$ and then $t \leq (nt)^*$ or $t \odot nt = 0 \in I$. So by hypothesis, we imply that $t \in I$. Thus I is n-fold Boolean ideal of A. □

By the above theorem, we have:

Corollary 2.15. *If I is a semi-maximal ideal of A, then I is a n-fold Boolean ideal of A.*

The following example shows that the converse of the above theorem is not true in general.

Example 2.16. In Example 2.3, we have $I = \{0, a\}$ is 2-fold Boolean ideal. Since $b \odot 1b = 0 \in I$ but $b \notin I$, hence I is not semi-maximal ideal of A.

Lemma 2.17. *A is a n+1-bounded MV-algebra if and only if it satisfies the following condition:*

$$x \wedge (nx)^* = 0, \text{ for all } x \in A.$$

Proof. Let $x \wedge (nx)^* = 0$, for all $x \in A$. Hence $x^* \vee (nx) = 1$. So

$$1 = x^* \vee (nx) = nx \oplus ((nx)^* \odot x^*) = nx \oplus (nx \oplus x)^* = nx \oplus ((n+1)x)^*.$$

It follows from Lemma 1.2 (3) that $(n+1)x \leq nx$. Thus A is a n+1-bounded MV-algebra. The converse is clear. □

The following example shows that MV-algebras are not in general n+1-bounded MV-algebras.

Example 2.18. We consider Chang's MV-algebra $A = \{0, c, 2c, 3c, \dots, 1-2c, 1-c, 1\}$ in [2] with operations as follows:
if $x = nc$ and $y = mc$, then $x \oplus y := (m+n)c$,
if $x = 1 - nc$ and $y = 1 - mc$, then $x \oplus y := 1$,
if $x = nc$ and $y = 1 - mc$ and $m \leq n$, then $x \oplus y := 1$,
if $x = nc$ and $y = 1 - mc$ and $n < m$, then $x \oplus y := 1 - (m-n)c$,
if $x = 1 - mc$ and $y = nc$ and $m \leq n$, then $x \oplus y := 1$,
if $x = 1 - mc$ and $y = nc$ and $n < m$, then $x \oplus y := 1 - (m-n)c$,
if $x = nc$, then $x^* := 1 - nc$,
if $x = 1 - nc$, then $x^* := nc$.

Since $c \wedge (nc)^* = c \wedge (1 - nc) = c \odot (c^* \oplus (1 - nc)) = c \odot 1 = c \neq 0$, hence A is not n+1-bounded MV-algebra.

Theorem 2.19. *The following conditions are equivalent:*
(i) $\{0\}$ is a n-fold Boolean ideal of A,
(ii) $x \odot nx = x$, for all $x \in A$.

Proof. $(i) \rightarrow (ii)$ Let $x \odot nx = x$, for all $x \in A$. Hence $x \wedge (nx)^* = x \odot (x^* \oplus (nx)^*) = x \odot (x \odot nx)^* = x \odot x^* = 0 \in \{0\}$. Thus $\{0\}$ is n-fold Boolean ideal of A.

$(ii) \rightarrow (i)$ Assume that $\{0\}$ is a n-fold Boolean ideal. Hence for all $x \in A$ holds $x \wedge (nx)^* = 0$. Hence $x \odot (x \odot nx)^* = x \odot (x^* \oplus (nx)^*) = x \wedge (nx)^* = 0$ or equivalently, $x \leq x \odot (nx) \leq x$. Thus $x \odot (nx) = x$, for all $x \in A$. □

By the above theorem, we have

Corollary 2.20. *A is a n+1-bounded MV-algebra if and only if* $\{0\}$ *is a n-fold Boolean ideal of A.*

We recall that I is maximal ideal of A if and only if A/I is locally finite MV-algebra [18].

