Journal of Hyperstructures 4 *(1) (2015), 69-83.*
ISSN: 2322-1666 print/2251-8436 online

DELTA BASIS FUNCTIONS AND THEIR APPLICATIONS FOR SOLVING TWO-DIMENSIONAL LINEAR FREDHOLM INTEGRAL EQUATIONS

FARSHID MIRZAEE, SEYEDE FATEMEH HOSEINI AND KHADIJEH MAHDAVI MOGHADAM

Abstract. In this paper an expansion method, based on two-dimensional delta functions (2D-DFs), is developed to find numerical solutions of two-dimensional linear Fredholm integral equations. The main characteristic behind this method is that this method reduce such problems to a system of algebraic equations. Since this approach does not need integration, all calculations can be easily implemented. Finally, we estimate the error of the method, and present two numerical examples to demonstrate the accuracy of the method.

Key Words: Two-dimensional delta functions, Operational matrix, Linear Fredholm integral equation.

2010 Mathematics Subject Classification: Primary: 45B05; Secondary: 41A10, 65G99.

1. Introduction

The integral equations method is widely used for solving many problems in mathematics, physics and engineering. Many numerical methods of high accuracy have been developed for solving integral equations [2, 6, 10, 11, 12]. In this work, we consider the linear two-dimensional

Received: 15 September 2013, Accepted: 18 June 2015. Communicated by Ali Taghavi Jelodar;
∗Address correspondence to Farshid Mirzaee; E-mail: f.mirzaee@malayeru.ac.ir

Fredholm integral equation (2D-FIE) of the second kind as

$$f(s,t) = g(s,t) + \lambda \int_0^{T_1} \int_0^{T_2} K(s,t,x,y) f(x,y)dxdy, \quad (x,y) \in \Omega = [0,T_1) \times [0,T_2), \tag{1.1}$$

where the functions $K(s,t,x,y)$ and $g(s,t)$ are known functions defined on $E = \Omega \times \Omega$ and Ω, respectively, and $f(s,t)$ is an unknown scalar valued function defined on Ω.

Many problems in physics and engineering fields can be transformed into two-dimensional Fredholm integral equations [2, 10]. Existence and uniqueness of the solution of 2D-FIEs were investigated by El-Borai and et al in [5]. They proved that under the following conditions, the solution of integral equation (1.1) is exist and unique. The conditions are

(i) $\left\{\int_o^{T_1}\int_o^{T_2}\int_o^{T_1}\int_o^{T_2} |K(s,t,x,y)|^2 dsdtdxdy\right\}^{\frac{1}{2}} \leqslant \varepsilon$, where ε is small enough.

(ii) The given function $g(s,t)$ and its partial derivatives with respect to $s,\ t$ are continuous and its normality in $L^2[0,T_1] \times L^2[0,T_2]$ is given by

$$\left\{\int_o^{T_1}\int_o^{T_2} |g(s,t)|^2 dsdt\right\}^{\frac{1}{2}} = \mathcal{M}.$$

(iii) The unknown function $f(s,t)$ satisfies Lipschitz condition for the arguments $s,\ t$, where its norm is considered in $L^2[0,T_1] \times L^2[0,T_2]$.

(iv) $|\lambda| < \frac{1}{\mathcal{M}}$.

Although several numerical methods for approximating the solutions of one-dimensional integral equations were presented [9, 16, 17, 18, 22, 21], for two-dimensional ones only a few have been discussed in the literature [1, 8, 7, 13, 14, 20]. Here, a set of two-dimensional delta functions are used for solving two-dimensional integral equations. These functions are extensions of one-dimensional delta functions (1D-DFs) that were introduced by Roodaki et al. [19] for solving systems of integral equations. One-dimensional delta functions (1D-DFs) were defined by using the well-known triangular orthogonal functions [19].

In this article, firstly we review some properties of one and two-dimensional delta basis functions and their operational matrices. Then we utilize them for solving two-dimensional Fredholm integral equations in Section 4. In Section 5, we provide the error analysis for the method. Numerical examples are given in Section 6 and finally, we conclude the article in Section 7.

