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ON $(\in, \in \vee q)-$ANTI FUZZY BI-IDEALS OF ORDERED SEMIGROUP

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Abstract. In this paper, we introduce the notions of $(\bar{\in}, \bar{\in} \vee \bar{q})-$fuzzy bi-ideals, $(\in, \in \vee q)-$antifuzzy bi-ideals and $(\bar{\in}, \bar{\in} \vee \bar{q})-$antifuzzy bi-ideals of an ordered semigroup. We characterize $(\in, \in \vee q)-$antifuzzy bi-ideals by $(\bar{\in}, \bar{\in} \vee \bar{q})-$fuzzy bi-ideals, lower level sets and upper level sets. We characterize $(\bar{\in}, \bar{\in} \vee \bar{q})-$antifuzzy bi-ideals by $(\in, \in \vee q)-$fuzzy bi-ideals, lower level sets and upper level sets. We characterize regular ordered semigroups through $(\bar{\in}, \bar{\in} \vee \bar{q})-$antifuzzy bi-ideals and $(\in, \in \vee q)-$fuzzy bi-ideals.

Key Words: Fuzzy algebra, $(\bar{\in}, \bar{\in} \vee \bar{q})-$fuzzy bi-ideals, $(\in, \in \vee q)-$antifuzzy bi-ideals, $(\bar{\in}, \bar{\in} \vee \bar{q})-$antifuzzy bi-ideals.
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1. Introduction

Biswas introduced the concept of an antifuzzy subgroup of a group in [2] and studied the basic properties of a group in terms of antifuzzy subgroups. Hong and Jun[6] modified Biswas idea and applied it into BCK-algebra. Recently Shabir and Nawas studied antifuzzy ideals of a semigroup and gave some interesting properties [19]. Mordeson et.al. in [18] presented an up to date account of fuzzy subsemigroups and fuzzy ideals of a semigroup. The topic of these investigations belongs to the theoretical soft computing (fuzzy structures). Indeed, it is well known that semigroups are basic structures in many applied branches like automata and formal languages, coding theory, finite state machines and

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others. Due to these possibilities of applications, semigroups and related structures are presently extensively investigated in fuzzy settings (see e.g., monograph [18]). In particular (fuzzy) regular ordered semigroups, being union of groups etc., play an important role in the mentioned applications. The foregoing facts were the main motivation for the present investigation. P.Dheena and G.Mohanraj [3] investigated the relationship between fuzzy ideals and antifuzzy ideals in intuitionistic fuzzy ideals of semiring. G.Mohanraj [15] studied the relationship between $(\in, \in \vee q)-$fuzzy right ideals and $(\bar{\in}, \bar{\in} \vee \bar{q})-$antifuzzy right ideals. G.Mohanraj [16] characterized regular semiring using antifuzzy right ideal and fuzzy right ideals. Fuzzy sets in ordered semigroups were first considered by Kehayopulu and Tsingelis in [9]. The concept of a fuzzy bi-ideal B of a semigroup S was introduced by Good and Hughes in [5], which is a subsemigroup of S having the property $BSB \subseteq B$. Kehayopulu and Tsingelis in [10] studied the relation of bi-ideals and fuzzy bi-ideals of an ordered semigroup.

In this paper, we introduce the notions of $(\bar{\in}, \bar{\in} \vee \bar{q})-$fuzzy bi-ideals, $(\in, \in \vee q)-$antifuzzy bi-ideals and $(\bar{\in}, \bar{\in} \vee \bar{q})-$antifuzzy bi-ideals of an ordered semigroup. We find necessary and sufficient conditions for the existence of $(\bar{\in}, \bar{\in} \vee \bar{q})-$fuzzy bi-ideals, $(\in, \in \vee q)-$antifuzzy bi-ideals and $(\bar{\in}, \bar{\in} \vee \bar{q})-$antifuzzy bi-ideals. We characterize $(\in, \in \vee q)-$antifuzzy bi-ideals by$(\bar{\in}, \bar{\in} \vee \bar{q})-$fuzzy bi-ideals, lower level sets and upper level sets. We characterize $(\bar{\in}, \bar{\in} \vee \bar{q})-$antifuzzy bi-ideals by $(\in, \in \vee q)-$fuzzy bi-ideals, lower level sets and upper level sets. We characterize regular ordered semigroups through $(\bar{\in}, \bar{\in} \vee \bar{q})-$antifuzzy bi-ideals and $(\in, \in \vee q)-$fuzzy bi-ideals.

2. Preliminaries

Definition 2.1. *A non-empty set S together with binary operation "." is said to be a semigroup if*
1. *If $a, b \in S$, then $ab \in S$, and*
2. *$(a.b).c = a.(b.c)$ for all $a, b, c \in S$.*

Definition 2.2. *By an ordered semigroup (po-semigroup), we mean an algebraic structure $(S, ., \leq)$ in which the following conditions are satisfied:*

(OS1) *$(S, .)$ is a semigroup.*
(OS2) *$(S, \leq)$ is a poset.*
(OS3) *$a \leq b \Rightarrow xa \leq xb$ and $ax \leq bx$ for all $a, b, x \in S$.*

For $A \subseteq S$, we denote $(A] := \{t \in S | t \leq h \ for \ some \ h \in A\}$. If $A = \{a\}$, then we write $(a]$ instead of $\{(a]\}$. For non-empty subsets A, B of S, we denote $AB := \{ab | a \in A, b \in B\}$.
Let $(S, ., \leq)$ be an ordered semigroup. A non-empty subset A of S is called a subsemigroup of S if $A^2 \subseteq A$.

Definition 2.3. [10] *Let $(S, ., \leq)$ be an ordered semigroup. A non-empty subset A of S is called a bi-ideal of S if*
1. $(\forall a \in S)(\forall b \in A)(a \leq b \Rightarrow a \in A)$.
2. $A^2 \subseteq A$.
3. $ASA \subseteq A$.

By a fuzzy set μ of S, we mean a mapping $\mu : S \longrightarrow [0, 1]$.

Definition 2.4. *A fuzzy set μ in a set S of the form*

$$\mu(y) = \begin{cases} t \in (0,1] & \text{if } y = x, \\ 0 & \text{if } y \neq x. \end{cases}$$

is said to be a fuzzy point with support x and value t and is denoted by x_t.

To say that $x_t \in \mu$(resp.$x_t q \mu$) means that $\mu(x) \geq t$(resp.$\mu(x) + t > 1$,) and in this case, x_t is said to belong to(resp. be quasi-coincident with) a fuzzy set μ. To say that $x_t \in \vee q \mu$(resp.$x_t \in \wedge q \mu$) means that $x_t \in \mu$ or $x_t q \mu$(resp.$x_t \in \mu$ and $x_t q \mu$). We denote $x_t \bar{\in} \mu$ if $\mu(x) < t$ and $x_t \bar{q} \mu$ if $\mu(x) + t \leq 1$. We denote $x_t \overline{\in \vee q} \mu$ if $\mu(x) < t$ and $\mu(x) + t \leq 1$.

