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A TOPOLOGY ON BCK-MODULES VIA PRIME SUB-BCK-MODULES

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Abstract. In this paper, by considering the notion of prime BCK-sub-modules of BCK-modules, we introduce a topology on prime BCK-sub-modules of BCK-modules. Moreover, the notion of top BCK-modules, semi-prime sub -BCK-modules and extraordinary sub- BCK-modules are introduced. Finally the relationships between them are studied.

Key Words: BCK-algebra, BCK-modules, Top BCK-modules.
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1. Introduction

Every module is an action of a ring on a certain group. This is, indeed, a source of motivation to study the action of certain algebraic structures on groups. A BCK-module is an action of a BCK-algebra on commutative group. In 1994, the notion of BCK-modules was introduced by M. Aslam, H. A.S.Abujabal and A.B. Taheem [2]. They established isomorphism theorems and studied other properties of BCK-modules. The theory of BCK-modules was further developed by Z. Perveen and M. Aslam[10]. Now, in this paper we introduce the concept of a topology on prime BCK-sub-modules of BCK-modules, the notion of top BCK-modules, semi-prime sub -BCK-modules and extraordinary sub-BCK-modules. Also the relationships between them are studied.

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2. Preliminaries

Let us to begin this section with the definition of a BCK-algebra.

Definition 2.1.[8] Let X be a set with a binary operation $*$ and a constant 0. Then $(X, *, 0)$ is called a BCK- algebra if it satisfies the following axioms:
(BCK1)$((x * y) * (x * z)) * (z * y) = 0$,
(BCK2) $(x * (x * y)) * y = 0$,
(BCK3) $x * x = 0$,
(BCK4) $0 * x = 0$,
(BCK5) $x * y = y * x = 0$ imply that $x = y$, for all $x, y, z \in X$.
If there is an element 1 of a BCK- algebra X, satisfying $x * 1 = 0$, for all x in X, the element 1 is called unit of X. A BCK- algebra with unit is called to be bounded.

Definition 2.2.[8] A BCK- algebra $(X, *, 0)$ is called implicative, if $x = x * (y * x)$, for all x, y in X.

Definition 2.3.[1] Let $(X, *, 0)$ be a BCK-algebra, M be an abelian group under + and let $(x, m) \longrightarrow x \cdot m$ be a mapping of $X \times M \longrightarrow M$ such that
(i) $(x \wedge y) \cdot m = x \cdot (y \cdot m)$,
(ii) $x \cdot (m_1 + m_2) = x \cdot m_1 + x \cdot m_2$,
(iii) $0 \cdot m = 0$,
for all $x, y \in X, m_1, m_2 \in M$,where $x \wedge y = y * (y * x)$.
The notion is defined by Definition 2.3.,that is a left X-module(a left BCK-module on X).
If X is bounded, then the following additional condition holds:
(iv) $1 \cdot m = m$.
A right X-module can be defined similarly.

Example 2.4.[1] Let A be a non-empty set and $X = P(A)$ be the power set of A. Then X is a bounded commutative BCK-algebra with $x \wedge y = x \cap y$, for all $x, y \in X$. Define $x + y = (x \cup y) \cap (x \cap y)'$, the symmetric difference. Then $M = (X, +)$ is an abelian group with empty set $\emptyset$ as an identity element and $x + x = \emptyset$. Define $x \cdot m = x \cap m$, for any $x, m \in X$. Then simple calculations show that :
(i) $(x \wedge y) \cdot m = (x \cap y) \cap m = x \cap (y \cap m) = x \cdot (y \cdot m)$,
(ii) $x \cdot (m_1 + m_2) = x \cdot m_1 + x \cdot m_2$,

(iii) $0 \cdot m = \emptyset \cap m = \emptyset = 0$,
(iv) $1 \cdot m = A \cap m = m$. Thus X itself is an X-module.

Lemma 2.5.[1] Let $(X, *, 0)$ be a bounded implicative BCK- algebra and let $x + y = (x * y) \vee (y * x)$, then we have:
(i) $(X, +)$ forms a commutative group,
(ii) Any ideal I of X consisting of two elements forms an X- module.

Definition 2.6.[9] Let M be a left BCK- module over X and N be a submodule of M. Then N is said to be prime sub-BCK-module of M, if $N \neq M$ and $x \cdot m \in N$, implies that $m \in N$ or $x.M \subseteq N$, for any x in X and any m in M.