2. A review of one-dimensional delta basis functions

One-dimensional delta functions (1D-DFs), were introduced by Roodaki et al. [19]. They used these functions for solving systems of integral equations.

Definition 2.1. In an m-set of one-dimensional delta functions (1D-DFs) over interval $[0, 1)$, the ith component function is defined as

$$\Delta_i(s) = \begin{cases} \frac{(s-(i-1)h)}{h} & (i-1)h \leqslant s < ih \\ \frac{1-(s-ih)}{h} & ih \leqslant s < (i+1)h \\ 0 & \text{otherwise,} \end{cases}$$

where $i = 0, 1, 2, \ldots, m$ with a positive integer value for m, and $h = \frac{1}{m}$.

So the 1D-DFs vector $\Delta(s)$ can be defined as

$$\Delta(s) = [\Delta_0(s), \Delta_1(s), \ldots, \Delta_m(s)]^T,$$

where

$$\begin{aligned} \Delta_0(s) &= T1_0(s), \\ \Delta_i(s) &= T1_i(s) + T2_{i-1}(s), \ i = 1, 2, \ldots, m-1, \\ \Delta_m(s) &= T2_{m-1}(s), \end{aligned}$$

where $T1_i(s), T2_i(s)$ are the ith triangular orthogonal functions (TFs) defined in [19]. Thus

$$\Delta(s) = \begin{pmatrix} T1(s) \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ T2(s) \end{pmatrix},$$

where $T1(s)$ and $T2(s)$ are defined in [4].
Furthermore

$$\sum_{i=0}^{m} \Delta_i(s) = \sum_{i=0}^{m-1} [T1_i(s) + T2_i(s)] = \sum_{i=0}^{m-1} \Phi_i(s) = 1.$$

where $\Phi_i(s)$ is the ith block-pulse function [4].

The following properties are presented by Roodaki et al.

$$\Delta(s).\Delta^T(s) \simeq diag(\Delta(s)). \tag{2.1}$$

On the other hand,

$$\begin{aligned}
\int_0^1 \Delta_0(s).\Delta_i(s)ds &= \begin{cases} \frac{h}{3} & i=0 \\ \frac{h}{6} & i=1 \\ 0 & i=2,3,\dots,m, \end{cases} \\
\int_0^1 \Delta_i(s).\Delta_j(s)ds &= \begin{cases} \frac{2h}{3} & i=j \\ \frac{h}{6} & i-j=\pm1,\ i,j=1,2,\dots,m-1 \\ 0 & otherwise, \end{cases} \\
\int_0^1 \Delta_m(s).\Delta_i(s)ds &= \begin{cases} \frac{h}{3} & i=m \\ \frac{h}{6} & i=m-1 \\ 0 & i=0,1,2,\dots,m-2. \end{cases}
\end{aligned}$$

Therefore from above equations, we have

$$\int_0^1 \Delta(s).\Delta^T(s)ds = D,$$

where D is the following three-diagonal matrix

$$D = \begin{pmatrix} \frac{h}{3} & \frac{h}{6} & 0 & \dots & 0 \\ \frac{h}{6} & \frac{2h}{3} & \frac{h}{6} & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & \frac{h}{6} & \frac{2h}{3} & \frac{h}{6} \\ 0 & \dots & 0 & \frac{h}{6} & \frac{h}{3} \end{pmatrix}.$$

Now, assume that X is an $(m+1)$-vector. It can be concluded from Eq. (2.1) that

$$\begin{aligned}
\Delta(s).\Delta^T(s).X &\simeq diag(\Delta(s)).X \\
&= diag(X).\Delta(s) \\
&= \tilde{X}.\Delta(s),
\end{aligned}$$

where $\tilde{X}$ is an $(m+1)\times(m+1)$ diagonal matrix.

Let $f(s)$ be a function over $[0,1)$. The expansion of f with respect to 1D-DFs can be written as

$$f(s) \simeq \sum_{i=0}^{m} c_i\Delta_i(s) = C^T.\Delta(s),$$

where $c_i = f(ih),\ i = 0, 1, \ldots, m$. The vector C is called the 1D-DFs coefficient vector.

