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# THE STABILITY OF A GENERAL QUADRATIC FUNCTIONAL EQUATION IN FUZZY BANACH SPACES

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Abstract. The generalized Hyers-Ulam-Rassias stability proposition in respect of the quadratic functional equation namely
$f(x+y+z)+f(x-y)+f(x-z) = f(x-y-z)+f(x+y)+f(x+z)$
is what is taken into account to be dealt with in this paper.

**Key Words:** Fuzzy Banach space, Quadratic functional equation, Generalized Hyers-Ulam-Rassias stability.

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## 1. Introduction

The stability of functional equation raised a dispute which was made by S. M. Ulam [24] in 1940 that entwined the stability of group homomorphisms. Let $G_1$ be a group and let $G_2$ be a metric group with the metric $d(\cdot,\cdot)$. Given $\varepsilon > 0$ does there exist a $\delta > 0$ such that if a mapping $h : G_1 \longrightarrow G_2$ satisfies the inequality $d(h(xy), h(x)h(y)) < \delta$ for all $x, y \in G_1$, then a homomorphism $H : G_1 \longrightarrow G_2$ exists with $d(h(x), H(x)) < \varepsilon$ for all $x \in G_1$? Hyers in the next year put a rejoinder of the controversy for the Cauchy functional equation with statements "if $\delta > 0$ and $f : E \longrightarrow E_1$ with $E$ and $E_1$ Banach spaces, such that $\|f(x+y) - f(x) - f(y)\| \leqslant \delta$ for all $x, y \in E$ then there exists a unique $g : E \longrightarrow E_1$ such that $g(x+y) = g(x) + g(y)$ and $\|f(x) - g(x)\| \leqslant \delta$ for all $x, y \in E$". T. Aoki [1] prepared a widespread shape to the stability

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result that turned up in [9] for additive mapping which Hyers stability again took a panoramic form at the hand of Th. M. Rassias in 1978 for linear mapping by considering an unbounded Cauchy difference in [18].

The quadratic function $f(x) = cx^2$ satisfies the functional equation $f(x+y) + f(x-y) = 2f(x) + 2f(y)$ which is why the equation is redefined as the quadratic functional equation. In particular a solution of the quadratic functional equation is called a quadratic mapping. The aforementioned functional equation in favour of which a generalized Hyers-Ulam stability theorem has been formulated and justified by Skof [23] for the function $f : X \to Y$, where $X$ is a normed space and $Y$ is a Banach space. In [5] P. W. Cholewa showed that the result of Skof is true even when $X$ is an abelian group. S. Czerwik turned out the just-mentioned quadratic functional equation of the Hyers-Ulam-Rassias stability in [6]. The result got a generalized figure in the hands of Th.M. Rassias in [19] and C. Borelli and G. L. Forti in [3]. Two variables forms that emerged out of all of the above papers as regards stability theorem affirmed for quadratic functional equation. However as far as the paper [10] is concerned what the authors have accomplished is that of the Hyers-Ulam-Rasssias stability for the quadratic functional equation

$$f(x+y+z)+f(x-y)+f(x-z) = f(x-y-z)+f(x+y)+f(x+z) \tag{1.1}$$

in three variables. Fuzzy set theory, formulated by Zadeh [26] in 1965, is accepted as a potential mechanism for giving a standardized mould to what appear to be ambivalent and enigmatic. It is fuzzy theory that works in the position where classical theories fail to act upto. Fuzzy norm on linear spaces is the concept initiated by Katsaras [11] in 1984. Some mathematicians subsequently analysed fuzzy metrics and norms on linear spaces from different stand point [8, 12, 7, 13, 25]. Definition of fuzzy norm provided by Cheng and Moderson [4] was reshaped by Bag and Samanta [2] in 2003. Since then it has been a matter of consideration [17, 14, 15, 21] of what effects several fuzzy stability provided out of various functional equations.

In this article, it an achievement for our part to be had of the generalized Hyers-Ulam-Rassias type stability for the functional equation (1.1) in fuzzy Banach spaces.

## 2. Preliminaries

We recall some notations and basic definitions which will be needed in the sequel.

**Definition 2.1.** [22] A binary operation $*:[0,1]\times[0,1]\longrightarrow[0,1]$ is continuous $t$-norm if $*$ satisfies the following conditions:
$(i)$ $*$ is commutative and associative;
$(ii)$ $*$ is continuous;
$(iii)$ $a*1=a$ for every $a\in[0,1]$;
$(iv)$ $a*b\leq c*d$ whenever $a\leq c$, $b\leq d$ and $a,b,c,d\in[0,1]$.

Through out this article, we further assume that $a*a=a\forall a\in[0,1]$.

**Definition 2.2.** [16] The 3-tuple $(X,N,*)$ is called a fuzzy normed linear space if X is a real linear space,$*$is a continuous t - norm and $N$ is a fuzzy set in $X\times(0,\infty)$ satisfying the following conditions :
$(i)$ $N(x,t)>0$ ;
$(ii)$ $N(x,t)=1$ if and only if $x=0$ ;
$(iii)$ $N(cx,t)=N\left(x,\frac{t}{|c|}\right)$ if $c\neq 0$ ;
$(iv)$ $N(x,s)*N(y,t)\leqslant N(x+y,s+t)$ ;
$(v)$ $N(x,\cdot):(0,\infty)\rightarrow(0,1]$ is continuous for all $x,y\in X$ and $t,s>0$ .
Note that $N(x,t)$ can be thought of as the degree of nearness between $x$ and null vector 0 with respect to $t$ .

*Example* 2.3. Let $X=[0,\infty)$ , $a*b=ab$ for every $a,b\in[0,1]$ and $\|\cdot\|$ be the usual metric defined on $X$. Define $N(x,t)=e^{-\frac{\|x\|}{t}}$ for all $x\in X$. Then clearly $(X,N,*)$ is a fuzzy normed linear space.

*Example* 2.4. Let $(X,\|\cdot\|)$ be a normed linear space, and let $a*b=ab$ or $a*b=\min\{a,b\}$ for all $a,b\in[0,1]$ . Let $N(x,t)=\frac{t}{t+\|x\|}$ for all $x\in X$ and $t>0$ . Then $(X,N,*)$ is a fuzzy normed linear space and this fuzzy norm $N$ induced by $\|\cdot\|$ is called the standard fuzzy norm.

