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Lw^*wc AND Rw^*wc AND WEAK AMENABILITY OF BANACH ALGEBRAS

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ABSTRACT. We introduce some new concepts as $left - weak^* - weak$ convergence property [Lw^*wc-property] and $right-weak^*-weak$ convergence property [Rw^*wc-property] for Banach algebra A. Suppose that A^* and A^{**}, respectively, have Rw^*wc-property and Lw^*wc-property, then if A^{**} is weakly amenable, it follows that A is weakly amenable. Let $D : A \to A^*$ be a surjective derivation. If D'' is a derivation, then A is Arens regular.

Key Words: Amenability, weak amenability, Derivation, Arens regularity, Topological centers, Module actions, $Left - weak^* - to - weak$ convergence.
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1. Introduction

Let A be a Banach algebra and let B be a Banach $A - bimodule$. A derivation from A into B is a bounded linear mapping $D : A \to B$ such that

$$D(xy) = xD(y) + D(x)y \ for \ all \ x, \ y \in A.$$

The space of all continuous derivations from A into B is denoted by $Z^1(A, B)$.

Easy examples of derivations are the inner derivations, which are given for each $b \in B$ by

$$\delta_b(a) = ab - ba \ for \ all \ a \in A.$$

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The space of inner derivations from A into B is denoted by $N^1(A,B)$. The Banach algebra A is amenable, when for every Banach $A-bimodule$ B, the only derivation from A into B^* is inner. It is clear that A is amenable if and only if $H^1(A,B^*)=Z^1(A,B^*)/N^1(A,B^*)=\{0\}$. The concept of amenability for a Banach algebra A, introduced by Johnson in 1972, has proved to be of enormous importance in Banach algebra theory, see [13]. A Banach algebra A is said weakly amenable, if every derivation from A into A^* is inner. Equivalently, A is weakly amenable if and only if $H^1(A,A^*)=Z^1(A,A^*)/N^1(A,A^*)=\{0\}$. The concept of weak amenability was first introduced by Bade, Curtis and Dales in [2] for commutative Banach algebras, and was extended to the noncommutative case by Johnson in [14]. In this paper, for Banach $A-module$ B, we introduce new concepts as $left-weak^*-weak$ convergence property [Lw^*wc-property] and $right-weak^*-weak$ convergence property [Rw^*wc-property] with respect to A and we show that if A^* and A^{**}, respectively, have Rw^*wc-property and Lw^*wc-property and A^{**} is weakly amenable, then A is weakly amenable. We have also some conclusions regarding Arens regularity of Banach algebras. We introduce some notations and definitions that we used throughout this paper.

Let A be a Banach algebra and A^*, A^{**}, respectively, be the first and second dual of A. For $a\in A$ and $a'\in A^*$, we denote by $a'a$ and aa' respectively, the functionals in A^* defined by $\langle a'a,b\rangle=\langle a',ab\rangle=a'(ab)$ and $\langle aa',b\rangle=\langle a',ba\rangle=a'(ba)$ for all $b\in A$. The Banach algebra A is embedded in its second dual via the identification $\langle a,a'\rangle$ - $\langle a',a\rangle$ for every $a\in A$ and $a'\in A^*$. We say that a bounded net $(e_\alpha)_{\alpha\in I}$ in A is a left bounded approximate identity ($=LBAI$) [resp. right bounded approximate identity ($=RBAI$)] if, for each $a\in A$, $e_\alpha a\longrightarrow a$ [resp. $ae_\alpha\longrightarrow a$].

Let X,Y,Z be normed spaces and $m:X\times Y\to Z$ be a bounded bilinear mapping. Arens in [1] offers two natural extensions m^{***} and m^{t***t} of m from $X^{**}\times Y^{**}$ into Z^{**} as follows

1. $m^*:Z^*\times X\to Y^*$,given by $\langle m^*(z',x),y\rangle=\langle z',m(x,y)\rangle$ where $x\in X$, $y\in Y$, $z'\in Z^*$,
2. $m^{**}:Y^{**}\times Z^*\to X^*$, given by $\langle m^{**}(y'',z'),x\rangle=\langle y'',m^*(z',x)\rangle$ where $x\in X$, $y''\in Y^{**}$, $z'\in Z^*$,
3. $m^{***}:X^{**}\times Y^{**}\to Z^{**}$, given by $\langle m^{***}(x'',y''),z'\rangle=\langle x'',m^{**}(y'',z')\rangle$ where $x''\in X^{**}$, $y''\in Y^{**}$, $z'\in Z^*$.