Theorem 2.21. *If A is a totally ordered MV-algebra, then any n-fold Boolean ideal of A is maximal ideal of A and A/I is a locally finite MV-algebra.*

Proof. Let A be a totally ordered MV-algebra. Assume that I is n-fold Boolean ideal and let $x \in A$ be an element that $x \notin I$. From Theorem 2.14, we obtain $x \odot nx \notin I$, hence $x \leq (nx)^*$ or equivalently $x \odot nx = 0 \in I$, which is a contradiction. So we necessarily have $(nx)^* \leq x$. Therefore $nx \oplus x = 1$ and so $(n+1)x = 1$, hence $((n+1)x)^* = 0 \in I$. It follows from Lemma 1.5 that I is a maximal ideal of A. Hence A/I is a locally finite MV-algebra. □

By Theorem 2.21, we have the following result:

Corollary 2.22. *A totally ordered MV-algebra is a locally finite if* $\{0\}$ *is a n-fold Boolean ideal. A totally ordered n+1-bounded MV-algebra is a locally finite.*

Theorem 2.23. *An ideal I of A is a n-fold Boolean ideal if and only if A/I is a n+1-bounded MV-algebra.*

Proof. Suppose that I is a n-fold Boolean ideal. Hence for $x \in A$, we have

$$\begin{aligned}
x \wedge (nx)^* \in I \quad &\Leftrightarrow \quad (x \wedge (nx)^*)/I = 0/I, \\
&\Leftrightarrow \quad x/I \wedge (nx)^*/I = 0/I, \\
&\Leftrightarrow \quad x/I \wedge (x^*)^n/I = 0/I, \\
&\Leftrightarrow \quad x/I \wedge (x^*/I)^n = 0/I, \\
&\Leftrightarrow \quad x/I \wedge ((x/I)^*)^n = 0/I, \\
&\Leftrightarrow \quad x/I \wedge (n(x/I))^* = 0/I, \text{ for all } x/I \in A/I.
\end{aligned}$$

Hence A/I is a n+1-bounded MV-algebra. □

Theorem 2.24. *I is a maximal and n-fold Boolean ideal if and only if I is a prime and n-fold Boolean ideal of A.*

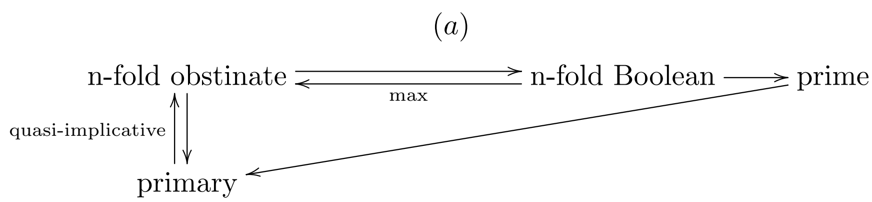
Proof. Let I be a prime and n-fold Boolean ideal of A. Also suppose that $x \notin I$. Then we have $x \wedge (nx)^* \in I$, for any $x \in A$. Since I is prime,

$x \in I$ or $(nx)^* \in I$. Since $x \notin I$, $(nx)^* \in I$. It follows from Lemma 1.5 that I is a maximal ideal of A. □

By Theorem 1.11 and Theorem 2.23, we conclude the following corollary:

Corollary 2.25. *I is a n-fold Boolean ideal if and only if A/I is a $n+1$-bounded MV-algebra if and only if A/I is a subdirect product of algebras $Ł_k$, with $2 \leq k \leq n$.*

Remark 2.26. In the following diagram, relationships among n-fold Boolean ideals and the other ideals in MV-algebras are described [8].



3. Fuzzy n-fold obstinate ideals in MV-algebras

Definition 3.1. Let μ be a fuzzy ideal in A. μ is called a fuzzy n-fold obstinate ideal if it satisfies

$$\mu(x \odot (ny)^*) \wedge \mu(y \odot (nx)^*) \geq (1 - \mu(x)) \wedge (1 - \mu(y)), \text{ for all } x, y \in A.$$

In particular, fuzzy 1-fold obstinate ideals are fuzzy obstinate ideals.

Lemma 3.2. *A fuzzy ideal μ of an MV-algebra A is a fuzzy obstinate ideal if and only if it satisfies the following condition:*

$$\mu((nx)^*) \geq 1 - \mu(x), \textit{ for all } x \in A.$$

Proof. Suppose that μ is a fuzzy n-fold obstinate ideal of A. Since $x \leq 1$, by fuzzy ideal properties, we obtain $1 - \mu(x) \leq 1 - \mu(1)$ and we conclude that

$\mu((nx)^*) = \mu((nx)^* \odot 1) \geq min\{\mu((nx)^* \odot 1), \mu(x \odot (n1)^*)\} \geq min\{1 - \mu(x), 1 - \mu(1)\} = 1 - \mu(x)$.