*Remark* 2.5. In fuzzy normed linear space $(X,N,*)$ , for all $x\in X$, $N(x,\cdot)$ is non- decreasing with respect to the variable $t$.

**Definition 2.6.** [20] Let $(X,N,*)$ be a fuzzy normed linear space. A sequence $\{x_n\}$ in $X$ is said to be convergent or converge if there exists an $x\in X$ such that $\lim\limits_{n\to\infty}N(x_n-x,t)=1$ . In this case, $x$ is called the limit of the sequence $\{x_n\}$ and we denote it by $N-\lim\limits_{n\to\infty}x_n=x$ .

**Definition 2.7.** [20] Let $(X,N,*)$ be a fuzzy normed linear space. A sequence $\{x_n\}$ in $X$ is called Cauchy sequence if for each $\varepsilon>0$ and $t>0$ there exists an $n_0\in\mathbb{N}$ such that for all $n\geqslant n_0$ and all $p>0$, we have $N(x_{n+p}-x_n,t)>1-\varepsilon$ .

## 3. The Generalized Hyers-Ulam Stability of The Functional Equation (1.1)

In this section, let $X$ be a real vector space and $(Y, N)$ be a fuzzy Banach space. Let $\psi : X^3 \to \mathbb{R}^+$ be a given function and the induced function $\Psi : X^2 \to \mathbb{R}^+$ be defined by
$\Psi(x,y) := \frac{1}{2}\left[\psi\left(\frac{x}{2}, y, \frac{x}{2}\right) + \psi\left(-\frac{x}{2}, -y, -\frac{x}{2}\right) + \psi\left(\frac{x}{2}, y, -\frac{x}{2}\right) + \psi\left(-\frac{x}{2}, -y, \frac{x}{2}\right)\right]$
for all $x, y \in X$.

**Theorem 3.1.** *Let $\psi : X^3 \to \mathbb{R}^+$ be a function such that the series $\sum\limits_{i=0}^{\infty} \frac{\psi(2^i x, 2^i y, 2^i z)}{2^i}$ converges for all $x, y, z \in X$. Let $f : X \to Y$ be mapping such that*

$$\lim_{t\to\infty} N(f(x+y+z) + f(x-y) + f(x-z) - f(x-y-z) - f(x+y) - f(x+z), t\psi(x,y,z)) = 1 \tag{3.1}$$

*uniformly on $X^3$. Then there exist a unique quadratic function $Q : X \to Y$ and a unique additive function $A : X \to Y$ which satisfy (1.1) and the functions $Q$ and $A$ are given by*

$$Q(x) := N - \lim_{n\to\infty} \frac{1}{4^n}\left(\frac{f(2^n x) + f(-2^n x)}{2} - f(0)\right),$$

$$A(x) := N - \lim_{n\to\infty} \frac{1}{2^n}\left(\frac{f(2^n x) - f(-2^n x)}{2}\right)$$

*for all $x \in X$. If for some $\delta > 0, \alpha > 0$*

$$N(f(x+y+z) + f(x-y) + f(x-z) - f(x-y-z) - f(x+y) - f(x+z), \delta\psi(x,y,z)) \geq \alpha \tag{3.2}$$

*for all $x, y, z \in X$, then*

$$N\left(f(x) - Q(x) - A(x) - f(0), \delta\left(\frac{1}{4}\sum_{i=0}^{\infty} \frac{\Psi(2^i x, 2^i x)}{4^i} + \frac{1}{2}\sum_{i=0}^{\infty} \frac{\Psi(2^i x, 2^i x)}{2^i}\right)\right) \geq \alpha \tag{3.3}$$

*for all $x \in X$.*

*Proof.* Consider a function $F : X \to Y$ defined by $F(x) = f(x) - f(0), x \in X$.

Clearly,$F$ satisfies (3.1) and $F(0) = 0$. Let two functions $F_1, F_2 : X \to Y$ be defined by

$$F_1(x) = \frac{F(x) + F(-x)}{2}, F_2(x) = \frac{F(x) - F(-x)}{2}, x \in X.$$

From definitions $F_1$ is an even function and $F_2$ is an odd function. By (3.1) corresponding to a given $\epsilon > 0$ there exists some $t_0 > 0$ such that

$$\begin{aligned} N(f(x+y+z) + f(x-y) + f(x-z) - f(x-y-z)- \\ f(x+y) - f(x+z), t\psi(x,y,z)) \geq 1-\epsilon \end{aligned} \tag{3.4}$$

for all $x, y, z \in X$ and $t \geq t_0$. Replacing both $x$ and $z$ by $\frac{x}{2}$ in (3.4), we get

$$\begin{aligned} N\left(f(x+y) + f\left(\frac{x}{2} - y\right) + f(0) - f(-y)-\right. \\ \left. f\left(\frac{x}{2} + y\right) - f(x), t\psi\left(\frac{x}{2}, y, \frac{x}{2}\right)\right) \geq 1-\epsilon \end{aligned}$$

for all $x, y \in X$ and $t \geq t_0$. It implies that

$$\begin{aligned} N\left(F(x+y) + F\left(\frac{x}{2} - y\right) - F(-y) - F\left(\frac{x}{2} + y\right) -\right. \\ \left. F(x), t\psi\left(\frac{x}{2}, y, \frac{x}{2}\right)\right) \geq 1-\epsilon \end{aligned} \tag{3.5}$$

for all $x, y \in X$ and $t \geq t_0$. Now,

$$\begin{aligned} N\left(F_1(x+y) + F_1\left(\frac{x}{2} - y\right) - F_1(y) - F_1\left(\frac{x}{2} + y\right) -\right. \\ \left. F_1(x), \frac{t}{2}\left(\psi\left(\frac{x}{2}, y, \frac{x}{2}\right) + \psi\left(-\frac{x}{2}, -y, -\frac{x}{2}\right)\right)\right) \end{aligned} \tag{3.6}$$

$$\begin{aligned} &\geq N\left(F(x+y) + F\left(\frac{x}{2} - y\right) - F(-y) - F\left(\frac{x}{2} + y\right) - F(x),\right. \\ &\qquad \left. t\psi\left(\frac{x}{2}, y, \frac{x}{2}\right)\right) * N\left(F(-x-y) + F\left(-\frac{x}{2} + y\right) - F(y) -\right. \\ &\quad \left. F\left(-\frac{x}{2} - y\right) - F(-x), t\psi\left(-\frac{x}{2}, -y, -\frac{x}{2}\right)\right) \geq (1-\epsilon) * (1-\epsilon) \text{ by (3.5)} \\ &= (1-\epsilon) \end{aligned}$$

