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FACTORIZATION PROPERTIES AND GENERALIZATION OF MULTIPLIERS IN MODULE ACTIONS

KAZEM HAGHNEJAD AZAR

Abstract. In this paper, We establish some necessary and sufficient conditions for relationships between the topological centers of module actions and factorization properties of them and we give a new definition for multiplier in module actions with some results.

Key Words: Arens regularity, bilinear mapping, topological center, module action, factorization, weakly compact.

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1. Introduction

Hu, Neufang and Ruan in [15], have been studied multiplier on new class of Banach algebras. They showed that how a multiplier on Banach algebra A to be implemented by an element from A is determined by its behavior on A^* and A^{**}, respectively. In this paper, we study this topic on module actions with some conclusions. In 1951 Arens shows that the second dual A^{**} of Banach algebra A endowed with the either Arens multiplications is a Banach algebra, see [1]. The constructions of the two Arens multiplications in A^{**} lead us to definition of topological centers for A^{**} with respect to both Arens multiplications. The topological centers of Banach algebras and module actions have been studied in [3, 5, 6, 9, 15, 16, 17, 18, 19, 24, 25]. In this paper, we extend some problems

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∗Address correspondence to Kazem Haghnejad Azar; E-mail: haghnejad@uma.ac.ir

from [6, 16, 22] to the general criterion on module actions with some results in group algebras. The extension of bilinear maps on normed space and the concept of regularity of bilinear maps have been studied by [1, 2, 5, 6, 9]. We start by recalling these definitions as follows.
Let X, Y, Z be normed spaces and $m : X \times Y \to Z$ be a bounded bilinear mapping. Arens in [1] offers two natural extensions m^{***} and m^{t***t} of m from $X^{**} \times Y^{**}$ into Z^{**} as following
1. $m^* : Z^* \times X \to Y^*$, given by $\langle m^*(z', x), y\rangle = \langle z', m(x, y)\rangle$ where $x \in X$, $y \in Y$, $z' \in Z^*$,
2. $m^{**} : Y^{**} \times Z^* \to X^*$, given by $\langle m^{**}(y'', z'), x\rangle = \langle y'', m^*(z', x)\rangle$ where $x \in X$, $y'' \in Y^{**}$, $z' \in Z^*$,
3. $m^{***} : X^{**} \times Y^{**} \to Z^{**}$, given by $\langle m^{***}(x'', y''), z'\rangle = \langle x'', m^{**}(y'', z')\rangle$ where $x'' \in X^{**}$, $y'' \in Y^{**}$, $z' \in Z^*$.
The mapping m^{***} is the unique extension of m such that $x'' \to m^{***}(x'', y'')$ from X^{**} into Z^{**} is $weak^* - to - weak^*$ continuous for every $y'' \in Y^{**}$, but the mapping $y'' \to m^{***}(x'', y'')$ is not in general $weak^* - to - weak^*$ continuous from Y^{**} into Z^{**} unless $x'' \in X$. Hence the first topological center of m may be defined as following

$$Z_1(m) = \{x'' \in X^{**} : \ y'' \to m^{***}(x'', y'') \\ is\ weak^* - to - weak^*\ continuous\}.$$

Let now $m^t : Y \times X \to Z$ be the transpose of m defined by $m^t(y, x) = m(x, y)$ for every $x \in X$ and $y \in Y$. Then m^t is a continuous bilinear map from $Y \times X$ to Z, and so it may be extended as above to $m^{t***} : Y^{**} \times X^{**} \to Z^{**}$. The mapping $m^{t***t} : X^{**} \times Y^{**} \to Z^{**}$ in general is not equal to m^{***}, see [1], if $m^{***} = m^{t***t}$, then m is called Arens regular. The mapping $y'' \to m^{t***t}(x'', y'')$ is $weak^* - to - weak^*$ continuous for every $y'' \in Y^{**}$, but the mapping $x'' \to m^{t***t}(x'', y'')$ from X^{**} into Z^{**} is not in general $weak^* - to - weak^*$ continuous for every $y'' \in Y^{**}$. So we define the second topological center of m as

$$Z_2(m) = \{y'' \in Y^{**} : \ x'' \to m^{t***t}(x'', y'') \\ is\ weak^* - to - weak^*\ continuous\}.$$

It is clear that m is Arens regular if and only if $Z_1(m) = X^{**}$ or $Z_2(m) = Y^{**}$. Arens regularity of m is equivalent to the following

$$\lim_i \lim_j \langle z', m(x_i, y_j)\rangle = \lim_j \lim_i \langle z', m(x_i, y_j)\rangle,$$

whenever both limits exist for all bounded sequences $(x_i)_i \subseteq X$, $(y_i)_i \subseteq Y$ and $z' \in Z^*$, see [5].

