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UPPER BOUNDS AND ATTACHED PRIMES OF TOP LOCAL COHOMOLOGY MODULES DEFINED BY A PAIR OF IDEALS

SH. PAYROVI AND S. KARIMI

Abstract. Throughout R is a Noetherian local ring. In this paper we study cohomological dimension of an R-module M with respect to a pair of ideals and some of its relations with the attached prime ideals of M and the cohomological dimension of M with respect to an ideal. Furthermore, we generalize some results of [5] in particular, Theorem 2.8.

Key Words: Attached prime ideals, Top local cohomology modules.
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1. Introduction

Throughout R is a commutative Noetherian local ring with non-zero identity, I, J are two ideals of R and M is an R-module. For notations and terminologies not given in this paper, the reader is referred to [2] and [7], if necessary.

As a generalization of the usual local cohomology modules, the local cohomology modules with respect to a system of ideals was introduced by Bijan-Zadeh in [1]. As a special case of these extended modules, Takahashi, Yoshino and Yoshizawa in [7] defined the local cohomology modules with respect to a pair of ideals. To be more precise, let $W(I,J) = \{\mathfrak{p} \in (R) \ : \ I^n \subseteq \mathfrak{p} + J \ \text{ for some positive integer } \ n\}$. For

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∗Address correspondence to Shiroye Peyrovi, E-mail: shpayrovi@sci.ikiu.ac.ir

R-module M, the (I,J)-torsion submodule $\Gamma_{I,J}(M)$ of M, which consists of all elements x of M with $\mathrm{Supp}(Rx) \subseteq W(I,J)$, is considered. Let i be an integer. The local cohomology functor $H^i_{I,J}$ with respect to (I,J) is defined to be the i-th right derived functor of $\Gamma_{I,J}$. The i-th local cohomology module of M with respect to (I,J) is denoted by $H^i_{I,J}(M)$. When $J=0$, then $H^i_{I,J}$ coincides with the usual local cohomology functor H^i_I with the support in the closed subset $V(I)$.

In this paper we study some properties of cohomological dimension of an R-module M with respect to a pair of ideals I and J, $\mathrm{cd}(I,J,M)$, the supremum of all integers r for which $H^r_{I,J}(M) \neq 0$. When $J=0$ this integer coincides with $\mathrm{cd}(I,M)$.

Recall that for an R-module K, a prime ideal $\mathfrak{p}$ of R is said to be an associated prime ideal of K if $\mathfrak{p} = \mathrm{Ann}(x)$ for some element x of K. We denote the set of associated prime ideals of K by $\mathrm{Ass}_R(K)$. Furthermore, for an R-module K, a prime ideal $\mathfrak{p}$ of R is said to be an attached prime ideal of K if $\mathfrak{p} = \mathrm{Ann}(K/N)$ for some submodule N of K. We denote the set of attached prime ideals of K by $\mathrm{Att}_R(K)$. When K has a secondary representation, this definition agrees with the usual definition of attached primes. One of the main results of this paper is to relate these two items, the cohomological dimension of an R-module M with respect to a pair of ideals and the set of attached prime ideals of top local cohomology module of M. In 3.1 we relate these two items. We also relate the cohomological dimension with respect to a pair of ideals and the cohomological dimension with respect to an ideal in 2.1(ii), 2.4 and 3.4. For relate three concepts (cohomological dimension with respect to a pair of ideals, cohomological dimension with respect to an ideal and the set of attached prime ideals of an R-module M) we exhibit 3.9.

In addition, Proposition 3.2 in [4] shows the change of the cohomological dimension of an R-module under taking submodule. In this paper we state this change when we have inclusions on ideals I and J in 2.2 and 2.3. But for showing this relationship we need a generalization of [5, Theorem 2.8]. In [5, Theorem 2.8], M. T. Dibaei and R. Jafari showed, when R is complete, that if $T \subseteq \mathrm{Assh}(M)$, then there exists an ideal $\mathfrak{a}$ of R such that $T = \mathrm{Att}(H^n_{\mathfrak{a}}(M))$. In 3.6 we show that if T is a non-empty subset of $\mathrm{Att}(H^n_I(M))$, then there exists an ideal J of R such that $T = \mathrm{Att}(H^n_{I,J}(M))$.