Definition 2.5. *An ordered semigroup S is called regular if for every $a \in S$, there exists $x \in S$ such that $a \leq axa$.*

Definition 2.6. [10] *Let μ be a fuzzy subset of an ordered semigroup S. Then μ is said to be a fuzzy bi-ideal of S if*
1. $x \leq y \Rightarrow \mu(x) \geq \mu(y)$.
2. $\mu(xy) \geq \mu(x) \wedge \mu(y)$.
3. $\mu(xyz) \geq \mu(x) \wedge \mu(z), \ \forall x, y, z \in S$.

Definition 2.7. *Let μ be a fuzzy subset of an ordered semigroup S. A lower level set [upper level set] denoted by $L(\mu; t)$ $[U(\mu; t)]$ on S for $t \in (0, 1]$ is defined as:*

$$L(\mu; t) = \{x \in S | \mu(x) \leq t\} \ [U(\mu; t) = \{x \in S | \mu(x) \geq t\}]$$

Theorem 2.8. [13] *Let μ be a fuzzy subset of an ordered semigroup S. Then μ is a fuzzy bi-ideal of S if and only if $U(\mu;t)$ is a bi- ideal of S for all $t \in (0,1]$ whenever non-empty.*

3. $(\in, \in \vee q)-$Antifuzzy bi-ideal

Hereafter S denotes an ordered semigroup unless otherwise specified.

Definition 3.1. [12] *Let μ be a fuzzy subset of S. Then μ is said to be an antifuzzy bi-ideal of S if*
1. $(\forall x, y \in S)(x \leq y \Rightarrow \mu(x) \leq \mu(y))$
2. $(\forall x, y \in S)(\mu(xy) \leq \mu(x) \vee \mu(y))$
3. $(\forall x, y, z \in S)(\mu(xyz) \leq \mu(x) \vee \mu(z))$.

Definition 3.2. [7] *Let μ be a fuzzy subset of S. Then μ is called an $(\in, \in \vee q)-$fuzzy bi-ideal if*
$F1$. $(\forall x, y \in S)(\forall t \in (0,1])(x \leq y, y_t \in \mu \Rightarrow x_t \in \vee q\mu)$
$F2$. $(\forall x, y \in S)(\forall t \in (0,1])(x_t, y_r \in \mu \Rightarrow (xy)_{min\{t,r\}} \in \vee q\mu)$
$F3$. $(\forall x, y, z \in S)(\forall t \in (0,1]))(x_t, y \in S, z_r \in \mu \Rightarrow (xyz)_{min\{t,r\}} \in \vee q\mu)$.

Theorem 3.3. [7] *Let μ be a fuzzy subset of S. Then μ is an $(\in, \in \vee q)-$fuzzy bi-ideal of S if and only if*
$F4$. $(\forall x, y \in S)(x \leq y \Rightarrow \mu(x) \geq \mu(y) \wedge 0.5)$
$F5$. $(\forall x, y \in S)(\mu(xy) \geq \mu(x) \wedge \mu(y) \wedge 0.5)$
$F6$. $(\forall x, y, z \in S)(\mu(xyz) \geq \mu(x) \wedge \mu(z) \wedge 0.5)$.

Definition 3.4. [11] *For $a \in S$, we define $A_a = \{(y,z) \in S \times S / a \leq yz\}$*

Definition 3.5. *Let λ and μ be any fuzzy subsets of a semigroup S. Then the fuzzy product of λ and μ denoted by $\lambda \circ \mu$ is defined as follows:*

$$(\lambda \circ \mu)(a) = \begin{cases} \bigvee\limits_{(y,z) \in A_a} \{\lambda(y) \wedge \mu(z)\} & if \;\; A_a \neq \emptyset \\ 0 & if \;\; A_a = \emptyset. \end{cases}$$

Definition 3.6. *Let λ and μ be any fuzzy subsets and antifuzzy bi-ideals of a semigroup S. Then the $*$ product of λ and μ is defined by* [20] *$\lambda * \mu : S \to [0,1]$ denoted by $\lambda * \mu$ is defined as follows:*

$$(\lambda * \mu)(a) = \begin{cases} \bigwedge\limits_{(y,z) \in A_a} \{\lambda(y) \vee \mu(z)\} & if \;\; A_a \neq \emptyset \\ 1 & if \;\; A_a = \emptyset. \end{cases}$$

For an ordered semigroup S, the fuzzy set "0" and "1" are defined as follows:

$$0 : S \to [0, 1] \text{ by } 0(x) = 0 \text{ and } 1 : S \to [0, 1] \text{ by } 1(x) = 1,\ \forall x \in S.$$

Definition 3.7. [7] *The $\circ_{0.5}$ product of two fuzzy subsets λ and μ of S is defined as follows:*

$$(\lambda \circ_{0.5} \mu)(a) = \begin{cases} \sup\limits_{(y,z)\in A_a} \{\lambda(y) \wedge \mu(z) \wedge 0.5\} & if\ A_a \neq \emptyset \\ 0 & if\ A_a = \emptyset \end{cases}$$

Theorem 3.8. [7] *An ordered semigroup S is regular if and only if $\mu \circ_{0.5} 1 \circ_{0.5} \mu = \mu$, for every $(\in, \in \vee q)-$fuzzy bi-ideal μ of S.*

Definition 3.9. *Let μ be a fuzzy subset of S. μ is said to be an $(\in, \in \vee q)-$antifuzzy bi-ideal if*
AF1. $x \leq y, x_t \in \mu \Rightarrow y_t \in \vee q\mu$
AF2. $(xy)_t \in \mu \Rightarrow x_t \in \vee q\mu$ *or* $y_t \in \vee q\mu$
AF3. $(xyz)_t \in \mu \Rightarrow x_t \in \vee q\mu$ *or* $z_t \in \vee q\mu$, *for all $x, y, z \in S$.*

Theorem 3.10. *Let μ be fuzzy subset of S. Then μ is an $(\in, \in \vee q)-$antifuzzy bi-ideal if and only if*
AF4. $x \leq y \Rightarrow \mu(x) \wedge 0.5 \leq \mu(y)$
AF5. $\mu(xy) \wedge 0.5 \leq \mu(x) \vee \mu(y)$
AF6. $\mu(xyz) \wedge 0.5 \leq \mu(x) \vee \mu(z)$, *for all $x, y, z \in S$.*

Proof. $(AF1) \Leftrightarrow (AF4)$ Let us assume that AF1 holds. If there exists $x, y \in S$ with $x \leq y$ such that $\mu(y) < \mu(x) \wedge 0.5$, then choose a $t \in (0, 1)$ such that $\mu(y) < t < \mu(x) \wedge 0.5$. Then $x_t \in \mu$ but $y_t \bar{\in} \mu$. Moreover $t < 0.5$ and $\mu(y) < t$ imply $\mu(y) + t < 1$. Thus $y_t \bar{\in} \wedge \bar{q}\mu$ which contradicts to $y_t \in \vee q\mu$. Therefore $x \leq y \Rightarrow \mu(x) \wedge 0.5 \leq \mu(y)$, for all $x, y \in S$.