for all $x, y \in X$ and $t \geq t_0$. Replacing both $x$ and $z$ by $\frac{x}{2}$ and $-\frac{x}{2}$ respectively in (3.4), we get

$$N\left(f(y) + f\left(\frac{x}{2} - y\right) + f(x) - f(x-y) - f\left(\frac{x}{2} + y\right) -\right.$$

$$f(0), t\psi\left(\frac{x}{2}, y, -\frac{x}{2}\right)\right) \geq 1-\epsilon$$

It implies that

$$N\left(F(y)+F\left(\frac{x}{2}-y\right)+F(x)-F(x-y)-\right.$$
$$\left.F\left(\frac{x}{2}+y\right), t\psi\left(\frac{x}{2}, y, -\frac{x}{2}\right)\right) \geq 1-\epsilon \tag{3.7}$$

for all $x, y \in X$ and $t \geq t_0$. Now,

(3.8)

$$N\left(F_1(y)+F_1\left(\frac{x}{2}-y\right)+F_1(x)-F_1(x-y)-\right.$$
$$\left.F_1\left(\frac{x}{2}+y\right), \frac{t}{2}\left(\psi\left(\frac{x}{2}, y, -\frac{x}{2}\right)+\psi\left(-\frac{x}{2}, -y, \frac{x}{2}\right)\right)\right)$$
$$\geq N\left(F(y)+F\left(\frac{x}{2}-y\right)+F(x)-F(x-y)-F\left(\frac{x}{2}+y\right),\right.$$
$$\left.t\psi\left(\frac{x}{2}, y, -\frac{x}{2}\right)\right) * N\left(F(-y)+F\left(-\frac{x}{2}+y\right)+F(-x)-\right.$$
$$\left.F(-x+y)-F\left(-\frac{x}{2}-y\right), t\psi\left(-\frac{x}{2}, -y, \frac{x}{2}\right)\right) \geq (1-\epsilon)*(1-\epsilon) \text{ by } (3.7)$$
$$=(1-\epsilon)$$

for all $x, y \in X$ and $t \geq t_0$. Now,

$$N(F_1(x+y)+F_1(x-y)-2F_1(x)-2F_1(y), t\Psi(x,y)) \tag{3.9}$$
$$\geq N\left(F_1(x+y)+F_1\left(\frac{x}{2}-y\right)-F_1(y)-F_1\left(\frac{x}{2}+y\right)-F_1(x),\right.$$
$$\left.\frac{t}{2}\left(\psi\left(\frac{x}{2}, y, \frac{x}{2}\right)+\psi\left(-\frac{x}{2}, -y, -\frac{x}{2}\right)\right)\right) * N\left(F_1(y)+F_1\left(\frac{x}{2}-y\right)+\right.$$
$$\left.F_1(x)-F_1(x-y)-F_1\left(\frac{x}{2}+y\right), \frac{t}{2}\left(\psi\left(\frac{x}{2}, y, -\frac{x}{2}\right)+\psi\left(-\frac{x}{2}, -y, \frac{x}{2}\right)\right)\right)$$
$$\geq (1-\epsilon)*(1-\epsilon)[\text{ by } (3.6), (3.8)] = (1-\epsilon)$$

for all $x, y \in X$ and $t \geq t_0$.
Putting $y = x$ in (3.9), we get

$$N(F_1(2x)+F_1(0)-4F_1(x), t\Psi(x,x)) \geq 1-\epsilon$$

It implies that

$$N\left(F_1(x)-\frac{F_1(2x)}{4}, \frac{t}{4}\Psi(x,x)\right) \geq 1-\epsilon \tag{3.10}$$

for all $x \in X$ and $t \geq t_0$. By induction on positive integer $n$ we now show that

$$N\left(F_1(x) - \frac{F_1(2^n x)}{4^n}, \frac{t}{4}\sum_{i=0}^{n-1}\frac{\Psi(2^i x, 2^i x)}{4^i}\right) \geq 1-\epsilon \tag{3.11}$$

for all $x \in X$ and $t \geq t_0$. From (3.10) we see that (3.11) is true for $n = 1$. Let us assume that (3.11) is true for $n = k$, where $k \in \mathbb{N}$. Then

$$N\left(F_1(x) - \frac{F_1(2^k x)}{4^k}, \frac{t}{4}\sum_{i=0}^{k-1}\frac{\Psi(2^i x, 2^i x)}{4^i}\right) \geq 1-\epsilon \tag{3.12}$$

for all $x \in X$ and $t \geq t_0$. Now,

$$\begin{aligned}
&N\left(F_1(x) - \frac{F_1(2^{k+1} x)}{4^{k+1}}, \frac{t}{4}\sum_{i=0}^{k}\frac{\Psi(2^i x, 2^i x)}{4^i}\right)\\
&\geq N\left(F_1(x) - \frac{F_1(2^k x)}{4^k}, \frac{t}{4}\sum_{i=0}^{k-1}\frac{\Psi(2^i x, 2^i x)}{4^i}\right) *\\
&\qquad N\left(\frac{F_1(2^k x)}{4^k} - \frac{F_1(2^{k+1} x)}{4^{k+1}}, \frac{t}{4}\frac{\Psi(2^k x, 2^k x)}{4^k}\right)
\end{aligned}$$

$\geq (1-\epsilon)*(1-\epsilon)$[ by (3.10), (3.12)] $= (1-\epsilon)$

This completes the proof of (3.11). Putting $t = t_0$ and replacing $n$ and $x$ by $n$ and $2^n x$ respectively in (3.11), we get

$$N\left(F_1(2^n x) - \frac{F_1(2^{n+p} x)}{4^p}, \frac{t_0}{4}\sum_{i=0}^{p-1}\frac{\Psi(2^{n+i} x, 2^{n+i} x)}{4^i}\right) \geq 1-\epsilon \tag{3.13}$$

for all $n \geq 0$ and $p > 0$. Again,

$$\sum_{i=0}^{p-1}\frac{\Psi(2^{n+i} x, 2^{n+i} x)}{4^i} = \sum_{i=n}^{n+p-1}\frac{\Psi(2^i x, 2^i x)}{4^{i-n}} \tag{3.14}$$