3. Two-dimensional delta functions and their properties

In this section, 2D-DFs are defined by extending 1D-DFs that is the new basis idea in this paper.

Definition 3.1. An $(m_1+1)\times(m_2+1)$-set of 2D-DFs on the space $\Gamma = [0,1)\times[0,1)$ is defined by

$$\Delta_{i,j}(s,t) = \Delta_i(s).\Delta_j(t) = \tag{3.1}$$

$$\begin{cases} \left(\frac{s-(i-1)h_1}{h_1}\right)\left(\frac{t-(j-1)h_2}{h_2}\right) & (i-1)h_1 \leqslant s < ih_1,\ (j-1)h_2 \leqslant t < jh_2 \\ \left(\frac{s-(i-1)h_1}{h_1}\right)\left(\frac{(j+1)h_2-t}{h_2}\right) & (i-1)h_1 \leqslant s < ih_1,\ jh_2 \leqslant t < (j+1)h_2 \\ \left(\frac{(i+1)h_1-s}{h_1}\right)\left(\frac{t-(j-1)h_2}{h_2}\right) & ih_1 \leqslant s < (i+1)h_1,\ (j-1)h_2 \leqslant t < jh_2 \\ \left(\frac{(i+1)h_1-s}{h_1}\right)\left(\frac{(j+1)h_2-t}{h_2}\right) & ih_1 \leqslant s < (i+1)h_1,\ jh_2 \leqslant t < (j+1)h_2 \\ 0 & otherwise, \end{cases}$$

where m_1 and m_2 are arbitrary positive integers, $i = 0, 1, 2, \ldots, m_1,\ j = 0, 1, 2, \ldots, m_2,\ h_1 = \frac{1}{m_1}$ and $h_2 = \frac{1}{m_2}$.

Furthermore,

$$\sum_{i=0}^{m_1}\sum_{j=0}^{m_2}\Delta_{i,j}(s,t) = \sum_{i=0}^{m_1-1}\sum_{j=0}^{m_2-1}\Phi_{i,j}(s,t) = 1,$$

where

$$\Phi_{i,j}(s,t) = \begin{cases} 1 & ih_1 \leqslant s < (i+1)h_1,\ jh_2 \leqslant t < (j+1)h_2 \\ 0 & otherwise. \end{cases}$$

3.1. **Properties of 2D-DFs.** Obviously, $\Delta_{i,j}(s,t),\ i=0,1,2,\ldots,m_1,\ j=0,1,2,\ldots,m_2$ are disjoint

$$\Delta_{i_1,j_1}(s,t).\Delta_{i_2,j_2}(s,t)\simeq\begin{cases}\Delta_{i_1,j_1}(s,t) & i_1=j_1,\ i_2=j_2\\ 0 & otherwise.\end{cases}$$

Using Eq. (3.1), we can define 2D-DFs vector $\mathbf{\Delta}(s,t)$ as

$$\mathbf{\Delta}(s,t)=[\mathbf{\Delta}_0(s,t),\mathbf{\Delta}_1(s,t),\ldots,\mathbf{\Delta}_{m_1}(s,t)]^T, \tag{3.2}$$

where

$$\begin{aligned}\mathbf{\Delta}_0(s,t) &= [\Delta_{0,0}(s,t),\Delta_{0,1}(s,t),\ldots,\Delta_{0,m_2}(s,t)]^T,\\ \mathbf{\Delta}_1(s,t) &= [\Delta_{1,0}(s,t),\Delta_{1,1}(s,t),\ldots,\Delta_{1,m_2}(s,t)]^T,\\ &\vdots\\ \mathbf{\Delta}_{m_1}(s,t) &= [\Delta_{m_1,0}(s,t),\Delta_{m_1,1}(s,t),\ldots,\Delta_{m_1,m_2}(s,t)]^T.\end{aligned}$$

From the above representation, it follows that

$$\mathbf{\Delta}_p(s,t).\mathbf{\Delta}_q^T(s,t)\simeq\begin{cases}diag(\mathbf{\Delta}_p(s,t)) & p=q\\ 0 & p\neq q.\end{cases}$$