Finally we offer some results which are related to annihilator and attached prime ideals of top local cohomology modules. Note that for an Artinian R-module A, the set of all minimal elements of $\mathrm{Att} A$ is exactly the set of all minimal prime ideals containing $\mathrm{Ann} A$.

2. Upper bounds for local cohomology modules

Throughout this section R is a Noetherian local ring with maximal ideal $\mathfrak{m}$, I, J are ideals of R and M is a non-zero finitely generated R-module. Let $\mathrm{cd}(I, J, M)$ denote the supremum of all integers r for which $H^r_{I,J}(M) \neq 0$. We call this integer the cohomological dimension of M with respect to ideals I, J. In [4, Corollary 3.3] it is shown that

$$\mathrm{cd}(I, J, M) = \inf\{i \mid H^i_{I,J}(R/\mathfrak{p}) = 0 \ \text{for all}\ \mathfrak{p} \in \mathrm{Supp}_R(M)\} - 1.$$

The following are some basic facts about cohomological dimension of M with respect to ideals I, J.

Theorem 2.1. *Let M be a finitely generated R-module of dimension n and let $H^n_{I,J}(M) \neq 0$. Then there exists a*

(i) *submodule N of M such that $H^n_{I,J}(M) \cong H^n_I(M/N)$ and $\mathrm{cd}(I, J, N) < n$.*
(ii) *quotient module L of M such that $H^n_{I,J}(M) \cong H^n_I(L)$ and $\mathrm{cd}(I, J, M) = \mathrm{cd}(I, L)$.*

Proof. See [3, Theorem 2.3] and its proof. □

Corollary 2.2. *Let M be a finitely generated R-module of dimension n and let I_1, I_2 be two proper ideals of R such that $I_1 \subseteq I_2$. If $\mathrm{cd}(I_1, J, M) = n$, then $\mathrm{cd}(I_2, J, M) = n$. In particular, if $\mathrm{cd}(I, J, M) = n$, then $\mathrm{cd}(\mathfrak{m}, J, M) = n$.*

Proof. The result follows by [3, Theorem 2.5] and the fact that $\mathrm{cd}(I_2, J, M) \leq n$. □

Theorem 2.3. *Let M be a finitely generated R-module of dimension n and let J_1, J_2 be two ideals of R such that $J_1 \subseteq J_2$ and $\sqrt{I + J_1} = \mathfrak{m}$. Then $\mathrm{cd}(I, J_2, M) \leq \mathrm{cd}(I, J_1, M)$.*

Proof. In view of [7, Theorems 4.5, 4.7(2)] we have

$$\mathrm{cd}(I, J_2, M) \leq \dim M/J_2M \leq \dim M/J_1M = \mathrm{cd}(I, J_1, M).$$

□

Corollary 2.4. *Let M be a finitely generated R-module of dimension n. Then $\mathrm{cd}(\mathfrak{m}, J, M) \leq \mathrm{cd}(\mathfrak{m}, M)$.*

Proof. It follows easily by setting $J_1 = 0$ and $J_2 = J$ in Theorem 2.3. □

3. Attached prime ideals of top local cohomology modules

Recall that R is a Noetherian local ring with maximal ideal $\mathfrak{m}$, I, J are ideals of R and M is a non-zero finitely generated R-module of dimension n. The main results of this section is to relate the cohomological dimension of an R-module M with respect to a pair of ideals and the set of attached prime ideals of top local cohomology module of M.

Theorem 3.1. *Let M be a finitely generated R-module of dimension n. Then*

$$\mathrm{Att}_R H^n_{I,J}(M) = \{\mathfrak{p} \in \mathrm{Supp}(M) : \mathrm{cd}(I, J, R/\mathfrak{p}) = n\}.$$