On the other hand, let $x \leq y$ and $x_t \in \mu$. If $t \leq 0.5$, then $\mu(y) \geq \mu(x) \wedge 0.5 \geq t \wedge 0.5 = t$. Thus $y_t \in \mu$. If $t > 0.5$, then $\mu(y) \geq \mu(x) \wedge 0.5 \geq t \wedge 0.5 = 0.5$. Now, $\mu(y) \geq 0.5$ and $t > 0.5$ imply $\mu(y) + t > 1$. Thus $y_t q\mu$. Therefore $y_t \in \vee q\mu$.

$(AF2) \Leftrightarrow (AF5)$ Let us assume that AF2 holds. If there exists $x, y \in S$ such that $\mu(x) \vee \mu(y) < \mu(xy) \wedge 0.5$, then choose a $t \in (0, 1)$ such that $\mu(x) \vee \mu(y) < t < \mu(xy) \wedge 0.5$. Then $(xy)_t \in \mu$ but $x_t \bar{\in} \mu$ and $y_t \bar{\in} \mu$. Now, $\mu(x) < t$, $\mu(y) < t$ and $t < 0.5$ imply $\mu(x) + t < 1$ and $\mu(y) + t < 1$. Thus $x_t \overline{\in \vee q} \mu$ and $y_t \overline{\in \vee q} \mu$ which is a contradiction.

Conversely, let $(xy)_t \in \mu$. If $t \leq 0.5$, then $\mu(x) \vee \mu(y) \geq \mu(xy) \wedge 0.5 \geq t \wedge 0.5 = t$. Then $\mu(x) \vee \mu(y) \geq t$ implies $x_t \in \mu$ or $y_t \in \mu$.If $t > 0.5$, then $\mu(x) \vee \mu(y) \geq \mu(xy) \wedge 0.5 \geq t \wedge 0.5 = 0.5$. Then $\mu(x) \geq 0.5$ or $\mu(y) \geq 0.5$ and $t > 0.5$ imply $\mu(x) + t > 1$ or $\mu(y) + t > 1$. Thus $x_t q \mu$ or$y_t q \mu$. Therefore $x_t \in \vee q \mu$ or $y_t \in \vee q \mu$.

$(AF3) \Leftrightarrow (AF6)$ Assume that AF3 holds. If there exists $x, y, z \in S$ such that $\mu(xyz) \wedge 0.5 > \mu(x) \vee \mu(z)$, then choose a $t \in (0, 1)$ such that $\mu(xyz) \wedge 0.5 > t > \mu(x) \vee \mu(z)$. Then $(xyz)_t \in \mu$ but $x_t \bar{\in} \mu$ and $z_t \bar{\in} \mu$. Now, $\mu(x) < t$ and $\mu(z) < t$ imply $\mu(x) + t < 1$ and $\mu(z) + t < 1$. Thus $x_t \bar{q} \mu$ and $z_t \bar{q} \mu$. Therefore $x_t \overline{\in \vee q} \mu$ and $z_t \overline{\in \vee q} \mu$ which is a contradiction.

Conversely, let $(xyz)_t \in \mu$. If $t \leq 0.5$, then $\mu(x) \vee \mu(z) \geq \mu(xyz) \wedge 0.5 \geq t \wedge 0.5 = t$. Thus $x_t \in \mu$ or $z_t \in \mu$. If $t > 0.5$, then $\mu(x) \vee \mu(z) \geq \mu(xyz) \wedge 0.5 \geq t \wedge 0.5 = 0.5$. Now, $t > 0.5$ and $\mu(x) \geq 0.5$ or $\mu(z) \geq 0.5$ imply $\mu(x) + t > 1$ or $\mu(z) + t > 1$. Hence $x_t q \mu$ or $z_t q \mu$. Therefore $x_t \in \vee q \mu$ or $z_t \in \vee q \mu$. □

Remark 3.11. *(1) Every antifuzzy bi-ideal of S is an $(\in, \in \vee q)-$ antifuzzy bi-ideal of S.*
(2) Every $(\in, \in \vee q)-$antifuzzy bi-ideal need not be an antifuzzy bi-ideal of S by the following example.

Example 3.12. *Let $(S, \cdot, \leq)$ be an ordered semigroup where $S = \{a, b, c, d, e\}$. The order relation "$\leq$" is given by*

$$\leq := \{(a, a), (a, c), (a, d), (a, e)(b, b), (b, d), (b, e), (c, c), (c, e), (d, d), (d, e), (e, e)\}$$

and the multiplication is given as follows:

$\cdot$	a	b	c	d	e
a	a	d	a	d	d
b	a	b	a	d	d
c	a	d	c	d	e
d	a	d	a	d	d
e	a	d	c	d	e

$$\lambda(x) = \begin{cases} 0.8 & \text{if } x = b \\ 0.6 & \text{if } x = c \\ 0.5 & \text{if } x = \{d, e\} \\ 0.3 & \text{if } x = a. \end{cases}$$

Clearly λ is an $(\in, \in \vee q)-$antifuzzy bi-ideal. Since $c \leq e$, $0.6 = \lambda(c) > \lambda(e) = 0.5$, λ is not an antifuzzy bi-ideal. $b_{0.8} \in \lambda$, $c_{0.6} \in \lambda$ but $(b \cdot c)_{0.8 \wedge 0.6} = a_{0.6} \overline{\in \vee q} \lambda$ imply λ is not a $(\in, \in \vee q)-$fuzzy bi-ideal. $(b \cdot c)_{0.6} = a_{0.6} \bar{\in} \lambda$ but $b_{0.6} \in \wedge q \lambda$ and $c_{0.6} \in \wedge q \lambda$ imply λ is not a

$(\bar{\in}, \bar{\in} \vee \bar{q})-$fuzzy bi-ideal. $b \leq e, e_{0.6} \bar{\in} \lambda$ but $b_{0.6} \in \wedge q \lambda$ imply λ is not a $(\bar{\in}, \bar{\in} \vee \bar{q})-$antifuzzy bi-ideal.