Hence

$$\mathbf{\Delta}(s,t).\mathbf{\Delta}^T(s,t)\simeq\begin{pmatrix}diag(\mathbf{\Delta}_0(s,t)) & 0 & \ldots & 0\\ 0 & diag(\mathbf{\Delta}_1(s,t)) & \ldots & 0\\ \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & \ldots & diag(\mathbf{\Delta}_{m_1}(s,t))\end{pmatrix},$$

where 0 is the $(m_2+1)\times(m_2+1)$- zero matrix. So we have

$$\mathbf{\Delta}(s,t).\mathbf{\Delta}^T(s,t)\simeq diag(\mathbf{\Delta}(s,t))=\tilde{\mathbf{\Delta}}(s,t).$$

Also

$$\mathbf{\Delta}(s,t).\mathbf{\Delta}^T(s,t).X\simeq\tilde{\mathbf{\Delta}}(s,t).X=\tilde{X}.\mathbf{\Delta}(s,t),$$

where X is an $(m_1+1)(m_2+1)$-vector and $\tilde{X}=diag(X)$.

The disjointness property of $\mathbf{\Delta}_i(s,t)$ for $i=0,1,\ldots,m_1$ also implies that for every $(m_2+1)\times(m_2+1)$-matrix B,

$$\mathbf{\Delta}_i^T(s,t).B.\mathbf{\Delta}_i(s,t)\simeq\hat{B}.\mathbf{\Delta}_i(s,t),$$

where $\hat{B}$ is an (m_2+1)-vector with elements equal to the diagonal entries of B. Thus for every $(m_1+1)(m_2+1)\times(m_1+1)(m_2+1)$-matrix A, we have

$$\mathbf{\Delta}^T(s,t).A.\mathbf{\Delta}(s,t) \simeq \hat{A}.\mathbf{\Delta}(s,t),$$

which $\hat{A}$ is an $(m_1+1)(m_2+1)$-vector with elements equal to the diagonal entries of matrix A.

Also, we can carry out double integration of $\mathbf{\Delta}(s,t)$

$$\int_0^1 \int_0^1 \mathbf{\Delta}(s,t).\mathbf{\Delta}^T(s,t)dsdt = \mathbf{D}, \tag{3.3}$$

were

$$\mathbf{D} = D \otimes D,$$

where $\otimes$ denotes the kronecker product defined for arbitrary matrices A and B as $A \otimes B = (A_{i,j}B)$.

3.2. **Function expansion with 2D-DFs.** A function $f(s,t)$ on Γ may be extended using 2D-DFs as

$$f(s,t) \simeq \sum_{i=0}^{m_1} \sum_{j=0}^{m_2} c_{i,j}\Delta_{i,j}(s,t) = C^T.\mathbf{\Delta}(s,t), \tag{3.4}$$

where C is an $(m_1+1)(m_2+1)$-vector and $c_{i,j} = f(ih_1, jh_2)$ and $i = 0,1,2,\ldots,m_1,\ j = 0,1,2,\ldots,m_2$. The vector C is called the 2D-DFs coefficients vector.

Similarly, a function $K(s,t,x,y)$ on $\Gamma \times \Gamma$ can be approximated using 2D-DFs as follows

$$K(s,t,x,y) \simeq \mathbf{\Delta}^T(s,t) \begin{pmatrix} K(0,0,x,y) \\ K(0,h_2,x,y) \\ \vdots \\ K(0,m_2h_2,x,y) \\ K(h_1,0,x,y) \\ \vdots \\ K(h_1,m_2h_2,x,y) \\ \vdots \\ K(m_1h_1,0,x,y) \\ \vdots \\ K(m_1h_1,m_2h_2,x,y) \end{pmatrix}.$$