Since $\sum\limits_{i=0}^{\infty}\frac{\Psi(2^i x, 2^i x)}{4^i}$ converges, for a given $\delta > 0, \exists n_0 \in \mathbb{N}$ such that

$$\frac{t_0}{4}\sum_{i=n}^{n+p-1}\frac{\Psi(2^i x, 2^i x)}{4^i} < \delta \tag{3.15}$$

for all $n \geq n_0$ and $p > 0$. Now,

$$N\left(\frac{F_1(2^n x)}{4^n} - \frac{F_1(2^{n+p} x)}{4^{n+p}}, \delta\right)$$

$$\geq N\left(\frac{F_1(2^n x)}{4^n} - \frac{F_1(2^{n+p}x)}{4^{n+p}}, \frac{t_0}{4}\sum_{i=n}^{n+p-1}\frac{\Psi(2^i x, 2^i x)}{4^i}\right) \text{ by (3.15)}$$

$$= N\left(F_1(2^n x) - \frac{F_1(2^{n+p}x)}{4^p}, \frac{t_0}{4}\sum_{i=0}^{p-1}\frac{\Psi(2^{n+i} x, 2^{n+i} x)}{4^i}\right) \text{ by (3.14)}$$

$\geq 1-\epsilon$ by (3.13)

for all $n \geq n_0$ and $p > 0$. Hence the sequence $\left\{\frac{F_1(2^n x)}{4^n}\right\}$ is a Cauchy sequence in $Y$. Since $Y$ is fuzzy Banach space, the sequence $\left\{\frac{F_1(2^n x)}{4^n}\right\}$ converges to some $Q(x) \in Y$. So we can define a function $Q : X \to Y$ by

$$Q(x) := N-\lim_{n\to\infty}\frac{F_1(2^n x)}{4^n} = N-\lim_{n\to\infty}\frac{1}{4^n}\left(\frac{F(2^n x) + F(-2^n x)}{2}\right)$$

$$= N-\lim_{n\to\infty}\frac{1}{4^n}\left(\frac{f(2^n x) + f(-2^n x)}{2} - f(0)\right)$$

Thus for each $t > 0$ and $x \in X$,

$$\lim_{n\to\infty} N\left(\frac{F_1(2^n x)}{4^n} - Q(x), t\right) = 1. \tag{3.16}$$

Now we show that $Q$ satisfies (1.1). Let $x, y \in X$. Since $\lim_{n\to\infty}\frac{\psi(2^n x, 2^n y, 2^n z)}{4^n} = 0$, for fixed $t > 0$ and $0 < \epsilon < 1$ there exists $n_1 > n_0$ such that

$$\frac{t_0}{4^n}\frac{1}{2}[\psi(2^n x, 2^n y, 2^n z) + \psi(-2^n x, -2^n y, -2^n z)] < \frac{t}{7} \tag{3.17}$$

for all $n \geq n_1$. Now,

(3.18)

$$N(Q(x+y+z)+Q(x-y)+Q(x-z)-Q(x-y-z)-Q(x+y)-Q(x+z), t)$$

$$\geq N\left(Q(x+y+z) - \frac{F_1(2^n(x+y+z))}{4^n}, \frac{t}{7}\right)$$

$$*N\left(Q(x-y) - \frac{F_1(2^n(x-y))}{4^n}, \frac{t}{7}\right)$$

$$*N\left(Q(x-z) - \frac{F_1(2^n(x-z))}{4^n}, \frac{t}{7}\right)$$

$$*N\left(\frac{F_1(2^n(x-y-z))}{4^n} - Q(x-y-z), \frac{t}{7}\right)$$

$$*N\left(\frac{F_1(2^n(x+y))}{4^n}-Q(x+y),\frac{t}{7}\right)$$
$$*N\left(\frac{F_1(2^n(x+z))}{4^n}-Q(x+z),\frac{t}{7}\right)$$
$$*N\left(\frac{F_1(2^n(x+y+z))}{4^n}+\frac{F_1(2^n(x-y))}{4^n}+\frac{F_1(2^n(x-z))}{4^n}\right.$$
$$\left.-\frac{F_1(2^n(x-y-z))}{4^n}-\frac{F_1(2^n(x+y))}{4^n}-\frac{F_1(2^n(x+z))}{4^n},\frac{t}{7}\right)$$

Now putting $t=t_0$ and replacing $x,y,z$ by $2^nx,2^ny,2^nz$ respectively in (3.4), we get

$$N(f(2^n(x+y+z))+f(2^n(x-y))+f(2^n(x-z))-f(2^n(x-y-z))$$
$$-f(2^n(x+y))-f(2^n(x+z)),t_0\psi(2^nx,2^ny,2^nz))\geq 1-\epsilon.$$

This implies

(3.19) $N(F(2^n(x+y+z))+F(2^n(x-y))+F(2^n(x-z))-$
$F(2^n(x-y-z))-F(2^n(x+y))-F(2^n(x+z)),t_0\psi(2^nx,2^ny,2^nz))\geq 1-\epsilon$

for all $x,y,z\in X$. Now,

(3.20) $N(F_1(2^n(x+y+z))+F_1(2^n(x-y))+F_1(2^n(x-z))-$
$F_1(2^n(x-y-z))-F_1(2^n(x+y))-F_1(2^n(x+z)),$
$\frac{t_0}{2}(\psi(2^nx,2^ny,2^nz)+\psi(-2^nx,-2^ny,-2^nz)))$
$\geq N(F(2^n(x+y+z))+F(2^n(x-y))+F(2^n(x-z))-F(2^n(x-y-z))$
$-F(2^n(x+y))-F(2^n(x+z)),t_0\psi(2^nx,2^ny,2^nz))*$
$N(F(-2^n(x+y+z))+F(-2^n(x-y))+F(-2^n(x-z))-F(-2^n(x-y-z))$
$-F(-2^n(x+y))-F(-2^n(x+z)),t_0\psi(-2^nx,-2^ny,-2^nz))$
$\geq(1-\epsilon)*(1-\epsilon)[\text{ by (3.19)}]=(1-\epsilon).$