The mapping m^{***} is the unique extension of m such that $x''\to m^{***}(x'',y'')$ from X^{**} into Z^{**} is $weak^*-to-weak^*$ continuous for every $y'' \in Y^{**}$, but the mapping $y'' \to m^{***}(x'', y'')$ is not in general $weak^* - to - weak^*$ continuous from Y^{**} into Z^{**} unless $x'' \in X$. Hence the first topological center of m may be defined as following

$$Z_1(m) = \{x'' \in X^{**} : \ y'' \to m^{***}(x'', y'') \ is \ weak^* - to - weak^* \\ -continuous\}.$$

Let now $m^t : Y \times X \to Z$ be the transpose of m defined by $m^t(y, x) = m(x, y)$ for every $x \in X$ and $y \in Y$. Then m^t is a continuous bilinear map from $Y \times X$ to Z, and so it may be extended as above to $m^{t***} : Y^{**} \times X^{**} \to Z^{**}$. The mapping $m^{t***t} : X^{**} \times Y^{**} \to Z^{**}$ in general is not equal to m^{***}, see [1], if $m^{***} = m^{t***t}$, then m is called Arens regular. The mapping $y'' \to m^{t***t}(x'', y'')$ is $weak^* - to - weak^*$ continuous for every $y'' \in Y^{**}$, but the mapping $x'' \to m^{t***t}(x'', y'')$ from X^{**} into Z^{**} is not in general $weak^* - to - weak^*$ continuous for every $y'' \in Y^{**}$. So we define the second topological center of m as

$$Z_2(m) = \{y'' \in Y^{**} : \ x'' \to m^{t***t}(x'', y'') \ is \ weak^* - to - weak^* \\ -continuous\}.$$

It is clear that m is Arens regular if and only if $Z_1(m) = X^{**}$ or $Z_2(m) = Y^{**}$. Arens regularity of m is equivalent to the following

$$\lim_i \lim_j \langle z', m(x_i, y_j)\rangle = \lim_j \lim_i \langle z', m(x_i, y_j)\rangle,$$

whenever both limits exist for all bounded sequences $(x_i)_i \subseteq X$, $(y_i)_i \subseteq Y$ and $z' \in Z^*$, see [5, 20].

The regularity of a normed algebra A is defined to be the regularity of its algebra multiplication when considered as a bilinear mapping. Let a'' and b'' be elements of A^{**}, the second dual of A. By *Goldstin's* Theorem [4, P.424-425], there are nets $(a_\alpha)_\alpha$ and $(b_\beta)_\beta$ in A such that $a'' = weak^* - \lim_\alpha a_\alpha$ and $b'' = weak^* - \lim_\beta b_\beta$. So it is easy to see that for all $a' \in A^*$,

$$\lim_\alpha \lim_\beta \langle a', m(a_\alpha, b_\beta)\rangle = \langle a''b'', a'\rangle$$

and

$$\lim_\beta \lim_\alpha \langle a', m(a_\alpha, b_\beta)\rangle = \langle a''ob'', a'\rangle,$$

where $a''.b''$ and $a''ob''$ are the first and second Arens products of A^{**}, respectively, see [6, 17, 20].

The mapping m is left strongly Arens irregular if $Z_1(m) = X$ and m is right strongly Arens irregular if $Z_2(m) = Y$.

Regarding A as a Banach $A-bimodule$, the operation $\pi : A \times A \to A$ extends to π^{***} and π^{t***t} defined on $A^{**} \times A^{**}$. These extensions are known, respectively, as the first (left) and the second (right) Arens products, and with each of them, the second dual space A^{**} becomes a Banach algebra. In this situation, we shall also simplify our notations. So the first (left) Arens product of $a'', b'' \in A^{**}$ shall be simply indicated by $a''b''$ and defined by the three steps:

$$\langle a'a, b\rangle = \langle a', ab\rangle,$$
$$\langle a''a', a\rangle = \langle a'', a'a\rangle,$$
$$\langle a''b'', a'\rangle = \langle a'', b''a'\rangle.$$

for every $a, b \in A$ and $a' \in A^*$. Similarly, the second (right) Arens product of $a'', b'' \in A^{**}$ shall be indicated by $a''ob''$ and defined by :

$$\langle aoa', b\rangle = \langle a', ba\rangle,$$
$$\langle a'oa'', a\rangle = \langle a'', aoa'\rangle,$$
$$\langle a''ob'', a'\rangle = \langle b'', a'ob''\rangle.$$

for all $a, b \in A$ and $a' \in A^*$.