In the same way, each $K(ih_1, jh_2, x, y),\ i = 0, 1, 2, \dots, m_1,\ j = 0, 1, 2, \dots, m_2$ can be expanded by 2D-DFs with respect to independent variables x, y. Hence, the expansion of $K(s, t, x, y)$ can be written as

$$K(s,t,x,y) \simeq \mathbf{\Delta}^T(s,t) \begin{pmatrix} k_{0,0}^T \mathbf{\Delta}(x,y) \\ k_{0,1}^T \mathbf{\Delta}(x,y) \\ \vdots \\ k_{0,m_2}^T \mathbf{\Delta}(x,y) \\ k_{1,0}^T \mathbf{\Delta}(x,y) \\ \vdots \\ k_{1,m_2}^T \mathbf{\Delta}(x,y) \\ \vdots \\ k_{m_1,0}^T \mathbf{\Delta}(x,y) \\ \vdots \\ k_{m_1,m_2}^T \mathbf{\Delta}(x,y) \end{pmatrix} = \mathbf{\Delta}^T(s,t).k.\mathbf{\Delta}(x,y),$$

where $\mathbf{\Delta}(s,t)$ and $\mathbf{\Delta}(x,y)$ are 2D-DFs vectors of dimension $(m_1 + 1)(m_2 + 1)$ and $(m_3 + 1)(m_4 + 1)$, respectively, and k is a $(m_1 + 1)(m_2 + 1) \times (m_3 + 1)(m_4 + 1)$ 2D-DFs coefficient matrix as

$$k=\begin{pmatrix} k_{0,0}^T \\ \vdots \\ k_{0,m_2}^T \\ k_{1,0}^T \\ \vdots \\ k_{1,m_2}^T \\ \vdots \\ k_{m_1,0}^T \\ \vdots \\ k_{m_1,m_2}^T \end{pmatrix}=\begin{pmatrix} \begin{pmatrix} K(0,0,0,0) \\ \vdots \\ K(0,0,0,m_4h_4) \\ K(0,0,h_3,0)) \\ \vdots \\ K(0,0,h_3,m_4h_4)) \\ \vdots \\ K(0,0,m_3h_3,0) \\ \vdots \\ K(0,0,m_3h_3,m_4h_4) \end{pmatrix}^T \\ \vdots \\ \begin{pmatrix} K(0,m_2h_2,0,0) \\ \vdots \\ K(0,m_2h_2,0,m_4h_4) \\ K(0,m_2h_2,h_3,0) \\ \vdots \\ K(0,m_2h_2,h_3,m_4h_4) \\ \vdots \\ K(0,m_2h_2,m_3h_3,0) \\ \vdots \\ K(0,m_2h_2,m_3h_3,m_4h_4) \end{pmatrix}^T \\ \vdots \\ \begin{pmatrix} K(m_1h_1,0,0,0) \\ \vdots \\ K(m_1h_1,0,0,m_4h_4) \\ K(m_1h_1,0,h_3,0) \\ \vdots \\ K(m_1h_1,0,h_3,m_4h_4) \\ \vdots \\ K(m_1h_1,0,m_3h_3,0) \\ \vdots \\ K(m_1h_1,0,m_3h_3,m_4h_4) \end{pmatrix}^T \\ \vdots \\ \begin{pmatrix} K(m_1h_1,m_2h_2,0,0) \\ \vdots \\ K(m_1h_1,m_2h_2,0,m_4h_4) \\ K(m_1h_1,m_2h_2,h_3,0) \\ \vdots \\ K(m_1h_1,m_2h_2,h_3,m_4h_4) \\ \vdots \\ K(m_1h_1,m_2h_2,m_3h_3,0) \\ \vdots \\ K(m_1h_1,m_2h_2,m_3h_3,m_4h_4) \end{pmatrix}^T \end{pmatrix}.$$

4. Solving two-dimensional Fredholm integral equation of the second kind

In this section, we present a 2D-DFs method for solving 2D-FIE (1.1). Since any finite interval $[a, b)$ can be transformed to $[0, 1)$ by linear maps, it is supposed that $[0, T_1) = [0, T_2) = [0, 1)$, without any loss of generality.
Using Eq. (3.4), we can approximate function $g(s, t)$ as

$$g(s,t) \simeq G^T.\mathbf{\Delta}(s,t) = \mathbf{\Delta}^T(s,t).G, \tag{4.1}$$

where $\mathbf{\Delta}(s, t)$ is defined in Eq. (3.2), and G is $(m_1 + 1)(m_2 + 1)$-vector.