The first six terms on RHS of (3.18) tend to 1 as $n\to\infty$ and last term $\geq 1-\epsilon$ by (3.17) and (3.20). Hence

$$N(Q(x+y+z)+Q(x-y)+Q(x-z)-Q(x-y-z)-Q(x+y)-Q(x+z),t)$$
$$\geq 1-\epsilon$$

for all $t>0$,

$$\Rightarrow N(Q(x+y+z)+Q(x-y)+Q(x-z)-Q(x-y-z)-Q(x+y)-Q(x+z),t)$$
$$=1$$

for all $t > 0$. Therefore

$Q(x+y+z)+Q(x-y)+Q(x-z)-Q(x-y-z)-Q(x+y)-Q(x+z) = 0.$

This shows that $Q$ satisfies (1.1) i.e. $Q$ is quadratic. Let (3.2) hold for some $\delta > 0, \alpha > 0$. By using similar argument as in the beginning of the proof we can establish from (3.2) that

$$N\left(F_1(x) - \frac{F_1(2^n x)}{4^n}, \frac{\delta}{4}\sum_{i=0}^{n-1}\frac{\Psi(2^i x, 2^i x)}{4^i}\right) \geq \alpha \tag{3.21}$$

for all $x \in X$ and $n \in \mathbb{N}$. Let $t > 0$. Now,

$$N\left(F_1(x) - Q(x), \frac{\delta}{4}\sum_{i=0}^{n-1}\frac{\Psi(2^i x, 2^i x)}{4^i} + t\right)$$

$$\geq N\left(F_1(x) - \frac{F_1(2^n x)}{4^n}, \frac{\delta}{4}\sum_{i=0}^{n-1}\frac{\Psi(2^i x, 2^i x)}{4^i}\right) * N\left(\frac{F_1(2^n x)}{4^n} - Q(x), t\right)$$

Taking limit as $n \to \infty$ we get

$$N\left(F_1(x) - Q(x), \frac{\delta}{4}\sum_{i=0}^{\infty}\frac{\Psi(2^i x, 2^i x)}{4^i} + t\right) \geq \alpha * 1 = \alpha$$

For continuity of $N(x, \cdot)$ and taking limit as $t \to 0$ we get

$$N\left(F_1(x) - Q(x), \frac{\delta}{4}\sum_{i=0}^{\infty}\frac{\Psi(2^i x, 2^i x)}{4^i}\right) \geq \alpha \tag{3.22}$$

for all $x \in X$ and $n \in \mathbb{N}$. As the deduction of equations (3.6) and (3.8) instead of $F_1$ we get for $F_2$

$$\begin{aligned} N\Big(F_2(x+y) + F_2\left(\frac{x}{2} - y\right) + F_2(y) - F_2\left(\frac{x}{2} + y\right) - \\ F_2(x), \frac{t}{2}\left(\psi\left(\frac{x}{2}, y, \frac{x}{2}\right) + \psi\left(-\frac{x}{2}, -y, -\frac{x}{2}\right)\right)\Big) \geq 1 - \varepsilon \end{aligned} \tag{3.23}$$

and

$$\begin{aligned} N\Big(F_2(y) + F_2\left(\frac{x}{2} - y\right) + F_2(x) - F_2(x-y) - \\ F_2\left(\frac{x}{2} + y\right), \frac{t}{2}\left(\psi\left(\frac{x}{2}, y, -\frac{x}{2}\right) + \psi\left(-\frac{x}{2}, -y, \frac{x}{2}\right)\right)\Big) \geq 1 - \varepsilon \end{aligned} \tag{3.24}$$

for all $x, y \in X$ and $t \geq t_0$. Now,

$$N(F_2(x+y) + F_2(x-y) - 2F_2(x), t\Psi(x,y)) \tag{3.25}$$

$$\geq N\left(F_2(x+y)+F_2\left(\frac{x}{2}-y\right)+F_2(y)-F_2\left(\frac{x}{2}+y\right)-F_2(x),\right.$$
$$\left.\frac{t}{2}\left(\psi\left(\frac{x}{2},y,\frac{x}{2}\right)+\psi\left(-\frac{x}{2},-y,-\frac{x}{2}\right)\right)\right)*$$
$$N\left(F_2(y)+F_2\left(\frac{x}{2}-y\right)+F_2(x)-F_2(x-y)-F_2\left(\frac{x}{2}+y\right),\right.$$
$$\left.\frac{t}{2}\left(\psi\left(\frac{x}{2},y,-\frac{x}{2}\right)+\psi\left(-\frac{x}{2},-y,\frac{x}{2}\right)\right)\right)$$
$\geq (1-\epsilon)*(1-\epsilon)$[ by (3.23), (3.24)] $= (1-\epsilon)$

for all $x, y \in X$ and $t \geq t_0$. Putting $y = x$ in (3.25), we get

$$N(F_2(2x)-2F_2(x), t\Psi(x,x)) \geq 1-\epsilon \tag{3.26}$$

for all $x \in X$ and $t \geq t_0$. By induction on positive integer $n$ we now show that

$$N\left(\frac{F_2(2^n x)}{2^n}-F_2(x), \frac{t}{2}\sum_{i=0}^{n-1}\frac{\Psi(2^i x, 2^i x)}{2^i}\right) \geq 1-\epsilon \tag{3.27}$$

for all $x \in X$ and $t \geq t_0$. From (3.26) we see that (3.27) is true for $n = 1$. Let (3.27) be true for $n = k$, where $k \in \mathbb{N}$. Then

$$N\left(\frac{F_2(2^k x)}{2^k}-F_2(x), \frac{t}{2}\sum_{i=0}^{k-1}\frac{\Psi(2^i x, 2^i x)}{2^i}\right) \geq 1-\epsilon \tag{3.28}$$

for all $x \in X$ and $t \geq t_0$. Now,

$$N\left(\frac{F_2(2^{k+1} x)}{2^{k+1}}-F_2(x), \frac{t}{2}\sum_{i=0}^{k}\frac{\Psi(2^i x, 2^i x)}{2^i}\right)$$
$$\geq N\left(\frac{F_2(2^{k+1} x)}{2^{k+1}}-\frac{F_2(2^k x)}{2^k}, \frac{t}{2}\frac{\Psi(2^k x, 2^k x)}{2^k}\right)*$$
$$N\left(\frac{F_2(2^k x)}{2^k}-F_2(x), \frac{t}{2}\sum_{i=0}^{k-1}\frac{\Psi(2^i x, 2^i x)}{2^i}\right)$$
$\geq (1-\epsilon)*(1-\epsilon)$[ by (3.26), (3.28)] $= (1-\epsilon)$