The mapping m is left strongly Arens irregular if $Z_1(m) = X$ and m is right strongly Arens irregular if $Z_2(m) = Y$.
Let now B be a Banach $A - bimodule$, and let

$$\pi_\ell : \ A \times B \to B \ and \ \pi_r : \ B \times A \to B.$$

be the left and right module actions of A on B, respectively. Then B^{**} is a Banach $A^{**} - bimodule$ with module actions

$$\pi_\ell^{***} : \ A^{**} \times B^{**} \to B^{**} \ and \ \pi_r^{***} : \ B^{**} \times A^{**} \to B^{**}.$$

Similarly, B^{**} is a Banach $A^{**} - bimodule$ with module actions

$$\pi_\ell^{t***t} : \ A^{**} \times B^{**} \to B^{**} \ and \ \pi_r^{t***t} : \ B^{**} \times A^{**} \to B^{**}.$$

We may therefore define the topological centers of the left and right module actions of A on B as follows:

$$\begin{aligned}
Z_{B^{**}}(A^{**}) = Z(\pi_\ell) = \{a'' \in A^{**} : \ &the \ map \ b'' \to \pi_\ell^{***}(a'', b'') \ : \\
&B^{**} \to B^{**} \ is \ weak^* - to - weak^* \ continuous\} \\
Z^t_{B^{**}}(A^{**}) = Z(\pi_r^t) = \{a'' \in A^{**} : \ &the \ map \ b'' \to \pi_r^{t***}(a'', b'') \ : \\
&B^{**} \to B^{**} \ is \ weak^* - to - weak^* \ continuous\} \\
Z_{A^{**}}(B^{**}) = Z(\pi_r) = \{b'' \in B^{**} : \ &the \ map \ a'' \to \pi_r^{***}(b'', a'') \ : \\
&A^{**} \to B^{**} \ is \ weak^* - to - weak^* \ continuous\} \\
Z^t_{A^{**}}(B^{**}) = Z(\pi_\ell^t) = \{b'' \in B^{**} : \ &the \ map \ a'' \to \pi_\ell^{t***}(b'', a'') \ : \\
&A^{**} \to B^{**} \ is \ weak^* - to - weak^* \ continuous\}
\end{aligned}$$

We note also that if B is a left(resp. right) Banach $A - module$ and $\pi_\ell : \ A \times B \to B$ (resp. $\pi_r : \ B \times A \to B$) is left (resp. right) module action of A on B, then B^* is a right (resp. left) Banach $A - module$.
We write $ab = \pi_\ell(a, b)$, $ba = \pi_r(b, a)$, $\pi_\ell(a_1a_2, b) = \pi_\ell(a_1, a_2b)$,
$\pi_r(b, a_1a_2) = \pi_r(ba_1, a_2)$, $\pi_\ell^*(a_1b', a_2) = \pi_\ell^*(b', a_2a_1)$,
$\pi_r^*(b'a, b) = \pi_r^*(b', ab)$, for all $a_1, a_2, a \in A$, $b \in B$ and $b' \in B^*$ when there is no confusion.
Regarding A as a Banach $A - bimodule$, the operation $\pi : A \times A \to A$ extends to π^{***} and π^{t***t} defined on $A^{**} \times A^{**}$. These extensions are known, respectively, as the first(left) and the second (right) Arens products, and with each of them, the second dual space A^{**} becomes a Banach algebra. Recall that a left approximate identity ($= LAI$) [resp. right approximate identity ($= RAI$)] in Banach algebra A is a net $(e_\alpha)_{\alpha \in I}$ in A such that $e_\alpha a \longrightarrow a$ [resp. $ae_\alpha \longrightarrow a$]. We say that a net $(e_\alpha)_{\alpha \in I} \subseteq A$ is a approximate identity ($= AI$) for A if it is LAI and

RAI for A. If $(e_\alpha)_{\alpha\in I}$ in A is bounded and AI for A, then we say that $(e_\alpha)_{\alpha\in I}$ is a bounded approximate identity ($= BAI$) for A. Let A have a BAI. If the equality $A^*A = A^*$, $(AA^* = A^*)$ holds, then we say that A^* factors on the left (right). If both equalities $A^*A = AA^* = A^*$ hold, then we say that A^* factors on both sides.

2. Factorization property and Generalization of Multipliers

Definition 2.1. Let B be a left Banach $A-module$. Then B is said to be left weakly completely continuous ($= Lwcc$), if for each $a \in A$, the mapping $b \to \pi_\ell(a, b)$ from B into B is weakly compact. The definition of right weakly completely continuous ($= Rwcc$) is similar. We say that B is a weakly completely continuous ($= wcc$), if B is $Lwcc$ and $Rwcc$.