Definition 3.13. *Let μ be a fuzzy subset of S. Then μ is said to be a $(\bar{\in}, \bar{\in} \vee \bar{q})-$fuzzy bi-ideal if*
FN1. $x \leq y, x_t \bar{\in} \mu \Rightarrow y_t \bar{\in} \vee \bar{q} \mu$
FN2. $(xy)_t \bar{\in} \mu \Rightarrow x_t \bar{\in} \vee \bar{q} \mu$ or $y_t \bar{\in} \vee \bar{q} \mu$
FN3. $(xyz)_t \bar{\in} \mu \Rightarrow x_t \bar{\in} \vee \bar{q} \mu$ or $z_t \bar{\in} \vee \bar{q} \mu$, for all $x, y, z \in S$.

Theorem 3.14. *Let μ be fuzzy subset of S. Then μ is a $(\bar{\in}, \bar{\in} \vee \bar{q})-$fuzzy bi-ideal if and only if*
FN4. $x \leq y \Rightarrow \mu(x) \vee 0.5 \geq \mu(y)$
FN5. $\mu(xy) \vee 0.5 \geq \mu(x) \wedge \mu(y)$
FN6. $\mu(xyz) \vee 0.5 \geq \mu(x) \wedge \mu(z)$.

Proof. $(FN1) \Rightarrow (FN4)$ Assume that (FN1) holds. If there exists $x, y \in S$ with $x \leq y$ such that $\mu(x) \vee 0.5 < \mu(y)$, then choose a $t \in (0, 1)$ such that $\mu(x) \vee 0.5 < t < \mu(y)$. Then $x_t \bar{\in} \mu$ but $y_t \in \mu$. Moreover, $t > 0.5$ and $\mu(y) > t$ imply $\mu(y) + t > 1$. Thus $y_t \in \wedge q \mu$, which is a contradiction.

Conversely, let $x \leq y$ and $x_t \bar{\in} \mu$. If $t \leq 0.5$, then $\mu(y) \leq \mu(x) \vee 0.5 \leq t \vee 0.5 = 0.5$. Then $\mu(y) \leq 0.5$ and $t \leq 0.5$ imply $\mu(y) + t \leq 1$. Thus $y_t \bar{q} \mu$. If $t > 0.5$, then $\mu(y) \leq \mu(x) \vee 0.5 < t \vee 0.5 = t$. Then $y_t \bar{\in} \mu$. Thus $y_t \bar{\in} \vee \bar{q} \mu$.

$(FN2) \Leftrightarrow (FN5)$ Let us assume that $(FN2)$ holds. If there exists $x, y \in S$ such that $\mu(xy) \vee 0.5 < \mu(x) \wedge \mu(y)$, then choose a $t \in (0, 1)$ such that $\mu(xy) \vee 0.5 < t < \mu(x) \wedge \mu(y.)$ Then $(xy)_t \bar{\in} \mu$,but $x_t \in \mu$ and $y_t \in \mu$. Since $t > 0.5$, $\mu(x) + t > 1$ and $\mu(y) + t > 1$. Then $x_t \in \wedge q \mu$ and $y_t \in \wedge q \mu$ which is a contradiction.

Conversely, let $(xy)_t \bar{\in} \mu$. If $t \leq 0.5$, then $\mu(x) \wedge \mu(y) \leq \mu(xy) \vee 0.5 < t \vee 0.5 = 0.5$ Then $\mu(x) \leq 0.5$ or $\mu(y) \leq 0.5$ implies $\mu(x) + t \leq 1$ or $\mu(y) + t \leq 1$. Thus $x_t \bar{q} \mu$ or $y_t \bar{q} \mu$. If $t > 0.5$ then $\mu(x) \wedge \mu(y) \leq \mu(xy) \vee 0.5 < t \vee 0.5 = t$. Then $\mu(x) \wedge \mu(y) < t$ implies $\mu(x) < t$ or $\mu(y) < t$. Thus $x_t \bar{\in} \mu$ or $y_t \bar{\in} \mu$. Therefore $x_t \bar{\in} \vee \bar{q} \mu$ or $y_t \bar{\in} \vee \bar{q} \mu$.

$(FN3) \Leftrightarrow (FN6)$ Let us assume that $(FN3)$ holds. If there exists $x, y, z \in S$ such that $\mu(xyz) \vee 0.5 < \mu(x) \wedge \mu(z)$, then choose a $t \in (0, 1)$ such that $\mu(xyz) \vee 0.5 < t < \mu(x) \wedge \mu(z)$. Then $(xyz)_t \bar{\in} \mu$, but $x_t \in \mu$ and $z_t \in \mu$. Since $t > 0.5$, $\mu(x) + t > 1$ and $\mu(z) + t > 1$. Then $x_t q \mu$ and $z_t q \mu$. Thus $x_t \in \wedge q \mu$ and $z_t \in \wedge q \mu$. which is a contradiction.

Conversely, let $(xyz)_t \bar{\in} \mu$. If $t \leq 0.5$, then $\mu(x) \wedge \mu(z) \leq \mu(xyz) \vee 0.5 < t \vee 0.5 = 0.5$ Then $\mu(x) < 0.5$or $\mu(z) < 0.5$ implies $\mu(x) + t \leq 1$

or $\mu(z) + t \leq 1$. Thus $x_t \bar{q} \mu$ or $z_t \bar{q} \mu$. If $t > 0.5$ then $\mu(x) \wedge \mu(z) \leq \mu(xyz) \vee 0.5 < t \vee 0.5 = t$. Then $\mu(x) \wedge \mu(z) < t \Rightarrow \mu(x) < t$ or $\mu(z) < t$. Thus $x_t \bar{\in} \mu$ or$z_t \bar{\in} \mu$. Therefore $x_t \bar{\in} \vee \bar{q} \mu$ or $y_t \bar{\in} \vee \bar{q} \mu$. □

Remark 3.15. *(1)Every fuzzy bi-ideal of S is a* $(\bar{\in}, \bar{\in} \vee \bar{q})-$*fuzzy bi-ideal of S.*
(2) Every $(\bar{\in}, \bar{\in} \vee \bar{q})-$*fuzzy bi-ideal need not be a fuzzy bi-ideal of S by the following example.*

Example 3.16. *Consider the ordered semigroup* $(S, ., \leq)$ *as in the Example 3.12. Now, we define a fuzzy subset* μ *as follows:*

$$\mu(x) = \begin{cases} 0.8 & \text{if } x = a \\ 0.5 & \text{if } x = \{d, e\} \\ 0.4 & \text{if } x = c \\ 0.3 & \text{if } x = b. \end{cases}$$