Conversely, let $\mu((nx)^*) \geq 1 - \mu(x)$, for all $x \in A$. By Lemma 1.2, $(nx)^* \odot y \leq (nx)^*$ and $(ny)^* \odot x \leq (ny)^*$, we have

$$min\{1 - \mu(x), 1 - \mu(y)\} \leq 1 - \mu(x) \leq \mu((nx)^*) \leq \mu((nx)^* \odot y)$$

and $min\{1-\mu(x), 1-\mu(y)\} \leq 1-\mu(y) \leq \mu((ny)^*) \leq \mu((ny)^* \odot x)$. Thus

$$min\{1-\mu(x), 1-\mu(y)\} \leq min\{\mu((nx)^* \odot y), \mu((ny)^* \odot x)\}.$$

Hence μ is a fuzzy n-fold obstinate ideal of A. □

The following example shows that fuzzy n-fold obstinate ideals exist and a fuzzy ideal may not be a fuzzy obstinate ideal of A.

Example 3.3. Consider Example 2.2.

(i) Define a fuzzy set μ in A by $\mu(0) = 0.8$ and $\mu(1) = \mu(a) = \mu(b) = 0.5$. Obviously, μ is a n-fold fuzzy obstinate ideal on A, for $n \geq 1$.

(ii) Define a fuzzy set μ' in A by $\mu'(0) = 0.8$ and $\mu'(1) = \mu'(a) = \mu'(b) = 0.3$. Obviously, μ' is a fuzzy ideal in A. Since $\mu'(b^*) = \mu'(a) = 0.3 < 1 - \mu'(b) = 0.7$, hence μ' is not 1-fold obstinate ideal of A.

Lemma 3.4. *(Extension theorem of fuzzy n-fold obstinate ideals) Suppose that A is an MV-algebra and μ and ν are two non-constant fuzzy ideals such that $\mu \subseteq \nu$. If μ is a fuzzy n-fold obstinate ideal, then ν is also a fuzzy n-fold obstinate ideal of A.*

Proof. Let μ is a fuzzy n-fold obstinate ideal such that $\mu \subseteq \nu$. We show that ν is a fuzzy n-fold obstinate ideal. Since μ is a fuzzy n-fold obstinate ideal, $\mu((nx)^*) \geq 1 - \mu(x)$, for all $x \in A$.

Also, $\mu \subseteq \nu$, so $\mu(x) \leq \nu(x)$, for all $x \in A$. It follows that

$$\nu((nx)^*) \geq \mu((nx)^*) \geq 1 - \mu(x) \geq 1 - \nu(x).$$

Hence $\nu(nx)^* \geq 1-\nu(x)$, for all $x \in A$. Thus ν is a fuzzy n-fold obstinate ideal of A. □

Theorem 3.5. *Every fuzzy n-fold obstinate ideal is a fuzzy (n+1)-fold obstinate ideal of A.*

Proof. Let μ be a fuzzy n-fold obstinate ideal of A. We have

$$min\{\mu(x \odot (ny)^*), \mu(y \odot (nx)^*)\} \geq min\{1 - \mu(x), 1 - \mu(y)\}.$$

We show that μ is a fuzzy (n+1)-fold obstinate ideal, we need to prove that

$$min\{\mu(y \odot ((n+1)x)^*), \mu(x \odot ((n+1)y)^*)\} \geq min\{1 - \mu(x), 1 - \mu(y)\}.$$

Using Lemma 1.2, we have $y \odot ((n+1)x)^* \leq y \odot (nx)^*$ and $x \odot ((n+1)y)^* \leq x \odot (ny)^*$. Since μ is a fuzzy ideal, we obtaine $\mu(y \odot ((n+1)x)^*) \geq \mu(y \odot (nx)^*)$ and $\mu(x \odot (n+1)y)^*) \geq \mu(x \odot (ny)^*)$. By hypothesis, It follows that

$$min\{\mu(x \odot ((n+1)y)^*), \mu(y \odot ((n+1)x)^*)\} \geq min\{1 - \mu(x), 1 - \mu(y)\},$$

so μ is a fuzzy (n+1)-fold obstinate ideal of A. □

By finite induction, we can prove that every fuzzy n-fold obstinate ideal is a fuzzy (n+k)-fold obstinate ideal for any integer $k \geq 0$.

The following example shows that any fuzzy (n+1)-fold obstinate ideal may not be a fuzzy n-fold obstinate ideal of A.