Substituting Eqs. (3.4), (3.5) and (4.1) into the Eq. (1.1), we have

$$\begin{aligned} \mathbf{\Delta}^T(s,t).C &\simeq \mathbf{\Delta}^T(s,t).G + \lambda \int_0^1 \int_0^1 \mathbf{\Delta}^T(s,t).k.\mathbf{\Delta}(x,y)\mathbf{\Delta}^T(x,y).C dxdy \\ &= \mathbf{\Delta}^T(s,t).(G + \lambda k.\mathbf{D}.C). \end{aligned}$$

Therefore

$$C = G + \lambda.k.\mathbf{D}.C.$$

Therefore, we have the following system

$$G = (I - \lambda.k.\mathbf{D}).C, \tag{4.2}$$

that is a linear system of $(m_1 + 1)(m_2 + 1)$ algebraic equations with $(m_1 + 1)(m_2 + 1)$ unknown coefficients.

5. Convergence analysis

Assume that $(C[\Gamma], \|.\|)$ is the Banach space of all continuous functions on Γ with the norm

$$\|f(s,t)\| = \max_{(s,t)\in\Gamma} |f(s,t)|.$$

Furthermore, let for all $s, t, x, y \in [0, 1)$, $|K(s, t, x, y)| \leqslant M$, where M is a positive real number. We denote the error 2D-DFs by

$$e_{m_1,m_2} = \|f(s,t) - \bar{f}_{m_1,m_2}(s,t)\|,$$

where $f(s, t)$, $\bar{f}_{m_1,m_2}(s, t)$ show the exact and approximate solutions of the two-dimensional linear Fredholm integral equation, respectively. If we note to Eq. (4.2), we will see the coefficients $C_{i,j}$'s are not optimal. By using optimal coefficients, the representational error e_{m_1,m_2} can be reduced.

Theorem 5.1. *(Convergence). The solution of the two-dimensional linear Fredholm integral equation by using 2D-DFs approximation converges if* $0 < |\lambda|.M < 1$.

Proof. From definition of e_{m_1,m_2}, we have

$$\begin{aligned}
e_{m_1,m_2} &= \|f(s,t) - \bar{f}_{m_1,m_2}(s,t)\| \\
&= \max_{(s,t)\in\Gamma} |f(s,t) - \bar{f}_{m_1,m_2}(s,t)| \\
&= \max |g(s,t) + \lambda \int_0^1 \int_0^1 K(s,t,x,y) f(x,y) dxdy \\
&- \ g(s,t) + \lambda \int_0^1 \int_0^1 K(s,t,x,y) \bar{f}_{m_1,m_2}(x,y) dxdy \\
&\leqslant \max |\lambda| \int_0^1 \int_0^1 |K(s,t,x,y)|\ |f(x,y) - \bar{f}_{m_1,m_2}(x,y)| dxdy \\
&\leqslant |\lambda|.M \int_0^1 \int_0^1 \max |f(x,y) - \bar{f}_{m_1,m_2}(x,y)| dxdy \\
&= |\lambda|.M \|f(s,t) - \bar{f}_{m_1,m_2}(s,t)\| \\
&= |\lambda|.M.e_{m_1,m_2}.
\end{aligned}$$

Therefore,

$$(1 - |\lambda|.M) e_{m_1,m_2} \leqslant 0.$$

If $0 < |\lambda|.M < 1$ then, $e_{m_1,m_2} \to 0$ by increasing m_1, m_2. □

6. Numerical illustration

In this section, we present two examples and their numerical results to show the high accuracy of the solution obtained by 2D-DFs. Without any loss of generality, let $m_1 = m_2 = m$.

Example 6.1. ([15]) Consider the two-dimensional linear Fredholm integral equation

$$f(s,t) = g(s,t) + \int_0^1 \int_0^1 \frac{s}{(s+t)(1+x+y)} f(x,y) dxdy, \tag{6.1}$$

where $(s,t) \in \Gamma^2$ and $g(s,t) = \frac{1}{(1+s+t)^2} - \frac{s}{6(8+t)}$, with the exact solution $f(s,t) = \frac{1}{(1+s+t)^2}$.

Table 1 gives the comparison of the results of the absolute error functions obtained by the present method , block-pulse functions method [14] and the two-dimensional triangular orthogonal functions method [15] for $m = 32$.