Theorem 2.2. Let B be a left Banach $A - module$ and for all $a \in A$, L_a be the linear mapping from B into itself such that $L_a b = \pi_\ell(a, b)$ for all $b \in B$. Then $AB^{**} \subseteq B$ if and only if L_a is weakly compact.

Proof. Assume that $AB^{**} \subseteq B$. We take L_a^* as the adjoint of L_a. It is easy to show that $L_a^* b' = \pi_\ell^*(b', a)$ for all $b' \in B^*$. Then for every $b'' \in B^{**}$, we have

$$\langle L_a^{**} b'', b'\rangle = \langle b'', L_a^* b'\rangle = \langle b'', \pi_\ell^*(b', a)\rangle = \langle \pi_\ell^{**}(b'', b'), a\rangle$$

$$= \langle a, \pi_\ell^{**}(b'', b')\rangle = \langle \pi_\ell^{***}(a, b''), b'\rangle.$$

It follows that

$$L_a^{**} b'' = \pi_\ell^{**}(a, b'').$$

Let $(b'_\alpha)_\alpha \subseteq B^*$ such that $b'_\alpha \overset{w^*}{\to} b'$. Since $\pi_\ell^{***}(a, b'') \in B$, we have

$$\langle b'', L_a^* b'_\alpha\rangle = \langle L_a^{**} b'', b'_\alpha\rangle = \langle \pi_\ell^{***}(a, b''), b'_\alpha\rangle = \langle \pi_\ell^{***}(a, b''), b'\rangle = \langle b'', L_a^* b'\rangle.$$

We conclude that L_a^* is $weak^* - to - weak$ continuous, so L_a is weakly compact.
Conversely, assume that $b'' \in B^{**}$. Then by *Goldstine's* theorem [9, P.424-425], there is a net $(b_\alpha)_\alpha \subseteq B$ such that $b_\alpha \overset{w^*}{\to} b''$. Since for all $a \in A$, the operator L_a is weakly compact, there is a subnet $(b_{\alpha_\beta})_\beta$ from $(b_\alpha)_\alpha$ such that $(L_a(b_{\alpha_\beta}))_\beta$ is weakly convergence to some point

of B. Since $b_\alpha \stackrel{w^*}{\to} b''$, $(L_a(b_{\alpha_\beta}) = \pi(a, b_{\alpha_\beta}))_\beta$ is weakly convergence to $\pi_\ell^{***}(a, b'')$. Consequently, for all $b' \in B^*$, we have

$$\langle \pi_\ell^{***}(a, b''), b' \rangle = lim_\beta \langle b', \pi(a, b_{\alpha_\beta}) \rangle = lim_\beta \langle b', L_a b_{\alpha_\beta}) \rangle.$$

It follows that $\pi_\ell^{***}(a, b'') \in B$. □

Corollary 2.3. i) Suppose that B is a left Banach $A - module$. Then $AB^{**} \subseteq B$ if and only if B is $Lwcc$.
ii) Suppose that B is a right Banach $A - module$. Then $B^{**}A \subseteq B$ if and only if B is $Rwcc$.

Corollary 2.4. Let A be a WSC Banach algebra with a BAI. If A is Arens regular and A is a left ideal in its second dual, then A is reflexive.

Proof. Since A is a left ideal in A^{**}, by using proceeding corollary, A is $Lwcc$. Then by using Corollary 2.8 from [16], we are done. □

Example 2.5. i) Let G be a compact group. Then we know that $L^1(G)$ is a left ideal $L^1(G)^{**}$ (resp. $M(G)^{**}$), and so by Corollary 2.4 (resp. Corollary 2.3), $L^1(G)$ (resp. $M(G)$) is a $Lwcc$.
ii) Corollary 2.4 shows that if G is a finite group. Then $L^1(G)$ and $M(G)$ are reflexive.
iii) Let G be a locally compact Hausdorff group. Let X be a subsemigroup of G which is the clouser of an open subset, and which contains the identity e of G. Let Z be a closed two-sided proper ideal in X with the property that $X \setminus Z$ is relatively compact. Let S be the quotient of X obtained by identifying all points of Z, more formally, for $x, y \in X$ write $x \sim y$ if either $x = y$ or both $x \in Z$ and $y \in Z$ and write $S = X/\sim$. By using Corollary 3.3 from [21], $L^1(S)$ is an ideal $L^1(S)^{**}$, and so by using Corollary 2.3, $L^1(S)$ is a $Lwcc$.

Theorem 2.6. Let A be a WSC Banach algebra with a BAI. If A is Arens regular and A is a right ideal in its second dual, then A is reflexive.