2. Weak amenability of Banach algebras

In this section, for a Banach $A-module$ B, we introduce some new concepts as $left-weak^*-weak$ convergence property [Lw^*wc-property] and $right-weak^*-weak$ convergence property [Rw^*wc-property] with respect to A and we show that if A^* and A^{**}, respectively, have Rw^*wc-property and Lw^*wc-property and A^{**} is weakly amenable, then A is weakly amenable. We obtain some conclusions in the Arens regularity of Banach algebras.

Definition 2.1. Assume that B is a left Banach $A-module$. Let $a'' \in A^{**}$ and $(a_\alpha)_\alpha \subset A$ such that $a_\alpha \overset{w^*}{\to} a''$ in A^{**}. We say that $b' \in B^*$ has $left-weak^*-weak$ convergence property Lw^*wc-property with respect to A, if $b'a_\alpha \overset{w}{\to} b'a''$ in B^*.
When every $b' \in B^*$ has Lw^*wc-property with respect to A, we say that B^* has Lw^*wc-property. The definition of $right-weak^*-weak$ convergence property [$= Rw^*wc-$property] with respect to A is similar and if $b' \in B^*$ has $left-weak^*-weak$ convergence property and $right-weak^*-weak$ convergence property, then we say that $b' \in B^*$ has $weak^*-weak$ convergence property [$= w^*wc-$property].

By using [17, Lemma 3.1], it is clear that if A^* has Lw^*wc-property, then A is Arens regular.

Assume that B is a left Banach $A-module$. We say that $b'\in B^*$ has $left-weak^*-weak$ convergence property to zero Lw^*wc-property to zero with respect to A, if for every $(a_\alpha)_\alpha\subset A$, $b'a_\alpha\stackrel{w^*}{\rightarrow}0$ in B^* implies that $b'a_\alpha\stackrel{w}{\rightarrow}0$ in B^*.

Example 2.2. (1) Every reflexive Banach $A-module$ has w^*wc-property.

(2) Let Ω be a compact group and suppose that $A=C(\Omega)$ and $B=M(\Omega)$ (the measure algebra on σ-algebra of Ω). We know that $A^*=B$ and $\mu a_\alpha\in B$ whenever $(a_\alpha)_\alpha\subseteq A$ and $\mu\in B$. Suppose that $\mu a_\alpha\stackrel{w^*}{\rightarrow}0$, then for each $a\in A$, we have

$$\langle\mu a_\alpha,a\rangle=\langle\mu,a_\alpha*a\rangle=\int_\Omega(a_\alpha*a)d\mu\rightarrow 0.$$

We set $a=1_\Omega$. Then $\mu(a_\alpha)\rightarrow 0$. Now let $b'\in B^*$. Then

$$\langle b',\mu a_\alpha\rangle=\langle a_\alpha b',\mu\rangle=\int_\Omega a_\alpha b'd\mu\leq\parallel b'\parallel\mid\int_\Omega a_\alpha d\mu\mid=\parallel b'\parallel\mid\mu(a_\alpha)\mid\rightarrow 0.$$

It follows that $\mu a_\alpha\stackrel{w}{\rightarrow}0$, and so that μ has Rw^*wc-property to zero with respect to A.

Let now B be a Banach $A-bimodule$, and let

$$\pi_\ell:\ A\times B\rightarrow B\ and\ \pi_r:\ B\times A\rightarrow B.$$

be the left and right module actions of A on B, respectively. Then B^{**} is a Banach $A^{**}-bimodule$ with module actions

$$\pi_\ell^{***}:\ A^{**}\times B^{**}\rightarrow B^{**}\ and\ \pi_r^{***}:\ B^{**}\times A^{**}\rightarrow B^{**}.$$

Similarly, B^{**} is a Banach $A^{**}-bimodule$ with module actions

$$\pi_\ell^{t***t}:\ A^{**}\times B^{**}\rightarrow B^{**}\ and\ \pi_r^{t***t}:\ B^{**}\times A^{**}\rightarrow B^{**}.$$