Example 3.6. Let $A = \{0, 1, 2\}$ be a linearly ordered set (chain). A is an MV-algebra with operations $\wedge = min$, $x \oplus y = min\{2, x + y\}$ and $x \odot y = max\{0, x + y - 2\}$, for every $x, y \in A$ [14]. On the other hand A is an MV-algebra with the following operations:

$\oplus$	0	1	2
0	0	1	2
1	1	2	2
2	2	2	2

$*$	0	1	2
	2	1	0

Define a fuzzy set in A by $\mu(0) = 0.8$, $\mu(1) = 0.3$ and $\mu(2) = 0.3$.

Using Lemma 3.2, for $n = 2$, it is easy to chack that μ is a fuzzy 2-fold obstinate ideal of A but it is not a fuzzy 1-fold obstinate ideal of A because $0.3 = \mu(1^*) = \mu(1) \ngeq 1 - \mu(1) = 1 - 0.3 = 0.7$.

Theorem 3.7. *Let $f : X \to Y$ be onto MV-homomorphism. Then the preimage of a fuzzy n-fold obstinate ideal μ under f is also a fuzzy n-fold obstinate ideal of X.*

Proof. Suppose that μ is a fuzzy n-fold obstinate ideal of Y. Then for all $x, y \in X$.

We have

$$\begin{aligned} & min\{f^{-1}(\mu)(x \odot (ny)^*), f^{-1}(\mu)(y \odot (nx)^*)\}, \\ = \; & min\{\mu(f(x \odot (ny)^*), \mu(f(y \odot (nx)^*), \\ \geq \; & min\{1 - \mu(f(x)), 1 - \mu(f(y))\}, \\ = \; & min\{1 - f^{-1}(\mu)(x), 1 - f^{-1}(\mu)(y)\}. \end{aligned}$$

Thus $f^{-1}(\mu)$ is a fuzzy n-fold obstinate ideal of X. □

Proposition 3.8. *Let $f : X \to Y$ be an onto MV-homomorphism. The image $f(\mu)$ of a fuzzy n-fold obstinate ideal μ with a subproperty is also a fuzzy n-fold obstinate ideal of Y.*

Proof. It is sufficient to show that for all $y_1, y_2 \in Y$,

$$min\{f(\mu)(y_1 \odot (ny_2)^*), f(\mu)(y_2 \odot (ny_1)^*)\} \geq min\{1 - f(\mu)(y_1), 1 - f(\mu)(y_2)\}.$$

Let $y_1, y_2 \in Y$ and $x_1 \in f^{-1}(y_1)$, $x_2 \in f^{-1}(y_2)$ such that $1 - \mu(x_1) = 1 - sup_{t \in f^{-1}(y_1)} \mu(t)$ and $1 - \mu(x_2) = 1 - sup_{t \in f^{-1}(y_2)} \mu(t)$.

We have $f(\mu)(y_1 \odot (ny_2)^*) = sup_{t \in f^{-1}(y_1 \odot (ny_2)^*)} \mu(t) \geq \mu(x_1 \odot (nx)^*)$ and

$f(\mu)(y_2 \odot (ny_1)^*) = sup_{t \in f^{-1}(y_2 \odot (ny_1)^*)} \mu(t) \geq \mu(x_2 \odot (nx_1)^*)$. So

$$\begin{aligned} & min\{f(\mu)(y_1 \odot (ny_2)^*), f(\mu)(y_2 \odot (ny_1)^*)\}, \\ \geq \; & min\{\mu(x_1 \odot (nx_2)^*), \mu(x_2 \odot (nx_1)^*))\}, \\ \geq \; & min\{1 - \mu(x_1), 1 - \mu(x_2)\}. \end{aligned}$$

But $min\{1 - \mu(x_1), 1 - \mu(x_2)\} = min\{1 - f(\mu)(y_1), 1 - f(\mu)(y_2)\}$. We conclude that $f(\mu)$ is a fuzzy n-fold obstinate ideal of Y. □

Theorem 3.9. *A non-empty subset I of A is a n-fold obstinate ideal if and only if the characteristic function χ_I is a fuzzy n-fold obstinate ideal of A.*

Proof. Assume that I is a n-fold obstinate ideal of A. We will prove that χ_I is a fuzzy n-fold obstinate ideal of A.

