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ON COINCIDENCE POINTS OF GENERALIZED CONTRACTIVE PAIR MAPPINGS IN CONVEX METRIC SPACES

MOHAMMAD MOOSAEI* AND AZIZOLLAH AZIZI

Abstract. We obtain a contractive condition for the existence of coincidence points of a pair of self-mappings defined on a nonempty subset of a complete convex metric space. Moreover, we show that weakly compatible pairs have at least a common fixed point.

Key Words: Banach operator pair, Coincidence point, Common fixed point, Convex metric space, Fixed point, Weakly compatible pair.
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1. Introduction and preliminaries

W. Takahashi [22] introduced the notion of convexity in metric spaces and proved that all normed spaces and their convex subsets are convex metric spaces. He also gave some examples of convex metric spaces which are not embedded in any normed/Banach spaces. Afterward, many authors have studied fixed point theorems in convex metric spaces, for example see [2, 3, 4, 5, 6, 9, 10, 21].

In this paper, we introduce a generalized contractive condition for a pair of self-mappings and prove the existence of a coincidence point for such a pair in a complete convex metric space as well as we prove the existence of a common fixed point for weakly compatible mappings and Banach operator pairs.

We now review notations and definitions needed. We denote by $\mathbb{N}$ and

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∗Address correspondence to Mohammad Moosaei; E-mail: m.moosaei@basu.ac.ir

$\mathbb{R}$ the set of natural numbers and the set of real numbers, respectively. We also denote by I the identity mapping. Let K be a nonempty subset of a metric space (X, d), and let S, T be self-mappings of K. A point x of K is called (i) a fixed point of T if $Tx = x$; (ii) a common fixed point of the pair (S, T) if $Sx = Tx = x$; (iii) a coincidence point of the pair (S, T) if $Sx = Tx$. The set of fixed points of T is denoted by $F(T)$. The set of common fixed points(respectively, coincidence points)of the pair (S, T) is denoted by $F(S, T)$ (respectively, $C(S, T)$). Note that $C(I, T) = F(T)$. The mapping T is called (i) a contraction if there exists $k \in [0, 1)$ such that $d(Tx, Ty) \leq kd(x, y)$ for all $x, y \in K$; (ii) an S-contraction if there exists $k \in [0, 1)$ such that $d(Tx, Ty) \leq kd(Sx, Sy)$ for all $x, y \in K$. The pair (S, T) is said to be (i) commuting if $STx = TSx$ for all $x \in K$; (ii) R-weakly commuting [19] if there exists $R > 0$ such that $d(STx, TSx) \leq Rd(Sx, Tx)$ for all $x \in K$. If R=1, then the mappings are called weakly commuting [20]; (iii) compatible [16] if $\lim_{n\to\infty} d(STx_n, TSx_n) = 0$, whenever $\{x_n\}_{n=1}^{\infty}$ is a sequence in K such that $\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = x$ for some $x \in K$; (iv) weakly compatible if they commute on $C(S, T)$ i.e., $STx = TSx$ for all $x \in C(S, T)$ (see [8, 17] for more details). It is well known that commuting mappings are weakly commuting and weakly commuting mappings are R-weakly mappings. Moreover, R-weakly mappings are compatible and compatible mappings are weakly compatible.

The following example shows that the converse of the above results are not true in general.

Example 1.1. Let $X = \mathbb{R}$ with the usual metric $d(x, y) =| x - y |$ for all $x, y \in X$, we have:

(1) Let $K = [0, 1]$. Let $Sx = x^2$ and $Tx = \frac{x^2}{2}$ for all $x \in K$. It is trivial that S and T are weakly commuting but are not commuting.

(2) Let $K = [0, \infty]$, and consider $Sx = 2x - 1$ and $Tx = x^2$ for all $x \in K$. Then S and T are 2-weakly commuting but are not weakly commuting (see [19]).

(3) Let $K = X$, $Sx = x^3$, $Tx = 2x^3$, $x \in K$. Then S and T are compatible but are not R-weakly commuting (see [14, 15, 16] for more details).

(4) Let $K = [0, 10]$, and define self-mappings S and T of K by $S(1) = 1$, $S(x) = 4$ if $1 < x < 6$, $S(x) = 1$ if $6 \leq x \leq 10$, and $T(1) = 1$, $T(x) = 3$ if $1 < x < 6$, $T(x) = x - 5$ if $6 \leq x \leq 10$. For sequence $\{x_n\}_{n=1}^{\infty}$ defined by $x_n = 6 + \frac{1}{n}, n \geq 1$, we have $\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = 1$

but $\lim_{n\to\infty} d(STx_n, TSx_n) = 3 \neq 0$. So the mappings S and T are not compatible. It is easy to see that S and T are weakly compatible.

The ordered pair (S, T) is called a Banach operator pair if the set $F(T)$ is S-invariant, namely $S(F(T)) \subseteq F(T)$ (see [7]). It is easy to see that if the mappings S and T are commuting, then the pair (S, T) is a Banach operator pair but the converse is not true in general (see Example 1(ii) of [7]). If (S, T) is a Banach operator pair, then (T, S) need not be a Banach operator pair (see [7, 12]).

Definition 1.2. Let (X, d) be a metric space. A mapping $W\colon X \times X \times [0, 1] \to X$ is said to be a convex structure on X, if

$$d(u, W(x, y, \lambda)) \leq \lambda d(u, x) + (1 - \lambda)d(u, y)$$

for each $x, y, u \in X$ and $\lambda \in [0, 1]$ (see [22]). A metric space (X, d) together with a convex structure W is called a convex metric space. A nonempty subset K of X is said to be convex if $W(x, y, \lambda) \in K$ for all $x, y \in K$ and $\lambda \in [0, 1]$ (see [1, 22]).