Table 1: Absolute error for Example 1

(s,t)	Method of [14]	Method of [15]	Present method
(0.0,0.0)	1.58e - 01	0.00e+00	0.00e+00
(0.1,0.1)	3.26e - 01	7.58e - 03	1.34e - 04
(0.2,0.2)	2.73e - 01	9.54e - 03	1.99e - 04
(0.3,0.3)	2.42e - 01	9.60e - 03	2.21e - 04
(0.4,0.4)	3.18e - 01	9.03e - 03	4.53e - 04
(0.5,0.5)	3.35e - 01	3.00e - 04	2.29e - 04
(0.6,0.6)	1.69e - 01	4.30e - 04	1.90e - 04
(0.7,0.7)	1.14e - 01	2.24e - 03	6.80e - 04
(0.8,0.8)	2.91e - 01	2.62e - 03	7.11e - 04
(0.9,0.9)	2.13e - 01	2.81e - 03	4.64e - 05

Example 6.2. ([3]) Consider the two-dimensional linear Fredholm integral equation

$$f(s,t) = g(s,t) + \int_0^1 \int_0^1 (x \sin y + 1) f(x,y)\,dx dy \ , \tag{6.2}$$

where $(s,t) \in \Gamma^2$ and $g(s,t) = s \cos t - \frac{1}{6} \sin 1 (3 + \sin 1)$, with the exact solution $f(s,t) = s \cos t$.

Table 2, illustrates the error results for this example for $m = 8,\ 16$. Also, we compare the maximum absolute error computed by the present method, rationalized Haar functions method [3] and block-pulse functions method [14] in Table 3. It is obvious from Table 3 that the results obtained by the present method is better than that obtained in [3] and [14].

Table 2: Absolute error of present method for Example 2

(s,t)	m=8	m=16
(0.0,0.0)	2.72e - 03	8.31e - 04
(0.1,0.1)	2.60e - 03	7.84e - 04
(0.2,0.2)	2.36e - 03	7.69e - 04
(0.3,0.3)	2.19e - 03	7.41e - 04
(0.4,0.4)	2.27e - 03	6.58e - 04
(0.5,0.5)	2.73e - 03	8.31e - 04
(0.6,0.6)	2.10e - 03	5.98e - 04
(0.7,0.7)	1.72e - 03	6.66e - 04
(0.8,0.8)	1.69e - 03	6.54e - 04
(0.9,0.9)	2.05e - 03	5.70e - 04

Table 3: Approximate infinity-norm of absolute error for Example 2

m	Method of [3]	Method of [14]	Present method
8	6.09e - 02	4.39e - 01	3.69e - 03
16	3.09e - 02	3.99e - 01	9.30e - 04

7. Conclusion

In this study, we developed an efficient and computationally attractive method to solve the linear two-dimensional Fredholm integral equations. The method is based on the use of two-dimensional delta basis functions. Also, error analysis of the proposed method is provided in a theorem. The implementation of the current approach in analogy to existing methods is more convenient and accurate. An advantage of the considered method is that using 2D-TFs does not need any integration to evaluate the coefficients, therefore a lot of computational efforts have been reduced. The numerical examples that have been presented in the paper and the compared results support our claim. In addition, through the comparison with exact solution, we see that this method have good reliability and efficiency.

Acknowledgments

The authors are very grateful to referees and associate editor for their comments and suggestions which have improved the paper.

References

[1] A. Alipanah, S. Esmaeili, *Numerical solution of the two-dimensional Fredholm integral equations using Gaussian radial basis function*, J. Comput. Appl. Math., **235** (18) (2011) 5342-5347.

[2] K.E. Atkinson, *The Numerical Solution of Integral Equations of the Second Kind*, Cambridge Uni. Press (1997).

[3] E. Babolian, S. Bazm, P. Lima, *Numerical solution of nonlinear two-dimensional integral equations using rationalized Haar functions*, Commun. Nonl. Sci. Numer. Simul., **16** (2011) 1164-1175.