Proof. Assume that $\mathfrak{p} \in \mathrm{Supp}(M)$ and $\mathrm{cd}(I, J, R/\mathfrak{p}) = n$. Then $H^n_{I,J}(R/\mathfrak{p}) \neq 0$. If $J \nsubseteq \mathfrak{p}$, then $\dim R/(\mathfrak{p} + J) < n$ so in view of [7, Theorem 4.3] we have $H^n_{I,J}(R/\mathfrak{p}) = 0$ which is a contradiction. Thus $J \subseteq \mathfrak{p}$. Hence, $H^n_{I,J}(R/\mathfrak{p}) \cong H^n_I(R/\mathfrak{p})$ by [7, Corollary 2.5] since $R/\mathfrak{p}$ is a J-torsion R-module. Therefore, $H^n_I(R/\mathfrak{p}) \neq 0$ and by [3, Theorem 2.1] $\mathfrak{p} \in \mathrm{Att} H^n_{I,J}(M)$. Assume that $\mathfrak{p} \in \mathrm{Att} H^n_{I,J}(M)$. Then by [3, Theorem 2.1] and [7, Theorem 4.3] we have $H^n_I(R/\mathfrak{p}) \cong H^n_{I,J}(R/\mathfrak{p})$ so that $\mathrm{cd}(I, J, R/\mathfrak{p}) = n$. □

By setting $J = 0$ we obtain the following corollary which is a main result of [6].

Corollary 3.2. *Let M be a finitely generated R-module of dimension n. Then*

$$\mathrm{Att}_R H^n_I(M) = \{\mathfrak{p} \in \mathrm{Supp}(M) : \mathrm{cd}(I, R/\mathfrak{p}) = n\}.$$

Corollary 3.3. *Let M be a finitely generated R-module of dimension n. If $\mathrm{cd}(I, J, M) = n$, then there exists some $\mathfrak{p} \in \mathrm{Assh}(M)$ such that $J \subseteq \mathfrak{p}$.*

Proof. It follows easily from the proof of Theorem 3.1. □

Corollary 3.4. *Let M be a finitely generated R-module of dimension n. If $\mathrm{cd}(I, J, M) = n$, then $\mathrm{cd}(I, M) = n$*

Proof. By Theorem 3.1 and [3, Theorem 2.1] the result follows. □

Corollary 3.5. *Let M be a finitely generated R-module of dimension n and let J_1, J_2 be two ideals of R such that $J_1 \subseteq J_2$ and $\mathrm{cd}(I, J_2, M) = n$. Then $\mathrm{cd}(I, J_1, M) = n$.*

Proof. By [3, Theorem 2.1] we have $\mathrm{Att} H^n_{I,J_2}(M) \subseteq \mathrm{Att} H^n_{I,J_1}(M)$ since $J_1 \subseteq J_2$. Now by Theorem 3.1, $\mathrm{cd}(I, J_2, M) = n$ implies $\mathrm{Att} H^n_{I,J_2}(M) \neq \emptyset$. So that $\mathrm{Att} H^n_{I,J_1}(M) \neq \emptyset$. Now again using Theorem 3.1 we have the result. □

In the following by using [5, Theorem 2.8] we show that any subset T of $\mathrm{Assh}(M)$ can be written as two ways, as the set of attached prime ideals of top local cohomology module with respect to an ideal $H^n_{\mathfrak{a}}(M)$ and as the set of attached prime ideals of top local cohomology module with respect to a pair of ideals $H^n_{I,J}(M)$ for some ideals $\mathfrak{a}$, I and J of R.

Theorem 3.6. *Let R be complete and let M be a finitely generated R-module of dimension n and T be a non-empty subset of $\mathrm{Att} H^n_I(M)$. Then there exists an ideal J of R such that $T = \mathrm{Att} H^n_{I,J}(M)$.*

Proof. By assumption T is a non-empty subset of $\mathrm{Assh}(M)$. Thus there exists an ideal $\mathfrak{a}$ of R such that $T = \mathrm{Att} H^n_{\mathfrak{a}}(M)$ see [5, Theorem 2.8]. Set $J := \cap_{\mathfrak{p} \in \mathrm{Att} H^n_{\mathfrak{a}}(M)} \mathfrak{p}$. We show that $T = \mathrm{Att} H^n_{\mathfrak{a}}(M) = \mathrm{Att} H^n_{I,J}(M)$. Assume that $\mathfrak{q} \in \mathrm{Att} H^n_{I,J}(M)$. Then by [3, Theorem 2.1] it follows that $J \subseteq \mathfrak{q}$. Thus $\mathfrak{p} \subseteq \mathfrak{q}$ for some $\mathfrak{p} \in \mathrm{Att} H^n_{\mathfrak{a}}(M)$. Hence, this fact that $\mathfrak{p}$ is in $\mathrm{Assh}(M)$ shows that $\mathfrak{p} = \mathfrak{q}$ and so $\mathfrak{q} \in T = \mathrm{Att} H^n_{\mathfrak{a}}(M)$. Now, let $\mathfrak{q} \in T = \mathrm{Att} H^n_{\mathfrak{a}}(M)$. Then $J \subseteq \mathfrak{q}$ and also $\mathfrak{q} \in \mathrm{Att} H^n_I(M)$. Therefore, $\mathfrak{q} \in \mathrm{Att} H^n_{I,J}(M)$ by [3, Theorem 2.1]. □