Hence the proof of (3.27). Putting $t = t_0$ and replacing $n$ and $x$ by $p$ and $2^n x$ respectively in (3.27), we get

$$N\left(\frac{F_2(2^{n+p} x)}{2^p}-F_2(2^n x), \frac{t_0}{2}\sum_{i=0}^{p-1}\frac{\Psi(2^{n+i} x, 2^{n+i} x)}{2^i}\right) \geq 1-\epsilon. \tag{3.29}$$

for all $n \geq 0$ and $p > 0$. Again,

$$\sum_{i=0}^{p-1} \frac{\Psi(2^{n+i}x, 2^{n+i}x)}{2^i} = \sum_{i=n}^{n+p-1} \frac{\Psi(2^i x, 2^i x)}{2^{i-n}}. \tag{3.30}$$

Since $\sum\limits_{i=0}^{\infty} \frac{\Psi(2^i x, 2^i x)}{2^i}$ converges, for a given $\delta > 0$, there exists $n_2 \in \mathbb{N}$ such that

$$\frac{t_0}{2} \sum_{i=n}^{n+p-1} \frac{\Psi(2^i x, 2^i x)}{2^i} < \delta \tag{3.31}$$

for all $n \geq n_2$ and $p > 0$. Now,

$$N\left(\frac{F_2(2^{n+p}x)}{2^{n+p}} - \frac{F_2(2^n x)}{2^n}, \delta\right)$$

$$\geq N\left(\frac{F_2(2^{n+p}x)}{2^{n+p}} - \frac{F_2(2^n x)}{2^n}, \frac{t_0}{2} \sum_{i=n}^{n+p-1} \frac{\Psi(2^i x, 2^i x)}{2^i}\right)$$

$$= N\left(\frac{F_2(2^{n+p}x)}{2^p} - F_2(2^n x), \frac{t_0}{2} \sum_{i=0}^{p-1} \frac{\Psi(2^{n+i}x, 2^{n+i}x)}{2^i}\right) \text{ by (3.30)}$$

$\geq 1-\epsilon$ by (3.29)

for all $n \geq n_2$ and $p > 0$. Hence the sequence $\left\{\frac{F_2(2^n x)}{2^n}\right\}$ is a Cauchy sequence in $Y$. Since $Y$ is fuzzy Banach space, the sequence $\left\{\frac{F_2(2^n x)}{2^n}\right\}$ converges to some $A(x) \in Y$. So we can define a function $A : X \to Y$ by

$$A(x) := N\!-\lim_{n\to\infty} \frac{F_2(2^n x)}{2^n} = N\!-\lim_{n\to\infty} \frac{1}{2^n}\left(\frac{F(2^n x) - F(-2^n x)}{2}\right)$$

$$= N\!-\lim_{n\to\infty} \frac{1}{2^n}\left(\frac{f(2^n x) - f(-2^n x)}{2}\right)$$

Therefore for each $t > 0$ and $x \in X$,

$$\lim_{n\to\infty} N\left(\frac{F_1(2^n x)}{2^n} - A(x), t\right) = 1 \tag{3.32}$$

As before we can show that $A$ satisfies (1.1) for all $x, y, z \in X$ and

$$N\left(F_2(x) - A(x), \frac{\delta}{2} \sum_{i=0}^{\infty} \frac{\Psi(2^i x, 2^i x)}{2^i}\right) \geq \alpha \tag{3.33}$$

Now,

$$N\left(F_1(x)-Q(x)+F_2(x)-A(x),\delta\left(\frac{1}{4}\sum_{i=0}^{\infty}\frac{\Psi(2^ix,2^ix)}{4^i}+\frac{1}{2}\sum_{i=0}^{\infty}\frac{\Psi(2^ix,2^ix)}{2^i}\right)\right)$$

$$\geq N\left(F_1(x)-Q(x),\frac{\delta}{4}\sum_{i=0}^{\infty}\frac{\Psi(2^ix,2^ix)}{4^i}\right)*N\left(F_2(x)-A(x),\frac{\delta}{2}\sum_{i=0}^{\infty}\frac{\Psi(2^ix,2^ix)}{2^i}\right)$$

$\geq \alpha*\alpha$[ by (3.22), (3.33)] $=\alpha$

Hence

$$N\left(f(x)-Q(x)-A(x)-f(0),\delta\left(\frac{1}{4}\sum_{i=0}^{\infty}\frac{\Psi(2^ix,2^ix)}{4^i}+\frac{1}{2}\sum_{i=0}^{\infty}\frac{\Psi(2^ix,2^ix)}{2^i}\right)\right)\geq\alpha$$

for all $x\in X$. Since $\frac{\Psi(2^nx,2^ny)}{2^n}\to 0$ as $n\to\infty$, for fixed $t>0$ and $0<\epsilon<1$ there exists $n_3\in\mathbb{N}$ such that

$$t_0\frac{\Psi\left(2^n\left(\frac{x}{2}+\frac{y}{2}\right),2^n\left(\frac{y}{2}-\frac{x}{2}\right)\right)}{2^n}<\frac{t}{4} \tag{3.34}$$

for all $x,y\in X$ and $n\geq n_3$. Now,

$$N\left(2A\left(\frac{x+y}{2}\right)-A(x)-A(y),t\right)$$

$$\geq N\left(2A\left(\frac{x+y}{2}\right)-2\frac{F_2\left(2^n\left(\frac{x+y}{2}\right)\right)}{2^n},\frac{t}{4}\right)*N\left(\frac{F_2(2^nx)}{2^n}-A(x),\frac{t}{4}\right)*$$

$$N\left(\frac{F_2(2^ny)}{2^n}-A(y),\frac{t}{4}\right)*N\left(2\frac{F_2\left(2^n\left(\frac{x+y}{2}\right)\right)}{2^n}-\frac{F_2(2^nx)}{2^n}-\frac{F_2(2^ny)}{2^n},\frac{t}{4}\right)$$