Proof. Proof is similar to Corollary 2.4. □

Definition 2.7. Suppose that B is a left Banach $A - module$. Let $(e_\alpha)_\alpha \subseteq A$ be left approximate identity for A. We say that $(e_\alpha)_\alpha$ is $weak^*$ left approximate identity ($= W^*LAI$) for B^*, if for all $b' \in B^*$,

we have $\pi_\ell(e_\alpha, b') \overset{w^*}{\to} b'$. The definition of the $weak^*$ right approximate identity ($= W^*RAI$) is similar.
We say that $(e_\alpha)_\alpha$ is a $weak^*$ approximate identity ($= W^*AI$) for B^*, if B^* has $weak^*$ left and right approximate identity that are equal.

Ülger in [22] shows that for a Banach algebra A with a BAI, if A is a bisded ideal in its second dual, then $AA^* = A^*A$ and if A is Arens regular, then A^* factors on the both side. In the following, we extend these problems for module actions with some results in group algebras.
Let B be a left Banach $A-module$. Then, $b' \in B^*$ is said to be left weakly almost periodic functional if the set $\{\pi_\ell(b', a) : \ a \in A, \ \| a \| \leq 1\}$ is relatively weakly compact. We denote by $wap_\ell(B)$ the closed subspace of B^* consisting of all the left weakly almost periodic functionals in B^*. The definition of the right weakly almost periodic functional ($= wap_r(B)$) is the same.
By [5, 16, 20], the definition of $wap_\ell(B)$ is equivalent to the following

$$\langle \pi_\ell^{***}(a'', b''), b' \rangle = \langle \pi_\ell^{t***t}(a'', b''), b' \rangle$$

for all $a'' \in A^{**}$ and $b'' \in B^{**}$. Thus, we can write

$$wap_\ell(B) = \{b' \in B^* : \ \langle \pi_\ell^{***}(a'', b''), b' \rangle = \langle \pi_\ell^{t***t}(a'', b''), b' \rangle \\ for\ all\ a'' \in A^{**}, \ b'' \in B^{**}\}.$$

By using [20], $b' \in wap_\ell(B)$ if and only if for each sequence $(a_n)_n \subseteq A$ and $(b_m)_m \subseteq B$ and each $b' \in B^*$, we have

$$\lim_m \lim_n \langle b', \pi_\ell(a_n, b_m) \rangle = \lim_n \lim_m \langle b', \pi_\ell(a_n, b_m) \rangle,$$

whenever both the iterated limits exist.
It is clear that $wap_\ell(B) = B^*$ if and only if $Z^\ell_{A^{**}}(B^{**}) = B^{**}$.

Definition 2.8. Let B be a left Banach $A-module$ and A has a BAI as $(e_\alpha)_\alpha$. We introduce the following subspace of B^*.

$$\ell(B^*) = \{b' \in B^* : \ \pi_\ell^*(b', e_\alpha) \overset{w}{\to} b'\}.$$

Let B be a right Banach $A-module$ and A has a BAI as $(e_\alpha)_\alpha$. Such as proceeding definition, we introduce the following subspace of B^*.

$$\Re(B^*) = \{b' \in B^* : \ \pi_r^{t*}(b', e_\alpha) \overset{w}{\to} b'\}.$$

If $\ell(B^*) = B^*$ (resp. $\Re(B^*) = B^*$), then it is clear that $(e_\alpha)_\alpha \subseteq A$ is a weakly right (resp. left) approximate identity for B^*. Therefore by using Lemma 2.8, $\ell(B^*) = B^*$ (resp. $\Re(B^*) = AB^*$) if and only if B^* factors on the left (resp. right).

Theorem 2.9. Let B be a left Banach $A-module$ and A has a $RBAI$ as $(e_\alpha)_\alpha$. Then we have the following assertions.
i) $\ell(B^*) = B^*A$.
ii) $wap_\ell(B) \subseteq \ell(B^*)$, if B^* has W^*LAI as $A-module$ $(e_\alpha)_\alpha$.

Proof. i) Let $\pi_\ell : A \times B \to B$ be the left module action such that $\pi_\ell(a, b) = ab$ for all $a \in A$ and $b \in B$. Thus for every $a \in A$, $b' \in B^*$ and $b'' \in B^{**}$, we have

$$\langle b'', \pi_\ell^*(b'a, e_\alpha)\rangle = \langle b'', \pi_\ell^*(b', ae_\alpha)\rangle = \langle \pi_\ell^{**}(b'', b'), ae_\alpha\rangle \to \langle \pi_\ell^{**}(b'', b'), a\rangle$$
$$= \langle b'', \pi_\ell^*(b', a)\rangle = \langle b'', b'a\rangle.$$