For a Banach $A-bimodule$ B, we define the topological centers of the left and right module actions of A on B as follows:

$$Z^\ell_{A^{**}}(B^{**})=Z(\pi_r)=\{b''\in B^{**}:\ the\ map\ a''\rightarrow\pi_r^{***}(b'',a''):A^{**}\rightarrow B^{**} is\ weak^*-weak^*\ continuous\}$$

$$Z^{\ell}_{B^{**}}(A^{**}) = Z(\pi_\ell) = \{a'' \in A^{**} : \textit{ the map } b'' \to \pi_\ell^{***}(a'', b'') \ : \ B^{**} \to B^{**} \textit{ is weak}^* - \textit{weak}^* \textit{ continuous}\}$$

$$Z^{r}_{A^{**}}(B^{**}) = Z(\pi_\ell^t) = \{b'' \in B^{**} : \textit{ the map } a'' \to \pi_\ell^{t***}(b'', a'') \ : \ A^{**} \to B^{**} \textit{ is weak}^* - \textit{weak}^* \textit{ continuous}\}$$

$$Z^{r}_{B^{**}}(A^{**}) = Z(\pi_r^t) = \{a'' \in A^{**} : \textit{ the map } b'' \to \pi_r^{t***}(a'', b'') \ : \ B^{**} \to B^{**} \textit{ is weak}^* - \textit{weak}^* \textit{ continuous}\}.$$

Theorem 2.3. *i) Assume that B is a left Banach $A-$module. If $B^*A^{**} \subseteq B^*$, then B^* has Lw^*wc-property.*
ii) Assume that B is a right Banach $A-$module. If $A^{**}B^* \subseteq B^*$ and $Z^r(\pi_r) = Z^r_{A^{**}}(B^{**}) = B^{**}$, then B^* has Rw^*wc-property.

Proof. i) Assume that $a'' \in A^{**}$ and $(a_\alpha)_\alpha \subseteq A$ such that $a_\alpha \stackrel{w^*}{\to} a''$. Then for every $b'' \in B^{**}$, since $b'a'' \in B^*$, we have

$$< b'', b'a'' >=< a''b'', b' >= \lim_\alpha < a_\alpha b'', b' >= \lim_\alpha < b'', b'a_\alpha > .$$

It follows that $b'a_\alpha \stackrel{w}{\to} b'a''$.
ii) The proof is similar to (i). □

Theorem 2.4. *Let A be a Banach algebra and suppose that A^* and A^{**}, respectively, have Rw^*wc-property and Lw^*wc-property. If A^{**} is weakly amenable, then A is weakly amenable.*

Proof. Assume that $a'' \in A^{**}$ and $(a_\alpha)_\alpha \subseteq A$ such that $a_\alpha \stackrel{w^*}{\to} a''$. Then for each $a' \in A^*$, we have $a_\alpha a' \stackrel{w^*}{\to} a''a'$ in A^*. Since A^* has Rw^*wc-property, $a_\alpha a' \stackrel{w}{\to} a''a'$ in A^*. Then for every $x'' \in A^{**}$, we have

$$\langle x''a_\alpha, a' \rangle = \langle x'', a_\alpha a' \rangle \to \langle x'', a''a' \rangle = \langle x''a'', a' \rangle.$$

It follows that $x''a_\alpha \stackrel{w^*}{\to} x''a''$. Since A^{**} has Lw^*wc-property with respect to A, $x''a_\alpha \stackrel{w}{\to} x''a''$. If $D : A \to A^*$ is a bounded derivation, we extend it to a bounded linear mapping D'' from A^{**} into A^{***}. Suppose that $a'', b'' \in A^{**}$ and $(a_\alpha)_\alpha, (b_\beta)_\beta \subseteq A$ such that $a_\alpha \stackrel{w^*}{\to} a''$ and $b_\beta \stackrel{w^*}{\to} b''$. Since $x''a_\alpha \stackrel{w}{\to} x''a''$ for every $x'' \in A^{**}$, we have

$$\lim_\alpha \langle D''(b''), x''a_\alpha\rangle = \langle D''(b''), x''a''\rangle.$$

In the following we take limit on the *weak** topologies. Thus we have

$$\lim_\alpha \lim_\beta D(a_\alpha)b_\beta = D''(a'')b''.$$

Consequently, we have

$$D''(a''b'') = \lim_\alpha \lim_\beta D(a_\alpha b_\beta) = \lim_\alpha \lim_\beta D(a_\alpha)b_\beta + \lim_\alpha \lim_\beta a_\alpha D(b_\beta)$$
$$= D''(a'')b'' + a''D''(b'').$$