Let be X a convex metric space. The open balls and the closed balls are convex subsets of X. If $\{C_\alpha\}_{\alpha\in J}$ is a family of convex subsets of X, then $\bigcap_{\alpha\in J} C_\alpha$ is a convex subset of X (see [1, 22] for more details). All normed spaces and their convex subsets are convex metric spaces. But there are some examples of convex metric spaces which are not embedded in any normed space (see [22]).

Definition 1.3. Let K be a convex subset of a convex metric space X with the structure W. A self-mapping T of K is said to be affine if $T(W(x, y, \lambda)) = W(Tx, Ty, \lambda)$ for each $x, y \in K$ and $\lambda \in [0, 1]$ (see [13]).

2. Main results

The following lemma of [11] plays a basic role to prove Theorem 2.3.

Lemma 2.1. *Let X be a nonempty set and $f : X \to X$ a function. Then there exists a subset $E \subseteq X$ such that $f(E) = f(X)$ and $f : E \to X$ is one-to-one.*

Theorem 2.2. *Let K be a nonempty closed convex subset of a complete convex metric space X, and let T be a self-mapping of K. If T satisfies*

$$ad(x, Tx) + bd(y, Ty) + cd(Tx, Ty) \leq ed(x, y) \tag{2.1}$$

for each $x, y \in K$, where $(a, b, c, e) \in \mathbb{R}^4$ and

$$2b- \mid c \mid\leq e < 2(a + b + c)- \mid c \mid .$$

Then T has a fixed point. Moreover, if $e < c$, then T has a unique fixed point.

Proof. By Theorem 3.2 of [18] , $F(T)$ is nonempty. Suppose $e < c$ and $u, v \in F(T)$. Inequality (2.1) implies $(c-e)d(u,v) \leq 0$. Therefore, $u = v$. □

Theorem 2.3. *Let K be a nonempty subset of a complete convex metric space X, and let S and T be two self-mappings of K such that $S(K)$ is closed and convex as well as $T(K) \subseteq S(K)$. If (S,T) satisfies*

$$ad(Sx,Tx) + bd(Sy,Ty) + cd(Tx,Ty) \leq ed(Sx,Sy) \tag{2.2}$$

for each $x, y \in K$, where $(a,b,c,e) \in \mathbb{R}^4$ and

$$2b - \mid c \mid \leq e < 2(a+b+c) - \mid c \mid .$$

Then S and T have a coincidence point. Moreover, if $e < c$, then the restriction of the mapping S to $C(S,T)$ is a constant mapping.

Proof. By Lemma 2.1, there exists a subset $E \subseteq K$ such that $S(E) = S(K)$ and $S : E \to K$ is one-to-one. Now, define a function $H : S(E) \to S(E)$ by $H(Sx) = Tx$ for all $x \in E$. Since $S : E \to K$ is one-to-one, H is well-defined. From(2.2), we obtain

$$ad(Sx, H(Sx)) + bd(Sy, H(Sy)) + cd(H(Sx), H(Sy)) \leq ed(Sx, Sy)$$

for each $x, y \in E$. By Theorem 2.2, there exists $u \in E$ such that $H(Su) = Su$. Hence u is a coincidence point of (S,T). We next show that the mapping S on $C(S,T)$ is constant. Let $y \in K$ and $Sy = Ty$. Since $u, y \in C(S,T)$ and $e < c$, inequality (2.2) implies $Su = Ty$. □

Theorem 2.4. *Let K be a nonempty subset of a complete convex metric space X. Let S and T be two self-mappings of K such that $S(K)$ is a closed convex subset of X, and $T(K) \subseteq S(K)$. If (S,T) is a weakly compatible pair and satisfies in inequality (2.2) and $e < c$, then S and T have a common fixed point.*

Proof. By Theorem 2.3, $C(S,T)$ is nonempty. Let $u \in C(T,S)$; hence, $Su = Tu = v$. Since (S,T) is weakly compatible, $Sv = Tv$. From (2.2), we obtain $cd(v,Tv) \leq ed(v,Tv)$. Since $e < c$, we conclude that $Sv = Tv = v$. □

Theorem 2.5. *Let K be a nonempty subset of a complete convex metric space X. Let S and T be two self-mappings of K such that $F(S)$ is a nonempty closed convex subset of X. Assume that (S,T) satisfies* *in inequality (2.2), and (T, S) is a Banach operator pair. Then the mappings S and T have at least a common fixed point. In particular, if $e < c$, then the mappings S and T have a unique common fixed point.*

Proof. Since (T, S) is a Banach operator pair, we have $T(F(S)) \subseteq F(S)$. By Theorem 2.3, there exists $u \in F(S)$ such that $Tu = Su = u$. So $F(S, T)$ is nonempty. Let $e < c$ and $v \in F(S, T)$. From (2.2), we get $cd(u, v) \leq ed(u, v)$. This implies that $u = v$. Therefore, $F(S, T)$ is a singleton set. □

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Moohammad Moosaei
Department of Mathematics, Bu-Ali Sina University, P.O.Box 6517838695, Hamedan, Iran
Email: `m.moosaei@basu.ac.ir`

Azizollah Azizi
Department of Mathematics, Faculty of Science, Imam Khomeini International University, P.O.Box 34149-16818, Qazvin, Iran
Email: `azizi@SCI.ikiu.ac.ir`