It follow that $\pi_\ell^*(b'a, e_\alpha) \xrightarrow{w} b'a$ and so $b'a \in \ell(B^*)$. For reverse inclusion, since by $Cohen's$ factorization theorem, we have B^*A is a closed subspace of B^*, $\ell(B^*) \subseteq B^*A$.

ii) Let $b' \in wap_\ell(B)$. Since $(e_\alpha)_\alpha$ is W^*LAI for B^*, $\pi_\ell^*(b', e_\alpha) \xrightarrow{w^*} b'$. Also the set $\{\pi_\ell^*(b', e_\alpha) : \ \alpha \in I\}$ is relatively weakly compact which implies that $\pi_\ell^*(b', e_\alpha) \xrightarrow{w} b'$. □

Corollary 2.10. Let B be a left Banach $A-module$ and A has a $RBAI$ and $Z_{A^{**}}(B^{**}) = B^{**}$. If B^* has W^*LAI, then B^* factors on the left.

Example 2.11.
i) Let G be a finite group. Then, by using proceeding corollary, we conclude that $L^\infty(G)M(G) = RUC(G) = L^\infty(G)$.
ii) Let G be an infinite compact group. Since $L^1(G)$ has a BAI, $L^\infty(G)$ has a W^*BAI. By using Proposition 4.4 from [22], we know that $wap(L^1(G)) = C(G)$. Thus, by using proceeding theorem, $C(G) \subseteq \ell(L^\infty(G))$.

Theorem 2.12. Let B be a right Banach $A - module$ and A has a $LBAI$ as $(e_\alpha)_\alpha$. Then we have the following assertions.
i) $\Re(B^*) = AB^*$.
ii) $wap_r(B) \subseteq \Re(B^*)$, if B^* has W^*RAI $A - module$ $(e_\alpha)_\alpha$.

Proof. Proof is similar to Theorem 2.9. □

Corollary 2.13. Let B be a right Banach $A-module$ and A has a BAI as $(e_\alpha)_\alpha$. If B factors on the left, then $wap_r(B) \subseteq \Re(B^*)$.

Proof. Let $b \in B$ and $b' \in B^*$. Since B factors on the left, there are $a \in A$ and $y \in B$ such that $b = ya$. Then

$$\langle \pi_r^{t*}(b', e_\alpha), b\rangle = \langle b', \pi_r^t(e_\alpha, b)\rangle = \langle b', \pi_r(b, e_\alpha)\rangle = \langle b', \pi_r(ya, e_\alpha)\rangle$$
$$= \langle \pi_r^*(b', y), ae_\alpha\rangle \to \langle \pi_r^*(b', y), a\rangle = \langle b', ya\rangle = \langle b', b\rangle.$$

It follows that $\pi_r^{t*}(b', e_\alpha) \overset{w^*}{\to} b'$. Then by using Theorem 2.12, we are done. $\square$

Corollary 2.14. Let B be a right Banach $A-module$ and A has a BAI and $Z^t_{B^{**}}(A^{**}) = B^{**}$. If B^* has W^*RAI $A-module$, then B^* factors on the right.

Theorem 2.15. We have the following statements.

(1) Let B be a left Banach $A-module$. If $AB^{**} \subseteq B$, then $B^*A \subseteq wap_\ell(B)$.
(2) Let B be a right Banach $A-module$. If $B^{**}A \subseteq B$, then $AB^* \subseteq wap_r(B)$.

Proof. (1) By Corollary 2.3, we know that B is $Lwcc$. Let $a \in A$ and suppose that L_a is the mapping from B into itself by definition $L_a(b) = \pi_\ell(a, b)$ for each $b \in B$. By easy calculation, it is clear that $(L_a)^*(b') = \pi_\ell^*(b', a)$. Since L_a is weakly compact, $(L_a)^*$ is weakly compact. Then the set

$$\{(L_a)^*(\pi_\ell^*(b', x)) :\ x \in A_1\},$$

is weakly compact. Now let $x \in A_1$ and $y \in B$. Then we have the following equality

$$\langle (L_a)^*(\pi_\ell^*(b', x)), y\rangle = \langle \pi_\ell^*(b', x), L_a(y)\rangle = \langle \pi_\ell^*(b', x), \pi_\ell(a, y)\rangle$$
$$= \langle \pi_\ell^*(\pi_\ell^*(b', x), a), y\rangle\rangle.$$

It follows that the mapping $\pi_\ell^*(\pi_\ell^*(b', x), a)$ is weakly compact for each $a \in A$ and $b' \in B^*$. Hence $\pi_\ell^*(b', x) \in wap_\ell(B)$, and so $B^*A \subseteq wap_\ell(B)$.

(2) Proof is similar to proceeding proof.