Theorem 2.7.[9] Let M be a left BCK-module over X. Then P is a prime sub-BCK-module of M containing N if and only if $\frac{P}{N}$ is a prime BCK-submodule of X-module $\frac{M}{N}$.

Lemma 2.8.[9] Let P be a prime ideal of a lower semi-lattice X containing I. Then we have $\frac{P}{I}$ is a prime ideal of BCK- algebra $\frac{X}{I}$.

Example 2.9.[9] Assume $X = P(A)$, where $A = \{1, 2\}$ i.e. $X = \{\emptyset, \{1\}, \{2\}, \{1, 2\}\}$. consider $B = \{1\} \subseteq A$, then $P(B)$ is a prime BCK- submodule of $P(A)$.

Let M be a left BCK-module over X. $Spec(M)$ denote the collection of all prime BCK- sub-modules K of M .

3. Top BCK-module

Definition 3.1. Let M be a BCK-module on X. If N is a sub-BCK-module of M, then we define $V(N) = \{K \in Spec(M) | \ N \subseteq K\}$ and define $\tau(M) = \{V(N) | \ N \text{ is a sub-}BCK\text{-module of } M\}$.

Example 3.2 . Let M be a BCK-module on X. Then it is easy to see that $V(M) = \emptyset$, $V(\{0\}) = Spec(M)$.

Theorem 3.3. Let M be a BCK-module on X and N_i, for $i \in I$, be sub-BCK-modules of M. Then $\bigcap_{i\in I} V(N_i) = V(\sum_{i\in I} N_i)$.

Proof. Let $K \in \bigcap_{i\in I} V(N_i)$. Then $N_i \subseteq K$, for all $i \in I$. Hence $\sum_{i\in I} N_i \subseteq K$. So $K \in V(\sum_{i\in I} N_i)$. Therefore $\bigcap_{i\in I} V(N_i) \subseteq V(\sum_{i\in I} N_i)$.
Now let $K \in V(\sum_{i\in I} N_i)$. Then $\sum_{i\in I} N_i \subseteq K$. Since $N_i \subseteq \sum_{i\in I} N_i$, for $i \in I$, hence $N_i \subseteq K$, for $i \in I$. Then $K \in V(N_i)$, for $i \in I$. So $K \in \bigcap_{i\in I} V(N_i)$. Therefore $V(\sum_{i\in I} N_i) \subseteq \bigcap_{i\in I} V(N_i)$.

Remark 3.4. By Example 3.2 we see that $\emptyset \in \tau(M)$ and $Spec(M) \in \tau(M)$ and by Theorem 3.3 $\tau(M)$ is closed under arbitrary intersection of its elements ,but it could happen that $V(N_{i_k}) \in \tau(M)$, for $k = 1, ..., s$ and $\bigcup V(N_{i_k}) \notin \tau(M)$. So in generally $\tau(M)$ is not a topology on $Spec(M)$.

Definition 3.5. If $\tau(M)$ is closed under finite union of its elements, then $(M, +)$ is called a top X-module.

Example 3.6. Assume $A = \{1, 2\}$ and $X = P(A)$. By some calculation we see that $Spec(M) = \{\{\emptyset, \{1\}\}, \{\emptyset, \{2\}\}\}$ and so X is a top X-module.

Theorem 3.7. Let X be a BCK-algebra, $(M, +)$ be an abelian group and M be X-module. If N and K are sub-X-modules of M, then

(i) $K \subseteq N$ implies that $V(N) \subseteq V(K)$,

(ii) $V(N) \bigcup V(K) \subseteq V(N \bigcap K)$.

Proof. (i) Let $L \in V(N)$. Then $N \subseteq L$. Since $K \subseteq N$, therefore $L \in V(K)$.

(ii) Let $L \in V(N) \bigcup V(K)$. Then $N \subseteq L$ or $K \subseteq L$. Hence $N \bigcap K \subseteq L$. Therefore $L \in V(N \bigcap K)$. So $V(N) \bigcup V(K) \subseteq V(N \bigcap K)$.

Definition 3.8. Let M be a BCK-module on X. A sub-BCK-module

N of M is called semi-prime if N is the intersection of prime sub-X-module of M.

Example 3.9. In Example 3.6 we see that $\emptyset = \{\emptyset, \{1\}\} \bigcap \{\emptyset, \{2\}\}$ is semi-prime .

Definition 3.10. Let M be a BCK-module on X. Then a prime sub-BCK-module K is extraordinary if for semi-prime sub-modules N and L of M, $N \bigcap L \subseteq K$ implies $N \subseteq K$ or $L \subseteq K$.