Last term

$$=N\left(2F_2\left(2^n\left(\frac{x+y}{2}\right)\right)-F_2(2^nx)-F_2(2^ny),2^n\frac{t}{4}\right)$$

$$=N\left(F_2\left(2^n\left(\frac{x}{2}+\frac{y}{2}+\frac{y}{2}-\frac{x}{2}\right)\right)+F_2\left(2^n\left(\frac{x}{2}+\frac{y}{2}-\frac{y}{2}+\frac{x}{2}\right)\right)-\right.$$
$$\left.2F_2\left(2^n\left(\frac{x}{2}+\frac{y}{2}\right)\right),2^n\frac{t}{4}\right)$$

$$\geq N\left(F_2\left(2^n\left(\frac{x}{2}+\frac{y}{2}+\frac{y}{2}-\frac{x}{2}\right)\right)+F_2\left(2^n\left(\frac{x}{2}+\frac{y}{2}-\frac{y}{2}+\frac{x}{2}\right)\right)\right.$$
$$\left.-2F_2\left(2^n\left(\frac{x}{2}+\frac{y}{2}\right)\right),t_0\Psi\left(2^n\left(\frac{x}{2}+\frac{y}{2}\right),2^n\left(\frac{y}{2}-\frac{x}{2}\right)\right)\right)\text{ by (3.34)}$$

$\geq 1-\epsilon$ by (3.25)

Also first three terns tend to 1 as $n \to \infty$. Therefore

$$N\left(2A\left(\frac{x+y}{2}\right) - A(x) - A(y), t\right) \geq 1-\epsilon$$

for all $t > 0$. Thus

$$N\left(2A\left(\frac{x+y}{2}\right) - A(x) - A(y), t\right) = 1$$

for all $t > 0$.

$$\Rightarrow 2A\left(\frac{x+y}{2}\right) - A(x) - A(y) = 0$$

for all $x, y \in X$. Hence $A$ is Jensen additive. To prove the uniqueness of $Q$ let us assume that $Q'$ be another function satisfying (1.1) and (3.16 ). Now,

$$N(Q(x) - Q'(x), t)$$

$$\geq N\left(Q(x) - \frac{F_1(2^n x)}{4^n}, \frac{t}{2}\right) * N\left(Q'(x) - \frac{F_1(2^n x)}{4^n}, \frac{t}{2}\right).$$

Each term on the RHS tends to 1 as $n \to \infty$. Then $N(Q(x) - Q'(x), t) = 1$ for all $t > 0, x \in X$. Thus $Q(x) = Q'(x)$ for all $x \in X$. This shows that $Q$ is unique. In similar manner it can be shown that $A$ is unique. This completes the proof of the theorem. □

**Corollary 3.2.** *Let $p < 1$ be a real number and $U : \mathbb{R}^{+^3} \to \mathbb{R}^+$ be a function such that $U(tx, ty, tz) \leq t^p U(x, y, z)$ for all $t \neq 0$ and $x, y, z \in \mathbb{R}^+$. Let $f : X \to Y$ be mapping such that*

$$\lim_{t \to \infty} N(f(x+y+z) + f(x-y) + f(x-z) - f(x-y-z) -$$
$$f(x+y) - f(x+z), tU(\|x\|, \|y\|, \|z\|)) = 1$$

*uniformly on $X^3$. Then there exist a unique quadratic function $Q : X \to Y$ and a unique additive function $A : X \to Y$ which satisfy (1.1) and the functions $Q$ and $A$ are given by*

$$Q(x) := N - \lim_{n \to \infty} \frac{1}{4^n}\left(\frac{f(2^n x) + f(-2^n x)}{2} - f(0)\right),$$

$$A(x) := N - \lim_{n \to \infty} \frac{1}{2^n}\left(\frac{f(2^n x) - f(-2^n x)}{2}\right)$$

*for all $x \in X$. If for some $\delta > 0, \alpha > 0$*

$$N(f(x+y+z) + f(x-y) + f(x-z) - f(x-y-z) -$$

$$f(x+y) - f(x+z), \delta U(\|x\|, \|y\|, \|z\|)) \geq \alpha$$

*for all* $x, y, z \in X$*, then*

$$N\left(f(x) - Q(x) - A(x) - f(0), \delta\left(\frac{2}{4-2^p} U\left(\frac{\|x\|}{2}, \|x\|, \frac{\|x\|}{2}\right) + \frac{2}{2-2^p} U\left(\frac{\|x\|}{2}, \|x\|, \frac{\|x\|}{2}\right)\right)\right) \geq \alpha$$

*for all* $x \in X$*.*

*Proof.* Define $\psi(x, y, z) = U(\|x\|, \|y\|, \|z\|)$. Then

$$\sum_{i=0}^{\infty} \frac{\Psi(2^i x, 2^i x)}{4^i} = \frac{8}{4-2^p} U\left(\frac{\|x\|}{2}, \|x\|, \frac{\|x\|}{2}\right),$$

$$\sum_{i=0}^{\infty} \frac{\Psi(2^i x, 2^i x)}{2^i} = \frac{4}{2-2^p} U\left(\frac{\|x\|}{2}, \|x\|, \frac{\|x\|}{2}\right).$$

□

**Corollary 3.3.** *Let* $p < 1, \theta \geq 0$ *and let* $f : X \to Y$ *be mapping such that*

$$\lim_{t\to\infty} N(f(x+y+z)+f(x-y)+f(x-z)-f(x-y-z)- f(x+y) - f(x+z), t\theta(\|x\|^p + \|y\|^p + \|z\|^p)) = 1$$

*uniformly on* $X^3$*. Then there exist a unique quadratic function* $Q : X \to Y$ *and a unique additive function* $A : X \to Y$ *which satisfy (1.1) and the functions* $Q$ *and* $A$ *are given by*

$$Q(x) := N - \lim_{n\to\infty} \frac{1}{4^n}\left(\frac{f(2^n x) + f(-2^n x)}{2} - f(0)\right),$$

$$A(x) := N - \lim_{n\to\infty} \frac{1}{2^n}\left(\frac{f(2^n x) - f(-2^n x)}{2}\right)$$

*for all* $x \in X$*. If for some* $\delta > 0, \alpha > 0$

$$N(f(x+y+z)+f(x-y)+f(x-z)-f(x-y-z)- f(x+y) - f(x+z), \delta\theta(\|x\|^p + \|y\|^p + \|z\|^p) \geq \alpha$$