$\square$

Example 2.16.
i) Let G be a locally compact group and $1 \leq p \leq \infty$. We know that $L^p(G)$ is the left Banach $L^1(G) - module$ under convolution as multiplication. Assume that for all $f \in L^1(G)$, $L_f : L^p(G) \to L^p(G)$ be the linear mapping such that $L_f g = f * g$ whenever $g \in L^p(G)$. Then, since $L^1(G)L^p(G)^{**} = L^1(G)L^p(G) \subseteq L^p(G)$ for all $1 < p < \infty$, by Theorem 2.14, L_f is weakly compact.
It is the same that for all $\mu \in M(G)$, the mapping L_μ from $L^p(G)$ into itself with $L_\mu f = \mu * f$ is weakly compact whenever $1 < p < \infty$.
ii) Let G be an infinite compact group. Then we know that $L^1(G)$ is an ideal in its second dual, $L^1(G)^{**}$. Therefore, by using proceeding theorem and Proposition 3.3, from [22], we have $LUC(G) = L^\infty(G)L^1(G) \subseteq wap(L^1(G))$ and $RUC(G) = L^1(G)L^\infty(G) \subseteq wap(L^1(G))$. By using Proposition 4.4 from [22], since $wap(L^1(G)) = C(G)$, we conclude that $LUC(G) \cap RUC(G) \subseteq C(G)$, and so $LUC(G) \cap RUC(G) = C(G)$.

Let B be a Banach $A - bimodule$ and $a'' \in A^{**}$. We define the locally topological centers of the left and right module actions of a'' on B, respectively, as follows

$$Z^t_{a''}(B^{**}) = Z^t_{a''}(\pi^t_\ell) = \{b'' \in B^{**} : \ \pi_\ell^{t***t}(a'', b'') = \pi_\ell^{***}(a'', b'')\},$$
$$Z_{a''}(B^{**}) = Z_{a''}(\pi^t_r) = \{b'' \in B^{**} : \ \pi_r^{t***t}(b'', a'') = \pi_r^{***}(b'', a'')\}.$$

It is clear that

$$\bigcap_{a'' \in A^{**}} Z^t_{a''}(B^{**}) = Z^t_{A^{**}}(B^{**}) = Z(\pi^t_\ell),$$

$$\bigcap_{a'' \in A^{**}} Z_{a''}(B^{**}) = Z_{A^{**}}(B^{**}) = Z(\pi_r).$$

The definition of $Z^t_{b''}(A^{**})$ and $Z_{b''}(A^{**})$ for some $b'' \in B^{**}$ are the same.

Theorem 2.17. Let B be a Banach left $A - module$ and A has a $LBAI$ $(e_\alpha)_\alpha \subseteq A$ such that $e_\alpha \xrightarrow{w^*} e''$ in A^{**} where e'' is a left unit for A^{**}. Suppose that $Z^t_{e''}(B^{**}) = B^{**}$. Then, B factors on the right with respect to A if and only if e'' is a left unit for B^{**}.

Proof. Assume that B factors on the right with respect to A. Then for every $b \in B$, there are $x \in B$ and $a \in A$ such that $b = ax$. Then for every $b' \in B^*$, we have

$$\begin{aligned}\langle \pi_\ell^*(b', e_\alpha), b\rangle &= \langle b', \pi_\ell(e_\alpha, b)\rangle = \langle \pi_\ell^{***}(e_\alpha, b), b'\rangle \\ &= \langle \pi_\ell^{***}(e_\alpha, ax), b'\rangle = \langle \pi_\ell^{***}(e_\alpha a, x), b'\rangle \\ &= \langle e_\alpha a, \pi_\ell^{**}(x, b')\rangle = \langle \pi_\ell^{**}(x, b'), e_\alpha a\rangle \\ &\to \langle \pi_\ell^{**}(x, b'), a\rangle = \langle b', b\rangle.\end{aligned}$$

It follows that $\pi_\ell^*(b', e_\alpha) \stackrel{w^*}{\to} b'$ in B^*. Let $b'' \in B^{**}$ and $(b_\beta)_\beta \subseteq B$ such that $b_\beta \stackrel{w^*}{\to} b''$ in B^{**}. Since $Z^t_{e''}(B^{**}) = B^{**}$, for every $b' \in B^*$, we have the following equality

$$\begin{aligned}\langle \pi_\ell^{***}(e'', b''), b'\rangle &= \lim_\alpha \lim_\beta \langle b', \pi_\ell(e_\alpha, b_\beta)\rangle \\ &= \lim_\beta \lim_\alpha \langle b', \pi_\ell(e_\alpha, b_\beta)\rangle = \lim_\beta \langle b', b_\beta\rangle \\ &= \langle b'', b'\rangle.\end{aligned}$$