Clearly μ *is an* $(\bar{\in}, \bar{\in} \vee \bar{q})-$*fuzzy bi-ideal. Since* $c \leq e \Rightarrow 0.4 = \mu(c) < \mu(e) = 0.5$, μ *is not a fuzzy bi-ideal.* $c \leq e, e_{0.5} \in \mu$ *but* $c_{.5} \bar{\in} \wedge \bar{q} \mu$ *imply* μ *is not a* $(\in, \in \vee q)-$*fuzzy bi-ideal.* $(b \cdot a \cdot c)_{0.5} = a_{0.5} \in \mu$ *but* $b_{0.5} \bar{\in} \wedge \bar{q} \mu$ *and* $c_{0.5} \bar{\in} \wedge \bar{q} \mu$ *imply* μ *is not a* $(\in, \in \vee q)-$*antifuzzy bi-ideal.* $c_{0.5} \bar{\in} \mu, b_{0.4} \bar{\in} \mu$ *but* $(b \cdot c)_{0.4 \vee 0.5} = a_{0.5} \in \wedge q \mu$ *imply* μ *is not a* $(\bar{\in}, \bar{\in} \vee \bar{q})-$*antifuzzy bi-ideal.*

Definition 3.17. *Let* μ *be a fuzzy subset of S.* μ *is said to be an* $(\overline{\in}, \overline{\in} \vee \bar{q})-$*antifuzzy bi-ideal if*
AFN1. $x \leq y,\ y_t \overline{\in} \mu \Rightarrow x_t \bar{\in} \vee \bar{q} \mu$
AFN2. $x_t \bar{\in} \mu$ *and* $y_s \bar{\in} \mu \Rightarrow (xy)_{t \vee s} \bar{\in} \vee \bar{q} \mu$
AFN3. $x_t \bar{\in} \mu$ *and* $z_s \bar{\in} \mu \Rightarrow (xyz)_{t \vee s} \bar{\in} \vee \bar{q} \mu$, *for all* $x, y, z \in S$.

Theorem 3.18. *Let* μ *be fuzzy set of S.* μ *is an* $(\bar{\in}, \bar{\in} \vee \bar{q})-$*antifuzzy bi-ideal if and only if the following three statements hold.*
AFN4. $x \leq y \Rightarrow \mu(x) \leq \mu(y) \vee 0.5$
AFN5. $\mu(xy) \leq \mu(x) \vee \mu(y) \vee 0.5$
AFN6. $\mu(xyz) \leq \mu(x) \vee \mu(z) \vee 0.5$, *for all* $x, y, z \in S$.

Proof. $(AFN1) \Leftrightarrow (AFN4)$ Let us assume that AF1 holds. If there exists $x, y \in S$ with $x \leq y$ such that $\mu(x) > \mu(y) \vee 0.5$, then choose a $t \in (0, 1)$ such that $\mu(x) > t > \mu(y) \vee 0.5$. Then $y_t \bar{\in} \mu$ but $x_t \in \mu$. Now, $t > 0.5$ and $\mu(x) > t$ imply $\mu(x) + t > 1$. Thus $x_t q \mu$. Therefore $x_t \in \wedge q \mu$ which is a contradiction.

Conversely, let $x \leq y$ and $y_t \bar{\in} \mu$. If $t \leq 0.5$, then $\mu(x) \leq \mu(y) \vee 0.5 < t \vee 0.5 = 0.5$ Then $t \leq 0.5$ and $\mu(x) \leq 0.5$ imply $\mu(x) + t \leq 1$. Thus $x_t \bar{q} \mu$.

If $t > 0.5$, then $\mu(x) \leq \mu(y) \vee 0.5 < t \vee 0.5 = t$. Thus $x_t \bar{\in} \mu$. Therefore $x_t \bar{\in} \vee \bar{q}\mu$.

$(AFN2) \Leftrightarrow (AFN5)$ Let us assume that $(AFN2)$ holds. If there exists $x, y \in S$ such that $\mu(xy) > \mu(x) \vee \mu(y) \vee 0.5$, then choose a $t \in (0, 1)$ such that $\mu(xy) > t > \mu(x) \vee \mu(y) \vee 0.5$. Then $x_t \bar{\in} \mu$ and $y_t \bar{\in} \mu$ but $(xy)_t \in \mu$. Moreover, $t > 0.5$ and $\mu(xy) > t$ imply $\mu(xy) + t > 1$. Thus $(xy)_t \in \wedge q\mu$ which is a contradiction.

Conversely, let $x_t \bar{\in} \mu$ and $y_s \bar{\in} \mu$. Then $\mu(x) \vee \mu(y) < t \vee s$. If $t \vee s \leq 0.5$, then $\mu(xy) \leq \mu(x) \vee \mu(y) \vee 0.5 < t \vee s \vee 0.5 = 0.5$. Then $\mu(xy) \leq 0.5$ and $t \vee s \leq 0.5$ imply $\mu(xy) + t \vee s \leq 1$. Thus $(xy)_{t \vee s} \bar{q} \mu$. If $t \vee s > 0.5$, then $\mu(xy) \leq \mu(x) \vee \mu(y) \vee 0.5 < t \vee s \vee 0.5 = t \vee s$. Thus $(xy)_{t \vee s} \bar{\in} \mu$. Therefore $(xy)_{t \vee s} \bar{\in} \vee \bar{q}\mu$.

$(AFN3) \Leftrightarrow (AFN6)$ Assume that $(AFN3)$ holds. If there exists $x, y, z \in S$ such that $\mu(xyz) > \mu(x) \vee \mu(z) \vee 0.5$, then choose a $t \in (0, 1)$ such that $\mu(xyz) > t > \mu(x) \vee \mu(z) \vee 0.5$. Then $x_t \bar{\in} \mu$ and $z_t \bar{\in} \mu$ but $(xyz)_t \in \mu$. Now, $t > 0.5$ and $\mu(xyz) > t$ imply $\mu(xyz) + t > 1$. Thus $(xyz)_t \in \wedge q\mu$ which is a contradiction.

On the other hand, let $x_t \bar{\in} \mu$ and $z_s \bar{\in} \mu$. Then $\mu(x) \vee \mu(z) < t \vee s$. If $t \vee s \leq 0.5$, then $\mu(xyz) \leq \mu(x) \vee \mu(z) \vee 0.5 < t \vee s \vee 0.5 = 0.5$. Then $\mu(xyz) < 0.5$ and $t \vee s \leq 0.5$ imply $\mu(xyz) + t \vee s < 1$. Thus $(xyz)_{t \vee s} \bar{q} \mu$. If $t \vee s > 0.5$, then $\mu(xyz) \leq \mu(x) \vee \mu(z) \vee 0.5 < t \vee s \vee 0.5 = t \vee s$. Thus $(xyz)_{t \vee s} \bar{\in} \mu$. Therefore $(xyz)_{t \vee s} \bar{\in} \vee \bar{q}\mu$. □

Remark 3.19. *(1) Every antifuzzy bi-ideal of S is an $(\bar{\in}, \bar{\in} \vee \bar{q})-$antifuzzy bi-ideal of S. (2) Every $(\bar{\in}, \bar{\in} \vee \bar{q})-$antifuzzy bi-ideal need not be an antifuzzy bi-ideal of S by the following example.*