Corollary 3.7. *Let R be complete and let M be a finitely generated R-module of dimension n and T be a non-empty subset of $\mathrm{Assh}(M)$. Then $T = \mathrm{Att} H^n_{I,J}(M) = \mathrm{Att} H^n_{\mathfrak{a}}(M)$ for some ideals I, J and $\mathfrak{a}$ of R.*

Proof. It follows easily by [5, Theorem 2.8] and Theorem 3.6. □

Theorem 3.8. *Let R be complete and let M be a finitely generated R-module with $\dim M = \mathrm{cd}(I, J, M) = n$. Then for each $\mathfrak{p} \in \mathrm{Att} H^n_{I,J}(M)$ there exists $Q \in \mathrm{Supp}(M)$ with $\dim(R/Q) = 1$ such that $H^n_Q(R/\mathfrak{p}) \neq 0$.*

Proof. Set $T = \mathrm{Att} H^n_{I,J}(M)$. Then in view of Corollary 3.7 we have $T = \mathrm{Att} H^n_{\mathfrak{a}}(M)$ for some ideal $\mathfrak{a}$ of R. Now from [5, Proposition 2.1] there exists an integer r such that for all $1 \leq i \leq r$ there exists $Q_i \in \mathrm{Supp}(M)$ with $\dim(R/Q_i) = 1$ such that $\cap_{\mathfrak{p} \in T} \mathfrak{p} \not\subseteq Q_i$, in addition we may assume that $\mathfrak{a} = \cap_{i=1}^r Q_i$. Assume that $\mathfrak{p} \in \mathrm{Att} H^n_{I,J}(M)$. Then $\mathfrak{p} \in \mathrm{Att} H^n_{\mathfrak{a}}(M)$. In view of

$$\mathrm{Att} H^n_{\mathfrak{a}}(M) = \{\mathfrak{q} \in \mathrm{Supp}(M) : \mathrm{cd}(\mathfrak{a}, R/\mathfrak{q}) = n\}$$

we have $H^n_{\cap_{i=1}^r Q_i}(R/\mathfrak{p}) \neq 0$. Now from [2, 3.2.3] by setting $\mathfrak{b} = \cap_{i=1}^{r-1} Q_i$ and $\mathfrak{c} = Q_r$ we have the following long exact sequence

$$H^n_{\mathfrak{b}+\mathfrak{c}}(R/\mathfrak{p}) \to H^n_{\mathfrak{b}}(R/\mathfrak{p}) \oplus H^n_{\mathfrak{c}}(R/\mathfrak{p}) \to H^n_{\mathfrak{b}\cap\mathfrak{c}}(R/\mathfrak{p}) \to 0,$$

where $H^n_{\mathfrak{b}\cap\mathfrak{c}}(R/\mathfrak{p}) = H^n_{Q_1\cap\cdots\cap Q_r}(R/\mathfrak{p}) \neq 0$. So we have $H^n_{\mathfrak{b}}(R/\mathfrak{p}) \oplus H^n_{\mathfrak{c}}(R/\mathfrak{p}) \neq 0$. Therefore $H^n_{\mathfrak{b}}(R/\mathfrak{p}) \neq 0$ or $H^n_{\mathfrak{c}}(R/\mathfrak{p}) \neq 0$. If $H^n_{\mathfrak{c}}(R/\mathfrak{p}) \neq 0$ we are done. Otherwise from [2, 3.2.3] and by setting $\mathfrak{b} = \cap_{i=1}^{r-2} Q_i$ and $\mathfrak{c} = Q_{r-1}$ and repeat this method we can get the result. □