*for all* $x, y, z \in X$*, then*

$$N\left(f(x) - Q(x) - A(x) - f(0), \delta\left(\frac{2\theta(2+2^p)}{(4-2^p)2^p}\|x\|^p + \frac{2\theta(2+2^p)}{(2-2^p)2^p}\|x\|^p\right)\right) \geq \alpha$$

*for all* $x \in X$*.*

*Proof.* Define $\psi(x,y,z)=\theta(\|x\|^p+\|y\|^p+\|z\|^p)$ . Then

$$\sum_{i=0}^{\infty}\frac{\Psi(2^ix,2^ix)}{4^i}=\frac{8\theta(2+2^p)}{(4-2^p)2^p}\|x\|^p,\sum_{i=0}^{\infty}\frac{\Psi(2^ix,2^ix)}{2^i}=\frac{4\theta(2+2^p)}{(2-2^p)2^p}\|x\|^p.$$

□

**Corollary 3.4.** *Let $\theta\geq 0$ and $f:X\to Y$ be mapping such that*

$$\lim_{t\to\infty}N(f(x+y+z)+f(x-y)+f(x-z)-f(x-y-z)-f(x+y)-f(x+z),t\theta)=1$$

*uniformly on $X^3$. Then there exist a unique quadratic function $Q:X\to Y$ and a unique additive function $A:X\to Y$ which satisfy (1.1) and the functions $Q$ and $A$ are given by*

$$Q(x):=N-\lim_{n\to\infty}\frac{1}{4^n}\left(\frac{f(2^nx)+f(-2^nx)}{2}-f(0)\right),$$

$$A(x):=N-\lim_{n\to\infty}\frac{1}{2^n}\left(\frac{f(2^nx)-f(-2^nx)}{2}\right)$$

*for all $x\in X$. If for some $\delta>0,\alpha>0$*

$$N(f(x+y+z)+f(x-y)+f(x-z)-f(x-y-z)-f(x+y)-f(x+z),\delta\theta)\geq\alpha$$

*for all $x,y,z\in X$, then*

$$N\left(f(x)-Q(x)-A(x)-f(0),\frac{8\delta\theta}{3}\right)\geq\alpha$$

*for all $x\in X$.*

*Proof.* Define $\psi(x,y,z)=\theta$ . Then

$$\sum_{i=0}^{\infty}\frac{\Psi(2^ix,2^ix)}{4^i}=\frac{8\theta}{3},\sum_{i=0}^{\infty}\frac{\Psi(2^ix,2^ix)}{2^i}=4\theta.$$

□

**Theorem 3.5.** *Let $\psi:X^3\to\mathbb{R}^+$ be a function such that the series $\sum_{i=0}^{\infty}4^i\psi\left(\frac{x}{2^i},\frac{y}{2^i},\frac{z}{2^i}\right)$ converges for all $x,y,z\in X$. Let $f:X\to Y$ be mapping such that*

$$\lim_{t\to\infty}N(f(x+y+z)+f(x-y)+f(x-z)-f(x-y-z)-$$
$$f(x+y)-f(x+z),t\psi(x,y,z))=1$$

*uniformly on $X^3$. Then there exist a unique quadratic function $Q : X \to Y$ and a unique additive function $A : X \to Y$ which satisfy (1.1) and the functions $Q$ and $A$ are given by*

$$Q(x) := N - \lim_{n\to\infty} 4^n \left(\frac{1}{2}\left(f\left(\frac{x}{2^n}\right) + f\left(-\frac{x}{2^n}\right)\right) - f(0)\right),$$

$$A(x) := N - \lim_{n\to\infty} 2^{n-1} \left(f\left(\frac{x}{2^n}\right) - f\left(-\frac{x}{2^n}\right)\right)$$

*for all $x \in X$. If for some $\delta > 0, \alpha > 0$*

$$N(f(x+y+z)+f(x-y)+f(x-z)-f(x-y-z)- f(x+y) - f(x+z), \delta\psi(x, y, z)) \geq \alpha$$

*for all $x, y, z \in X$, then*

$$N\left(f(x) - Q(x) - A(x) - f(0), \delta\left(\frac{1}{4}\sum_{i=0}^{\infty} 4^i \Psi\left(\frac{x}{2^i}, \frac{x}{2^i}\right) + \frac{1}{2}\sum_{i=0}^{\infty} 2^i \Psi\left(\frac{x}{2^i}, \frac{x}{2^i}\right)\right)\right) \geq \alpha$$

*for all $x \in X$ .*

**Theorem 3.6.** *Let $\psi : X^3 \to \mathbb{R}^+$ be a function such that both the series $\sum_{i=0}^{\infty} \frac{\psi(2^i x, 2^i y, 2^i z)}{4^i}$ and $\sum_{i=0}^{\infty} 2^i \psi\left(\frac{x}{2^i}, \frac{y}{2^i}, \frac{z}{2^i}\right)$ converges for all $x, y, z \in X$. Let $f : X \to Y$ be mapping such that*

$$\lim_{t\to\infty} N(f(x+y+z)+f(x-y)+f(x-z)-f(x-y-z)- f(x+y) - f(x+z), t\psi(x, y, z)) = 1$$

*uniformly on $X^3$. Then there exist a unique quadratic function $Q : X \to Y$ and a unique additive function $A : X \to Y$ which satisfy (1.1) and the functions $Q$ and $A$ are given by*

$$Q(x) := N - \lim_{n\to\infty} \frac{1}{4^n}\left(\frac{f(2^n x) + f(-2^n x)}{2} - f(0)\right),$$

$$A(x) := N - \lim_{n\to\infty} 2^{n-1} \left(f\left(\frac{x}{2^n}\right) - f\left(-\frac{x}{2^n}\right)\right)$$

*for all $x \in X$. If for some $\delta > 0 and \alpha > 0$ we have*

$$N(f(x+y+z)+f(x-y)+f(x-z)-f(x-y-z)- f(x+y) - f(x+z), \delta\psi(x, y, z)) \geq \alpha$$

*for all* $x, y, z \in X$ *, then*

$$N\left(f(x) - Q(x) - A(x) - f(0), \delta\left(\frac{1}{4}\sum_{i=0}^{\infty}\frac{\Psi(2^i x, 2^i x)}{4^i} + \frac{1}{2}\sum_{i=0}^{\infty} 2^i \Psi\left(\frac{x}{2^i}, \frac{x}{2^i}\right)\right)\right) \geq \alpha$$

*for all* $x \in X$.

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