Consider the ordered semigroup $(S, ., \leq)$ as in the Example 3.12. Now, we define a fuzzy subset σ as follows:

$$\sigma(x) = \begin{cases} 0.5 & \text{if } x = \{a, b, c\} \\ 0.3 & \text{if } x = e \\ 0.2 & \text{if } x = d. \end{cases}$$

Clearly σ is an $(\bar{\in}, \bar{\in} \vee \bar{q})-$antifuzzy bi-ideal.Since $0.5 = \sigma(b) > \sigma(c) = 0.3$, σ is not an antifuzzy bi-ideal. $a_{0.5} \in \sigma$ and $a \leq d$ but $d_{0.5} \bar{\in} \wedge \bar{q}\sigma$ imply σ is not a $(\in, \in \vee q)-$antifuzzy bi-ideal. $e_{0.3} \in \sigma, b_{0.5} \in \sigma$ but $(b \cdot e)_{0.5 \wedge 0.3} = d_{0.3} \bar{\in} \wedge \bar{q}\sigma$ imply σ is not a $(\in, \in \vee q)-$fuzzy bi-ideal.

Lemma 3.20. *([16]) Let μ be a fuzzy subset of S. Then $U(\mu; t) = L(1 - \mu; 1 - t)$.*

Proof. Let $x \in S$. Then,

$$\begin{aligned} x \in U(\mu;t) &\Leftrightarrow \mu(x) \geq t \\ &\Leftrightarrow -\mu(x) \leq -t \\ &\Leftrightarrow 1-\mu(x) \leq 1-t \\ &\Leftrightarrow x \in L(1-\mu;1-t). \end{aligned}$$

Thus $U(\mu;t) = L(1-\mu;1-t)$ for all $t \in [0,1]$. □

Theorem 3.21. *Let μ be a fuzzy subset of S. Then the following statements are equivalent.*
1. *μ is a $(\in, \in \vee q)-$fuzzy bi-ideal of S.*
2. *$U(\mu;t)$ is a bi-ideal in S. for $t \in [0,0.5]$ whenever non-empty.*
3. *$L(1-\mu;t)$ is a bi-ideal in S. for $t \in [0.5,1]$ whenever non-empty.*
4. *$1-\mu$ is an $(\bar{\in}, \bar{\in} \vee \bar{q})-$antifuzzy bi-ideal of S.*

Proof. (1) $\Rightarrow$ (2) Let μ be a $(\in, \in \vee q)$ fuzzy bi-ideal of S. Let $y \in U(\mu;t)$ and $x \leq y$ for $t \in [0,0.5]$. By Theorem 3.3, $\mu(x) \geq \mu(y) \wedge 0.5 \geq t \wedge 0.5 = t$. Then $x \in U(\mu;t)$. Let $x, y \in U(\mu;t)$ for $t \in [0,0.5]$. Then by Theorem 3.3, $\mu(xy) \geq \mu(x) \wedge \mu(y) \wedge 0.5 \geq t \wedge 0.5 = t$ and $\mu(xzy) \geq \mu(x) \wedge \mu(y) \wedge 0.5 \geq t \wedge 0.5 = t$ Thus $xy, xzy \in U(\mu;t)$, for all $x, y \in U(\mu;t)$ and for all $z \in S$. Thus $U(\mu;t)$ is a bi-ideal in S.

(2) $\Rightarrow$ (3) Let $U(\mu;t)$ be a bi-ideal in S for $t \in [0,0.5]$. By Lemma 3.20, $L(1-\mu;1-t)$ is a bi-ideal for $t \in [0,0.5]$. Then $L(1-\mu;t)$ is a bi-ideal for $t \in [0.5,1]$.

(3) $\Rightarrow$ (4) Let $x \leq y$ and $1-\mu(y) \vee 0.5 = t$. By (3), $L(1-\mu;t)$ is a bi-ideal. Then $y \in L(1-\mu;t)$ implies $x \in L(1-\mu;t)$. Thus $1-\mu(x) \leq 1-\mu(y) \vee 0.5$. Let $x, y \in S$ and $1-\mu(x) \vee 1-\mu(y) \vee 0.5 = t$. Then $x, y \in L(1-\mu;t)$ and $L(1-\mu;t)$ is a bi-ideal. Therefore $xy, xzy \in L(1-\mu;t)$ for $z \in S$. Thus $1-\mu(xy) \leq 1-\mu(x) \vee 1-\mu(y) \vee 0.5$ and $1-\mu(xzy) \leq 1-\mu(x) \vee 1-\mu(y) \vee 0.5$. By Theorem 3.18, $1-\mu$ is an $(\bar{\in}, \bar{\in} \vee \bar{q}-$antifuzzy bi-ideal of S.

(4) $\Rightarrow$ (1) Let $1-\mu$ be a $(\bar{\in}, \bar{\in} \vee \bar{q})-$ antifuzzy bi-ideal of S. Now, $x \leq y$ implies $1-\mu(x) \leq 1-\mu(y) \vee 0.5$. Then $-\mu(x) \leq -\mu(y) \vee -0.5$ implies $\mu(x) \geq \mu(y) \wedge 0.5$. Now, $(1-\mu)(xy) \leq (1-\mu)(x) \vee (1-\mu)(y) \vee 0.5$ implies $-\mu(xy) \leq -\mu(x) \vee -\mu(y) \vee -0.5$. Then $\mu(xy) \geq \mu(x) \wedge \mu(y) \wedge 0.5$. Similarly, we prove that $\mu(xyz) \geq \mu(x) \wedge \mu(z) \wedge 0.5$. By Theorem 3.3, μ is a $(\in, \in \vee q)-$fuzzy bi-ideal of S. □

Theorem 3.22. *Let μ be a fuzzy subset of S. Then the following statements are equivalent.*

1. *μ is a $(\bar{\in}, \bar{\in} \vee \bar{q})-$fuzzy bi-ideal of S.*
2. *$U(\mu; t)$ is a bi-ideal in S for all $t \in (0.5, 1]$ whenever non-empty.*
3. *$L(1-\mu; t)$ is a bi-ideal in S for all $t \in [0, 0.5)$ whenever non-empty.*
4. *$1-\mu$ is an $(\in, \in \vee q)-$antifuzzy bi-ideal of S.*

Proof. (1) $\Rightarrow$ (2) Let μ be a $(\bar{\in}, \bar{\in} \vee \bar{q})-$fuzzy bi-ideal of S. Let $x \leq y$ and $y \in U(\mu; t)$, for $t \in (0.5, 1]$. Then $\mu(x) \geq \mu(y) \geq t > 0.5$ implies $\mu(x) \geq t$. Thus $x \in U(\mu; t)$. Let $x, y \in U(\mu; t)$, for $t \in (0.5, 1]$. Then $\mu(xy) \vee 0.5 \geq \mu(x) \wedge \mu(y) \geq t > 0.5$ and $\mu(xzy) \vee 0.5 \geq \mu(x) \wedge \mu(y) \geq t > 0.5$ imply $\mu(xy) \geq t$ and $\mu(xzy) \geq t$. Thus $xy, xzy \in U(\mu; t)$ for all $x, y \in U(\mu; t)$, for all $z \in S$. Therefore $U(\mu; t)$ is a bi-ideal in S for all $t \in (0.5, 1]$.