It follows that $\pi_\ell^{***}(e'', b'') = b''$, and so e'' is a left unit for B^{**}.
Conversely, let e'' be a left unit for B^{**} and suppose that $b \in B$. Thren for every $b' \in B^*$, we have

$$\begin{aligned}\langle b', \pi(e_\alpha, b)\rangle &= \langle \pi^{***}(e_\alpha, b), b'\rangle = \langle e_\alpha, \pi^{**}(b, b')\rangle = \langle \pi^{**}(b, b'), e_\alpha\rangle \\ &= \langle e'', \pi^{**}(b, b')\rangle = \langle \pi^{***}(e'', b), b'\rangle = \langle b', b\rangle.\end{aligned}$$

Then we have $\pi_\ell^*(b', e_\alpha) \stackrel{w}{\to} b'$ in B^*, and so by Cohen factorization theorem we are done. $\square$

Corollary 2.18. Let B be a Banach left $A-module$ and A has a $LBAI$ $(e_\alpha)_\alpha \subseteq A$ such that $e_\alpha \stackrel{w^*}{\to} e''$ in A^{**} where e'' is a left unit for A^{**}. Suppose that $Z^t_{e''}(B^{**}) = B^{**}$. Then $\pi_\ell^*(b', e_\alpha) \stackrel{w}{\to} b'$ in B^* if and only if e'' is a left unit for B^{**}.

For a Banach algebra A, we recall that a bounded linear operator $T : A \to A$ is said to be a left (resp. right) multiplier if, for all $a, b \in A$, $T(ab) = T(a)b$ (resp. $T(ab) = aT(b)$). We denote by $LM(A)$ (resp. $RM(A)$) the set of all left (resp. right) multipliers of A. The set $LM(A)$ (resp. $RM(A)$) is normed subalgebra of the algebra $L(A)$ of bounded linear operator on A.
Let B be a Banach left [resp. right] $A-module$ and $T \in \mathbf{B}(A, B)$. Then T is called extended left [resp. right] multiplier if $T(a_1a_2) =$
$\pi_r(T(a_1), a_2)$
$[resp.\ T(a_1a_2) = \pi_\ell(a_1, T(a_2))]$ for all $a_1, a_2 \in A$.

We show by $LM(A,B)$ [*resp.* $RM(A,B)$] all of the Left [resp. right] multiplier extension from A into B.
Let $a' \in A^*$. Then the mapping $T_{a'} : a \to a'a$ [*resp.* $R_{a'}\ a \to aa'$] from A into A^* is left [right] multiplier, that is, $T_{a'} \in LM(A,A^*)$ [$R_{a'} \in RM(A,A^*)$]. $T_{a'}$ is weakly compact if and only if $a' \in wap(A)$. So, we can write $wap(A)$ as a subspace of $LM(A,A^*)$.

Theorem 2.19. Let B be a Banach $A-bimodule$ with a BAI $(e_\alpha)_\alpha \subseteq A$. Then

(1) If $T \in LM(A,B)$, then $T(a) = \pi_r^{***}(b'',a)$ for some $b'' \in B^{**}$.
(2) If $T \in RM(A,B)$, then $T(a) = \pi_\ell^{***}(a,b'')$ for some $b'' \in B^{**}$.

Proof. (1) Since $(T(e_\alpha))_\alpha \subseteq B$ is bounded, it has weakly limit point in B^{**}. Let $b'' \in B^{**}$ be a weakly limit point of $(T(e_\alpha))_\alpha$ and without loss generally, take $T(e_\alpha) \xrightarrow{w} b''$. Then for every $b' \in B^*$ and $a \in A$, we have

$$\begin{aligned}\langle \pi_r^{***}(b'',a), b'\rangle &= \lim_\alpha \langle b', T(e_\alpha)a\rangle = \lim_\alpha \langle b', T(e_\alpha a)\rangle \\ &= \lim_\alpha \langle T^*(b'), e_\alpha a\rangle = \langle T^*(b'), a\rangle = \langle b', T(a)\rangle.\end{aligned}$$

It follows that $\pi_r^{***}(b'',a) = T(a)$.

(2) Proof is similar to (1).

□

In the proceeding theorem, if we take $B = A$, then we have the following statements

(1) If $T \in LM(A)$, then $T(a) = a''a$ for some $a'' \in A^{**}$.
(2) If $T \in RM(A)$, then $T(a) = aa''$ for some $a'' \in A^{**}$.