Since A^{**} is weakly amenable, there is $a''' \in A^{***}$ such that $D'' = \delta_{a'''}$. We conclude that $D = D'' \mid_A = \delta_{a'''} \mid_A$. Hence for each $x' \in A^*$, we have $D = x'a''' \mid_A - a''' \mid_A x'$. Take $a' = a''' \mid_A$. It follows that $H^1(A, A^*) = 0$. □

Theorem 2.5. *Let A be a Banach algebra and suppose that $D : A \to A^*$ is a surjective derivation. If D'' is a derivation, then we have the following assertions.*

1. *A^* and A^{**}, respectively, have w^*wc-property and Lw^*wc-property with respect to A.*
2. *For every $a'' \in A^{**}$, the mapping $x'' \to a''x''$ from A^{**} into A^{**} is weak* $-$ weak continuous.*
3. *A is Arens regular.*
4. *If A has $LBAI$, then A is reflexive.*

Proof. (1) Since D is surjective, D'' is surjective, and so by using [19, Theorem 2.2], we have $A^{***}A^{**} \subseteq D''(A^{**})A^{**} \subseteq A^*$. Suppose that $a'' \in A^{**}$ and $(a_\alpha)_\alpha \subseteq A$ such that $a_\alpha \stackrel{w^*}{\to} a''$. Then for each $x' \in A^*$, we have $x'a_\alpha \stackrel{w^*}{\to} x'a''$. Since $A^{***}A^{**} \subseteq A^*$, $x'a'' \in A^*$. Then for every $x'' \in A^{**}$, we have

$$\langle x'', x'a_\alpha\rangle = \langle x''x', a_\alpha\rangle \to \langle a'', x''x'\rangle = \langle x'a'', x''\rangle = \langle x'', x'a''\rangle.$$

It follows that $x'a_\alpha \stackrel{w}{\to} x'a''$ in A^*. Thus x' has Lw^*wc-property with respect to A. The proof that x' has Rw^*wc-property with respect to A is similar, and so A^* has w^*wc-property.

Suppose that $x''' \in A^{***}$. Since $A^{***}A^{**} \subseteq A^*$, $x''a_\alpha \overset{w^*}{\to} x''a''$ for each $x'' \in A^{**}$. Then

$$\langle x''', x''a_\alpha \rangle = \langle x'''x'', a_\alpha \rangle \to \langle x'''x'', a'' \rangle = \langle x''', x''a'' \rangle.$$

It follows that $x''a_\alpha \overset{w}{\to} x''a''$. Thus x'' has Lw^*wc-property with respect to A.

(2) Suppose that $(a''_\alpha)_\alpha \subseteq A^{**}$ and $a''_\alpha \overset{w^*}{\to} a''$. Let $x'' \in A^{**}$. Then for every $x''' \in A^{***}$, since $A^{***}A^{**} \subseteq A^*$, we have

$$\langle x''', x''a''_\alpha \rangle = \langle x'''x'', a''_\alpha \rangle \to \langle x'''x'', a'' \rangle = \langle x''', x''a'' \rangle.$$

(3) It follows from (2).

(4) Let $(e_\alpha)_\alpha \subseteq A$ be a $BLAI$ for A. Without loss generality, by using [4, page 146], there is a left unit e'' for A^{**} such that $e_\alpha \overset{w^*}{\to} e''$. Suppose that $(a''_\alpha)_\alpha \subseteq A^{**}$ and $a''_\alpha \overset{w^*}{\to} a''$. Then for every $a''' \in A^{***}$, since $A^{***}A^{**} \subseteq A^*$, we have

$$\langle a''', a''_\alpha \rangle = \langle a''', e''a''_\alpha \rangle = \langle a'''e'', a''_\alpha \rangle \to \langle a'''e'', a'' \rangle = \langle a''', a'' \rangle.$$

It follows that $a''_\alpha \overset{w}{\to} a''$. Consequently A is reflexive.

□

Corollary 2.6. *Let A be a Banach algebra and suppose that $D : A \to A^*$ is a surjective derivation. Then the following statements are equivalent.*

(1) *A^* and A^{**}, respectively, have Rw^*wc-property and Lw^*wc-property.*

(2) *For every $a'' \in A^{**}$, the mapping $x'' \to a''x''$ from A^{**} into A^{**} is $weak^* - weak$ continuous.*

Problem. Suppose that S is a compact semigroup. Dose $L^1(S)^*$ and $M(S)^*$ have Lw^*wc-property or Rw^*wc-property?

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