[4] E. Babolian, K. Maleknejad, M. Roodaki, H. Almasieh, *Two-dimensional triangular functions and their applications to nonlinear 2D Volterra-Fredholm integral equations*, Comput. Math. Appl., **60** (2010) 1711-1722.

[5] M.M. El-Borai, M. A. Abdou, M. Basseem, *An Analysis of two dimesional integral equations of the seconed kined*, Le Matematiche, **62** (2007) 15-39.

[6] L.M. Delves, J. L. Mohamed, *Computational methods for integral equations*, Cambridge Uni. Press, Cambridge (1985).

[7] G. Han, J. Wang, *Extrapolation of Nyström solution for two dimensional nonlinear Fredholm integral equations*, J. Comput. Appl. Math., **134** (2001) 259-268.

[8] R. Hanson, J. Phillips, *Numerical solution of two-dimensional integral equations using linear elements*, SIAM J. Numer. Anal., **15** (1978) 113-121.

[9] C. Hsio, *Hybrid function method for solving Fredholm and Volterra integral equations of the second kind*, J. Comput. Appl. Math., **230**(1) (2009) 59-68.

[10] A.J. Jerri, *Introduction to Integral Equations with Applications*, John Wiley and Sons, INC (1999).

[11] R.P. Kanwal, *Linear integral equations*, Academic Press, London (1971).

[12] R. Kress, *Linear integral equations*, Springer, Berlin (1999).

[13] F. Liang, F.R. Lin, *A fast numerical solution method for two dimensional Fredholm integral equations of the second kind based on piecewise polynomial interpolation*, Appl. Math. Comput., **216** (2010) 3073-3088.

[14] K. Maleknejad, S. Sohrabi, B. Baranji, *Application of 2D-BPFs to nonlinear integral equations*, Commun. Nonlin. Sci. Numer. Simul., **15** (2010) 527-535.

[15] F. Mirzaee, S. Piroozfar, *Numerical solution of the linear two-dimensional Fredholm integral equations of the second kind via two-dimensional triangular orthogonal functions*, J. King Saud Uni. Sci., **22** (2010) 185-193.

[16] T. Okayama, T. Matsuo, M. Sugihara, *Sinc-collocation methods for weakly singular Fredholm integral equations of the second kind*, J. Comput. Math. Appl., **234** (2010) 1211-1227.

[17] K. Orav-Puurand, A. Pedas, G. Vainikko, *Nyström type methods for Fredholm integral equations with weak singularities*, J. Comput. Appl. Math., **234** (2010) 2848-2858.

[18] Y. Ordokhani, M. Razzaghi, *Solution of nonlinear Volterra-Fredholm-Hammerstein integral equations via a collocation method and rationalized Haar functions*, Appl. Math. Lett., **21** (2008) 4-9.

[19] M. Roodaki, H. Almasieh, *Delta basis functions and their applications to systems of integral equations*, Comput. Math. Appl., **63** (2012) 100-109.

[20] W.J. Xie, F.R. Lin, *A fast numerical solution method for two dimensional Fredholm integral equations of the second kind*, Appl. Num. Math., **59** (2009) 1709-1719.

[21] A. Yildirim, *Homotopi perturbation method for the mixed Volterra-Fredholm integral equations*, chaos, Solot. Fract., **42** (2009) 2760-2764.

[22] S.A. Yousefi, A. Lotfi, Mehdi Dehghan, *He's variational iteration method for solving nonlinear mixed Volterra- Fredholm integral equations*, Comput. Math. Appl., **58** (2009) 2172-2176.

Farshid Mirzaee
Faculty of Mathematical Sciences and Statistics, Malayer University, P. O. Box 65719-95863, Malayer, Iran
Email: f.mirzaee@malayeru.ac.ir, f.mirzaee@iust.ac.ir

Seyede Fatemeh Hoseini
Faculty of Mathematical Sciences and Statistics, Malayer University, P. O. Box 65719-95863, Malayer, Iran
Email: fatemeh.hoseini@stu.malayeru.ac.ir

Khadijeh Mahdavi Moghadam
Faculty of Mathematical Sciences and Statistics, Malayer University, P. O. Box 65719-95863, Malayer, Iran
Email: khadije_mahdavi@yahoo.com