Definition 2.20. Let B be a Banach left $A-module$ and $b'' \in B^{**}$. Suppose that $(b_\alpha)_\alpha \subseteq B$ such that $b_\alpha \xrightarrow{w^*} b''$. We define the following set

$$\widetilde{Z}_{b''}(A^{**}) = \{a'' \in A^{**} :\ \pi_\ell^{***}(a'', b_\alpha) \xrightarrow{w^*} \pi_\ell^{***}(a'', b'')\},$$

which is subspace of A^{**}. It is clear that $Z_{b''}(A^{**}) \subseteq \widetilde{Z}_{b''}(A^{**})$, and so

$$Z_{B^{**}}(A^{**}) = \bigcap_{b''\in B^{**}} Z_{b''}(A^{**}) \subseteq \bigcap_{b''\in B^{**}} \widetilde{Z}_{b''}(A^{**}).$$

For a Banach right $A-module$, the definition of $\widetilde{Z}^t_{a''}(B^{**})$ is similar.

Theorem 2.21, see [15, Theorem 3]. Let B be a left Banach $A-module$ and $T \in \mathbf{B}(A, B)$. Consider the following statements.

(1) $T = \ell_b$, for some $b \in B$.
(2) $T^{**}(a'') = \pi_\ell^{***}(a'', b'')$ for some $b'' \in B^{**}$ such that $\widetilde{Z}_{b''}(A^{**}) = A^{**}$.
(3) $T^*(B^*) \subseteq BB^*$.

Then (1) $\Rightarrow$ (2) $\Rightarrow$ (3).
Assume that B has WSC. If we take $T \in RM(A, B)$ and B has a sequential BAI, then (1), (2) and (3) are equivalent.

Proof. (1) $\Rightarrow$ (2)
Let $T = \ell_b$, for some $b \in B$. Then $T^{**}(a'') = \ell_b^{**}(a'') = \pi_\ell^{***}(a'', b)$ for every $a'' \in A^{**}$, and so proof is hold.
(2) $\Leftrightarrow$ (3)
Take $a'' \in (BB^*)^\perp$. Assume that $b'' \in B^{**}$ and $(b_\alpha)_\alpha \subseteq B$ such that $b_\alpha \xrightarrow{w^*} b''$. For every $b' \in B^{**}$, we have the following equality

$$\langle a'', T^*(b')\rangle = \langle T^{**}(a''), b'\rangle = \langle \pi_\ell^{***}(a'', b''), b'\rangle = \lim_\alpha \langle \pi_\ell^{***}(a'', b_\alpha), b'\rangle$$
$$= \lim_\alpha \langle a'', \pi_\ell^{**}(b_\alpha, b')\rangle = 0.$$

It follows that $T^*(B^*) \subseteq BB^*$.
Take $T \in RM(A, B)$ and suppose that B is WSC with sequential BAI. It is enough, we show that (3) $\Rightarrow$ (1). Assume that $(e_n)_n \subseteq A$ is a BAI for B. Then for every $b' \in B^*$, we have

$$\mid \langle b', T(e_n)\rangle - \langle b', T(e_m)\rangle \mid=\mid \langle T^*(b'), e_n - e_m\rangle \mid=\mid \langle \pi_\ell^{**}(b, b'), e_n - e_m\rangle \mid$$
$$=\mid \langle b, \pi_\ell^{*}(b', e_n - e_m)\rangle \mid=\mid \langle b', \pi_\ell(e_n - e_m, b)\rangle \mid\rightarrow 0.$$

It follows that $(T(e_n))_n$ is weakly Cauchy sequence in B and since B is WSC, there is $b \in B$ such that $T(e_n) \xrightarrow{w} b$ in B. Let $a \in A$. Then for every $b' \in B^*$, we have

$$\langle b', \pi_\ell(a, b)\rangle = \langle \pi_\ell^*(b', a), b)\rangle = \lim_n \langle \pi_\ell^*(b', a), T(e_n)\rangle$$
$$= \lim_n \langle b', \pi_\ell(a, T(e_n))\rangle = \lim_n \langle b', T(ae_n)\rangle$$
$$= \lim_n \langle T^*(b'), ae_n\rangle = \langle T^*(b'), a\rangle$$
$$= \langle b', T(a)\rangle.$$

Thus $\ell_b(a) = \pi_\ell(a, b) = T(a)$. □

Example 2.22. Let G be a locally compact group. Then by convolution multiplication, $M(G)$ is a $L^1(G)-bimodule$. Let $f \in L^1(G)$ and $T(\mu) = \mu * f$ for all $\mu \in M(G)$. Then $T^*(L^\infty(G)) \subseteq M(G)M(G)^*$. Also if we take $T(\mu) = f*\mu$ for all $\mu \in M(G)$, then we have $T^*(L^\infty(G)) \subseteq M(G)^*M(G)$.

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Kazem Haghnejad Azar
Department of Mathematics and Applications, University of Mohaghegh Ardabili, P.O.Box 179, Ardabil, Iran
Email: haghnejadmath@yahoo.com