Let $x, y \in A$. We show that

$$min\{\chi_I(x \odot (ny)^*), \chi_I(y \odot (nx)^*)\} \geq min\{1 - \chi_I(x), 1 - \chi_I(y)\}.$$

If $x \in I$ or $y \in I$, we have $min\{1 - \chi_I(x), 1 - \chi_I(y)\} = 0$ and

$$min\{\chi_I(x \odot (ny)^*), \chi_I(y \odot (nx)^*)\} \geq min\{1 - \chi_I(x), 1 - \chi_I(y)\}.$$

If $x \notin I$ and $y \notin I$, then $min\{1 - \chi_I(x), 1 - \chi_I(y)\} = 1$, since I is a n-fold obstinate ideal of A, we obtain $x \odot (ny)^* \in I$ and $y \odot (nx)^* \in I$. So $min\{\chi_I(x \odot (ny)^*), \chi_I(y \odot (nx)^*)\} = 1$. We conclude that

$$min\{\chi_I(x \odot (ny)^*, \chi_I(y \odot (nx)^*)\} \geq min\{1 - \chi_I(x), 1 - \chi_I(y)\}.$$

Assume that χ_I is a fuzzy n-fold obstinate ideal of A, we prove that I is a n-fold obstinate ideal of A. Let $x, y \notin I$, we have $\chi_I(x) = 0 = \chi_I(y)$. Since χ_I is a fuzzy n-fold obstinate ideal of A, we have

$$min\{\chi_I(x \odot (ny)^*), \chi_I(y \odot (nx)^*)\} \geq min\{1 - \chi_I(x), 1 - \chi_I(y)\} = 1.$$

We obtain $\chi_I(x \odot (ny)^*) = \chi_I(y \odot (nx)^*)\} = 1$. Hence $x \odot (ny)^* \in I$ and $y \odot (nx)^* \in I$. □

Now, we describe the transfer principle [15] for fuzzy n-fold obstinate ideals in terms of level subsets:

Theorem 3.10. (i) *A fuzzy subset* μ *of an* MV*-algebra* A *is a fuzzy n-fold obstinate ideal of* A*, if* $\mu_t = \{x \in A : \mu(x) \geq t\}$ *is either empty or a n-fold obstinate ideal for every* $t \in [0, 1/2]$.
(ii) *If* $\mu_t \neq \emptyset$*, for any* $t \in (1/2, 1]$ *and* μ_t *is a n-fold obstinate ideal, then* μ *is a fuzzy n-fold obstinate ideal of* A.

Proof. (i) Assume that μ is a fuzzy n-fold obstinate ideal of A. Let $t \in [0, 1/2]$ and $x \in \mu_t$. Then $\mu(x) \geq t$. Since μ is a fuzzy ideal, $\mu(0) \geq \mu(x)$, therefore $0 \in \mu_t$. Let $x, y \notin \mu_t$. We show that $x \odot (ny)^* \in \mu_t$ and $y \odot (nx)^* \in \mu_t$. Since $x, y \notin \mu_t$, $\mu(x) < t$, $\mu(y) < t$ and μ is a fuzzy n-fold obstinate ideal of A, we have

$$\begin{aligned} \mu(x \odot (ny)^*) &\geq min\{\mu(x \odot (ny)^*)), \mu(y \odot (nx)^*)\} \\ &\geq min\{1 - \mu(x), 1 - \mu(y)\} \\ &\geq 1 - t \\ &\geq t. \end{aligned}$$

for every $t \in [0, 1/2]$. Also, by similarly, $\mu(y \odot (nx)^*) \geq t$. Hence $x \odot (ny)^* \in \mu_t$ and $y \odot (nx)^* \in \mu_t$. Thus μ_t is a n-fold obstinate ideal of A.
(ii) Assume that for every $t \in (1/2, 1]$, μ_t is a n-fold obstinate ideal of A. We will prove that μ is a fuzzy n-fold obstinate ideal of A. It is easy to prove that for all $x \in A$, $\mu(0) \geq \mu(x)$. Let $x, y \in A$. We show that $min\{\mu(x \odot (ny)^*), \mu(y \odot (nx)^*)\} \geq min\{1 - \mu(x), 1 - \mu(y)\}$. If not, there exist $a, b \in A$ such that

$min\{\mu(a \odot (nb)^*), \mu(b \odot (na)^*)\} < min\{1 - \mu(a), 1 - \mu(b)\}$. Setting

$t_0 = 1/2(min\{\mu(a \odot (nb)^*), \mu(b \odot (na)^*)\} + min\{1 - \mu(a), 1 - \mu(b)\})$.