Corollary 3.9. *Let R be complete and let M be a finitely generated R-module with $\dim M = \mathrm{cd}(I, J, M) = n$. Then for each $\mathfrak{p} \in \mathrm{Att}H^n_{I,J}(M)$ there exists $Q \in \mathrm{Supp}(M)$ with $\dim(R/Q) = 1$ such that $\mathrm{cd}(I, J, R/\mathfrak{p}) = \mathrm{cd}(I, R/\mathfrak{p}) = \mathrm{cd}(Q, R/\mathfrak{p})$.*

Proposition 3.10. *Let R be complete and let M be a finitely generated R-module of dimension n. Then*

(i) $V(\mathrm{Ann}H^n_{I,J}(M)) = V(\mathrm{Ann}H^n_I(M/JM))$.
(ii) $V(\mathrm{Ann}H^n_{I,J}(M)) = V(\mathrm{Ann}H^n_{\mathfrak{a}}(M))$ *for some ideal* $\mathfrak{a}$ *of* R.
(iii) $J = \cap_{\mathfrak{p}\in \mathrm{Att}H^n_{I,J}(M)}\mathfrak{p} = \sqrt{\mathrm{Ann}H^n_{I,J}(M)}$.

Proof. (i) It is well known that for an Artinian R-module A the set of all minimal elements of $\mathrm{Att}A$ is exactly the set of all minimal prime ideals containing $\mathrm{Ann}A$. So that

$$\begin{aligned} V(\mathrm{Ann}H^n_{I,J}(M)) &= \cup_{\mathfrak{p}\in Min V(\mathrm{Ann}H^n_{I,J}(M))} V(\mathfrak{p}) \\ &= \cup_{\mathfrak{p}\in Min\mathrm{Att}H^n_{I,J}(M)} V(\mathfrak{p}) = \cup_{\mathfrak{p}\in Min\mathrm{Att}H^n_I(M/JM)} V(\mathfrak{p}) \\ &= \cup_{\mathfrak{p}\in Min V(\mathrm{Ann}H^n_I(M/JM))} V(\mathfrak{p}) = V(\mathrm{Ann}H^n_I(M/JM)), \end{aligned}$$

where by $\mathfrak{p} \in Min\mathrm{Att}H^n_{I,J}(M)$ we mean that $\mathfrak{p}$ is a minimal element of $\mathrm{Att}H^n_{I,J}(M)$.
(ii) It follows from Corollary 3.7.
(iii) It follows from (i) and Theorem 3.6. □

Corollary 3.11. *Let R be complete and let M be a non-zero finitely generated R-module of dimension n such that $\mathrm{Supp}(M/JM) = \mathrm{Att}H^n_{I,J}(M)$. Then $V(\mathrm{Ann}H^n_{I,J}(M)) = V(\mathrm{Ann}(M/JM))$.*

Proof. It follows by the proof of Proposition 3.10 (i). □

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References

[1] M. H. Bijan-Zadeh, Torsion theories and local cohomology over commutative Noetherian rings, *J. London Math. Soc. (2)*, **19**, (1979), 402-410.

[2] M. P. Brodmann and R. Y. Sharp, *Local cohomology: an algebraic introduction with geometric applications*, Cambridge University Press, Cambridge, (1998).

[3] L. Chu, Top local cohomology modules with respect to a pair of ideals, *Proc. Amer. Math. Soc.*, **139**, (2011), 777-782.

[4] L. Z. Chue and Q. Wang, Some result on local cohomology modules defined by a pair of ideals, *J. Math. Kyoto Univ.* **49** (2009) 193-200. MR2531134 (2010e:13013)

[5] M. T. Dibaei and R. Jafari, Top local cohomology modules with specified attached primes, *Algebra Colloq.*, **15**, (2008), 341-344.

[6] M. T. Dibaei and S. Yassemi, Attached primes of the top local cohomology modules with respect to an ideal, *Arch. Math.*, **84**, (2005), no. 4, 292-297.

[7] R. Takahashi, Y. Yoshino and T. Yoshizawa, Local cohomology based on a nonclosed support defined by a pair of ideals, *J. Pure Appl. Algebra*, **213**, (2009), 582-600.

Sh. Payrovi
Department of Mathematics, Imam Khomeini International University , P.O.Box 34149-1-6818, Qazvin, Iran
Email: `shpayrovi@sci.ikiu.ac.ir`

S. Karimi
Faculty of Science, Payame Noor University, Tehran, Iran.
Email: `susan_karimi2@yahoo.com`