(2) $\Rightarrow$ (3) Let us assume that $U(\mu; t)$ is a bi-ideal for all $t \in (0.5, 1]$. By Lemma 3.20, $L(1-\mu; 1-t)$ is a bi-ideal for all $t \in (0.5, 1]$. Therefore $L(1-\mu; t)$ is a bi-ideal for all $t \in [0, 0.5)$.

(3) $\Rightarrow$ (4) Let us assume that (3) holds. Let $x \leq y$ and $1-\mu(y) = t$. If $t \geq 0.5$, then $(1-\mu)(x) \wedge 0.5 \leq t = (1-\mu)(y)$. If $t < 0.5$, then $L(1-\mu; t)$ is a bi-ideal. Then $x \leq y$ and $y \in L(1-\mu; t)$ imply $x \in L(1-\mu; t)$. Thus $1-\mu(x) \wedge 0.5 \leq 1-\mu(x) \leq t = 1-\mu(y)$. Let $x, y \in S$ and $1-\mu(x) \vee 1-\mu(y) = t$. If $t \geq 0.5$, then $1-\mu(xy) \wedge 0.5 \leq t = 1-\mu(x) \vee 1-\mu(y)$ and $1-\mu(xyz) \wedge 0.5 \leq t = 1-\mu(x) \vee 1-\mu(y), \forall z \in S$. If $t < 0.5$, then $L(1-\mu; t)$ is a bi-ideal. Then $x, y \in L(1-\mu; t)$ implies $xy, xzy \in L(1-\mu; t), \forall z \in S$. Thus $1-\mu(xy) \wedge 0.5 \leq 1-\mu(xy) \leq t = 1-\mu(x) \vee 1-\mu(y)$ and $1-\mu(xzy) \wedge 0.5 \leq 1-\mu(xzy) \leq t = 1-\mu(x) \vee 1-\mu(y)$ for all $x, y, z \in S$. By Theorem 3.10, $1-\mu$ is an $(\in, \in \vee q)-$antifuzzy bi-ideal of S.

(4) $\Rightarrow$ (1) Let $1-\mu$ be an $(\in, \in \vee q)-$antifuzzy bi-ideal of S. Let $x \leq y$. Then by Theorem 3.10, $(1-\mu)(x) \wedge 0.5 \leq (1-\mu)(y)$. Now, $(1-\mu)(x) \wedge 0.5 \leq (1-\mu)(y)$ implies$-\mu(x) \wedge -0.5 \leq -\mu(y)$. Then $\mu(x) \vee 0.5 \geq \mu(y)$. By Theorem 3.10, $(1-\mu)(xy) \wedge 0.5 \leq (1-\mu)(x) \vee (1-\mu)(y)$ and $(1-\mu)(xyz) \wedge 0.5 \leq (1-\mu)(x) \vee (1-\mu)(z)$, for all $x, y, z \in S$. Then $-\mu(xy) \wedge -0.5 \leq -\mu(x) \vee -\mu(y)$ and $-\mu(xyz) \wedge -0.5 \leq -\mu(x) \vee -\mu(z)$ imply $\mu(xy) \vee 0.5 \geq \mu(x) \wedge \mu(y)$ and $\mu(xyz) \vee 0.5 \geq \mu(x) \wedge \mu(z)$. By Theorem 3.14, μ is an $(\bar{\in}, \bar{\in} \vee \bar{q})$ fuzzy bi-ideal of S. □

Remark 3.23. *(1) For any $(\bar{\in}, \bar{\in} \vee \bar{q})-$fuzzy bi-ideal μ, $U(\mu; 0.5)$ need not be a bi-ideal.*
By Example 3.12, $U(\mu; 0.5) = \{a, d, e\}$ is not a bi-ideal of an ordered semigroup S, since $c \leq e, e \in U(\mu; 0.5)$ but $c \notin U(\mu; 0.5)$.
(2) For any $(\in, \in \vee q)-$antifuzzy bi-ideal, $L(\lambda; 0.5)$ need not a bi-ideal. By Example 3.12, $L(\lambda; 0.5) = \{a, d, e\}$ is not a bi-ideal.

4. Regular ordered semigroup

Definition 4.1. *Let λ and μ be fuzzy subsets of S. Then the $*_{0.5}$ product of λ and μ is defined as follows:*

$$(\lambda *_{0.5} \mu)(a) = \begin{cases} \bigwedge\limits_{(y,z)\in A_a} \{\lambda(y) \vee \mu(z) \vee 0.5\} & if\, A_a \neq \emptyset. \\ 1 & if\;\, A_a = \emptyset. \end{cases}$$

Note: $0.5 \leq (\lambda *_{0.5} \mu)(x) \leq 1$, *for all* $x \in S$.

Lemma 4.2. *Let μ be an $(\bar{\in}, \bar{\in} \vee \bar{q})-$antifuzzy bi-ideal of S. Then* $\mu *_{0.5} 0 *_{0.5} \mu \geq \mu$.

Proof. Let $a \in S$. If $A_a = \phi$, then $(\mu *_{0.5} 0 *_{0.5} \mu)(a) = 1 \geq \mu(a)$.
If $A_a \neq \emptyset$, then

$$\begin{aligned}(\mu *_{0.5} 0 *_{0.5} \mu)(a) &= \inf_{(y,z)\in A_a} \{(\mu *_{0.5} 0)(y) \vee \mu(z) \vee 0.5\} \\ &= \inf_{(y,z)\in A_a} \{\{\inf_{(p,q)\in A_y} (\mu(p) \vee 0(q) \vee 0.5)\} \vee \mu(z) \vee 0.5\} \\ &= \inf_{(y,z)\in A_a} \{\inf_{(p,q)\in A_y} \{(\mu(p) \vee 0.5)\} \vee \mu(z) \vee 0.5\} \\ &= \inf_{(y,z)\in A_a} \inf_{(p,q)\in A_y} \{\mu(p) \vee \mu(z) \vee 0.5\}\end{aligned}$$