We have $min\{\mu(a \odot (nb)^*), \mu(b \odot (na)^*)\} < t_0 < min\{1 - \mu(a), 1 - \mu(b)\}$. We conclude that $\mu(a \odot (nb)^*) < t_0$ or $\mu(b \odot (na)^*) < t_0$. Also, $t_0 < 1 - \mu(a)$ and $t_0 < 1 - \mu(b)$. We consider two cases:
Case 1. If $t_0 > \frac{1}{2}$, then we conclude that $\mu(a) < 1 - t_0 < t_0$ and $\mu(b) < 1 - t_0 < t_0$. Also, since $\mu(a \odot (nb)^*) < t_0$ or $\mu(b \odot (na)^*) < t_0$, hence $a \odot (nb)^* \notin \mu^{t_0}$ or $b \odot (na)^* \notin \mu^{t_0}$, for $a \notin \mu^{t_0}$ and $b \notin \mu^{t_0}$, which is a contradiction.
Case 2. If $t_0 \leq \frac{1}{2}$, since $1 - t_0 \geq \frac{1}{2}$, then $\mu(a \odot (nb)^*) < t_0 \leq \frac{1}{2} \leq 1 - t_0$ or $\mu(b \odot (na)^*) < t_0 \leq \frac{1}{2} \leq 1 - t_0$. Also, $\mu(a) < 1 - t_0$ and $\mu(b) < 1 - t_0$. Hence $a \odot (nb)^* \notin \mu^{1-t_0}$ or $b \odot (na)^* \notin \mu^{1-t_0}$, for $a \notin \mu^{1-t_0}$ and $b \notin \mu^{1-t_0}$, which is a contradiction.
Therefore μ is a fuzzy n-fold obstinate ideal of A. □

Corollary 3.11. *Let μ be a fuzzy ideal of an MV-algebra A. The level ideal $I = \{x \in A : \mu(x) = \mu(0)\}$ is a n-fold obstinate ideal of A if μ is a fuzzy n-fold obstinate ideal of A with $\mu(0) \in [0, 1/2]$.*

In the following theorem, we investigate the relation between fuzzy n-fold obstinate ideals and the fuzzy n-fold Boolean ideals of A.

Theorem 3.12. *Let μ be a fuzzy n-fold obstinate ideal of A such that $\mu(0) \leq 1/2$. Then μ is a fuzzy n-fold Boolean ideal of A.*

Proof. Let μ be a fuzzy n-fold obstinate ideal of A. It is sufficient to show that $\mu(x \wedge (nx)^*) = \mu(0)$. Since $0 \leq x \wedge (nx)^*$, by fuzzy ideal property, $\mu(0) \geq \mu(x \wedge (nx)^*)$. Since μ is a fuzzy n-fold obstinate ideal of A and $x \wedge (nx)^* \leq (nx)^*$,

$$\mu(x \wedge (nx)^*) \geq \mu((nx)^*) \geq 1 - \mu(x) \geq 1 - \mu(0) \geq \mu(0).$$

Hence $\mu(x \wedge (nx)^*) = \mu(0)$. Thus μ is a fuzzy n-fold Boolean ideal of A. $\square$

The following example, shows that the converse of the above theorem is not true, in general.

Example 3.13. Let $A = \{0, a, b, c, d, 1\}$. where $0 < a, c < d < 1$ and $0 < a < b < 1$. Define $\oplus$ and $*$ as follows:

$\oplus$	0	a	b	c	d	1
0	0	a	b	c	d	1
a	a	b	b	d	1	1
b	b	b	b	1	1	1
c	c	d	1	c	d	1
d	d	1	1	d	1	1
1	1	1	1	1	1	1

$*$	0	a	b	c	d	1
	1	d	c	b	a	0

Then $(A, \oplus, \odot, *, 0, 1)$ is an MV-algebra [14]. Define μ fuzzy set in A by $\mu(0) = \mu(1) = \mu(a) = \mu(b) = \mu(c) = \mu(d) = 0.4$. μ is a fuzzy n-fold Boolean ideal but is not fuzzy 1-fold obstinate ideal of A, since $\mu(a^*) = \mu(d) = 0.4 < 1 - \mu(a) = 1 - 0.4 = 0.6$.

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