Now $a \leq yz$ and $y \leq pq$ implies $a \leq yz \leq pqz$. Since $a \leq pqz$ and μ is an $(\bar{\in}, \bar{\in} \vee \bar{q})-$antifuzzy bi-ideal of S, $\mu(a) \leq \mu(pqz) \leq \mu(p) \vee \mu(z) \vee 0.5$.
Then, $\mu(a) = \inf\limits_{(y,z)\in A_a} \inf\limits_{(p,q)\in A_y} \mu(a) \leq \inf\limits_{(y,z)\in A_a} \inf\limits_{(p,q)\in A_y} \{\mu(p) \vee \mu(z) \vee 0.5\}$
$= (\mu *_{0.5} 0 *_{0.5} \mu)(a)$. Hence $\mu \leq \mu *_{0.5} 0 *_{0.5} \mu$. □

Lemma 4.3. *Let λ, μ and σ be fuzzy subsets of S. Then*

$$1 - (\lambda *_{0.5} (\mu *_{0.5} \sigma)) = (1 - \lambda) \circ_{0.5} [(1 - \mu) \circ_{0.5} (1 - \sigma)].$$

Proof. Let $a \in S$. If $A_a = \emptyset$, then $[\lambda *_{0.5} (\mu *_{0.5} \sigma)](a) = 1$ and
$[(1 - \lambda) \circ_{0.5} (1 - \mu) \circ_{0.5} (1 - \sigma)](a) = 0$.
Thus $1 - [\lambda *_{0.5} (\mu *_{0.5} \sigma)](a) = 0 = [(1 - \lambda) \circ_{0.5} (1 - \mu) \circ_{0.5} (1 - \sigma) \circ_{0.5}](a)$.
If $A_a \neq \emptyset$, then

$$\begin{aligned}[\lambda *_{0.5} (\mu *_{0.5} \sigma)](a) &= \inf_{(y,z)\in A_a} \{\lambda(y) \vee (\mu *_{0.5} \sigma)(z) \vee 0.5\} \\ &= \inf_{(y,z)\in A_a} \{\lambda(y) \vee \\ &\qquad \{\inf_{(p,q)\in A_z} \{\mu(p) \vee \sigma(q) \vee 0.5\}\} \vee 0.5.\}\end{aligned}$$

Now,

$$-[\lambda *_{0.5} (\mu *_{0.5} \sigma)](a) = \sup_{(y,z)\in A_a} \{-\lambda(y) \wedge \{ \sup_{(p,q)\in A_z} \{-\mu(p) \wedge -\sigma(q) \wedge -0.5\}\} \wedge -0.5.\}$$

Then,

$$\begin{aligned} 1 - [\lambda *_{0.5} (\mu *_{0.5} \sigma)](a) &= \sup_{(y,z)\in A_a} \{1 - \lambda(y) \wedge \{ \sup_{(p,q)\in A_z} \{1 - \mu(p) \wedge \\ &\quad 1 - \sigma(q) \wedge 1 - 0.5\}\} \wedge 1 - 0.5\} \\ &= \sup_{(y,z)\in A_a} \{(1 - \lambda)(y) \wedge \{ \sup_{(p,q)\in A_z} \{(1 - \mu)(p) \wedge \\ &\quad (1 - \sigma)(q) \wedge 0.5\}\} \wedge 0.5\} \\ &= \sup_{(y,z)\in A_a} \{(1 - \lambda)(y) \wedge [(1 - \mu) \circ_{0.5} (1 - \sigma)](z) \\ &\quad \wedge 0.5\} \\ &= \{(1 - \lambda) \circ_{0.5} [(1 - \mu) \circ_{0.5} (1 - \sigma)]\}(a). \end{aligned}$$

Thus $1 - [\lambda *_{0.5} (\mu *_{0.5} \sigma)] = (1 - \lambda) \circ_{0.5} [(1 - \mu) \circ_{0.5} (1 - \sigma)]$. □

Theorem 4.4. *Let S be an ordered semigroup. Then the following statements are equivalent.*
1. *S is regular.*
2. *$\mu *_{0.5} 0 *_{0.5} \mu = \mu$, for all $(\bar{\in}, \bar{\in} \vee \bar{q})-$antifuzzy bi-ideals μ of S.*
3. *$\mu \circ_{0.5} 1 \circ_{0.5} \mu = \mu$, for all $(\in, \in \vee q)-$fuzzy bi-ideals μ of S.*

Proof. (1) $\Rightarrow$ (2) Let S be an ordered regular semigroup. Let $a \in S$. Since S is regular, there is a $x \in S$ such that $a \leq axa$. Then $a \leq ax(axa)$. Thus $(a, xaxa) \in A_a$ and $A_a \neq \emptyset$. Then

$$\begin{aligned} (\mu *_{0.5} 0 *_{0.5} \mu)(a) &= \bigwedge_{(y,z)\in A_a} \{\mu(y) \vee (0 *_{0.5} \mu)(z) \vee 0.5\} \\ &\leq \{\mu(a) \vee (0 *_{0.5} \mu)(xaxa) \vee 0.5\} \\ &= \mu(a) \vee \{ \bigwedge_{(t,r)\in A_{xaxa}} 0(t) \vee \mu(r) \vee 0.5\} \vee 0.5 \\ &\leq \mu(a) \vee \{0(xax) \vee \mu(a) \vee 0.5\} \vee 0.5 \\ &= \mu(a) \vee \mu(a) \vee 0.5 \\ &= \mu(a) \vee 0.5. \end{aligned}$$

Then $0 \leq 0.5 \leq (\mu *_{0.5} 0 *_{0.5} \mu)(a) \leq \mu(a) \vee 0.5 \leq \mu(a)$.Therefore $(\mu *_{0.5} 0 *_{0.5} \mu) \leq \mu$. By Lemma 4.2, $(\mu *_{0.5} 0 *_{0.5} \mu) \geq \mu$. Hence $(\mu *_{0.5} 0 *_{0.5} \mu) = \mu$.

(2) $\Rightarrow$ (3) Let us assume that (3) holds. Let μ be $(\in, \in \vee q)$-fuzzy bi-ideals of S.

Then by Theorem 3.21, $1-\mu$ is an $(\bar{\in}, \bar{\in} \vee \bar{q})$ anti fuzzy bi-ideals of S. By (2), $(1-\mu) *_{0.5} 0 *_{0.5} (1-\mu) = 1-\mu$.Then,

$$\begin{aligned} \mu &= 1-(1-\mu) \\ &= 1-[(1-\mu) *_{0.5} 0 *_{0.5} (1-\mu)] \\ &= (1-(1-\mu)) \circ_{0.5} 1 \circ_{0.5} (1-(1-\mu)) \; [By\ Lemma\ 4.3] \\ &= \mu \circ_{0.5} 1 \circ_{0.5} \mu. \end{aligned}$$

Thus (3) holds.

(3) $\Rightarrow$ (1) The proof follows from the Theorem 3.8. □

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