Example 3.11. Assume $A = \{1, 2\}$ and $X = P(A)$. By some calculation we see that $P(\{1\})$ is an extraordinary sub-X-module .

Theorem 3.12. Let $(M, +)$ be BCK-module on X and N be a sub-BCK-module of M. Then $V(rad(N)) = V(N)$, where $rad(N) = \bigcap_{i \in I} N_i$, N_i is a prime sub-X-modules of M such that $N \subseteq N_i$.

Proof. Since $N \subseteq rad(N)$, then by Theorem 3.7 we get that $V(rad(N)) \subseteq V(N)$. Now let $K \in V(N)$. Then $N \subseteq K$. Since K is a prime sub-BCK-module, by definition of rad(N) we have $K \in V(rad(N))$, so $V(N) \subseteq V(rad(N))$. Therefore $V(rad(N)) = V(N)$.

Theorem 3.13. Let X be a BCK-algebra and $(M, +)$ be X-module. Then the following conditions are equivalent:
(a) $(M, +)$ is a top X-module,
(b) every prime sub X-module of $(M, +)$ is an extraordinary sub-X-module ,
(c) for semi-prime sub-modules N and L of M,$V(N) \bigcup V(L) = V(N \bigcap L)$.

Proof.(**a** $\to$ **b**). If *spec*$(M) = \emptyset$, then it is obvious . If *spec*$(M) \neq \emptyset$,we show that every prime sub-X-module K of M is an extraordinary. Let N and L be semi-prime sub -X-module of M such that $N \bigcap L \subseteq K$. Since M is top X-module, there exists sub-X-module T of M such that $V(N) \bigcup V(L) = V(T)$. Since N is a semi prime sub-X-module, then we have a family $\{K_i \mid i \in I\}$ of prime X-modules of M such that $N = \bigcap_{i \in I} K_i$. Therefore $K_i \in V(N)$, for $i \in I$, so $T \subseteq \bigcap_{i \in I} K_i = N$. Similarly $T \subseteq L$. Therefore $T \subseteq L \bigcap N$. By Theorem 3.7 $V(N) \bigcup V(L) \subseteq$

$V(N\bigcap L)\subseteq V(T)=V(N)\bigcup V(L)$. Hence, $V(N)\bigcup V(L)=V(N\bigcap L)$. Then $K\in V(N)$ or $K\in V(L)$. Therefore $N\subseteq K$ or $L\subseteq K$.

(**b** $\to$ **c**). Let N and L be semi- prime of submodules of an X-module M. By Theorem 3.7, $V(N)\bigcup V(L)\subseteq V(N\bigcap L)$. Now we show that $V(N\bigcap L)\subseteq V(N)\bigcup V(L)$. If $K\in V(N\bigcap L)$, then $(N\bigcap L)\subseteq K$. Since K is an extraordinary, then $N\subseteq K$ or $L\subseteq K$. Hence $K\in V(N)$ or $K\in V(L)$. Therefore $K\in V(N)\bigcup V(L)$.

(**c** $\to$ **a**). Let S and T be arbitrary sub-X-modules of M . If $V(S)=\emptyset$ or $V(T)=\emptyset$, then $V(S)\bigcup V(T)=V(T)$ or $V(S)\bigcup V(T)=V(S)$, respectively. If $V(S)\neq\emptyset$ and $V(T)\neq\emptyset$, then by Theorem 3.14 we get that $V(S)\bigcup V(T)=V(rad(S))\bigcup V(rad(T))=V(rad(S)\bigcap rad(T))$. Hence $(M,+)$ is a top X-module.

Theorem 3.14. Let X be a bounded implicative BCK-algebra , $(M,+)$ be a top X-module and N be a sub-X-module of M. Then $\frac{M}{N}$ is a top X-module. In particular every homomorphic image of a top X-module is a top X-module.

Proof. Let $\frac{K}{N}$ be a prime sub-X-module of $\frac{M}{N}$, $\frac{S}{N}$ and $\frac{S'}{N}$ be sub-semi prime modules of $\frac{M}{N}$ such that $\frac{S}{N}\bigcap\frac{S'}{N}\subseteq\frac{K}{N}$. Then $S\bigcap S'\subseteq K$. By Theorem 3.13 $S\subseteq K$ or $S'\subseteq K$. Therefore $\frac{S}{N}\subseteq\frac{K}{N}$ or $\frac{S'}{N}\subseteq\frac{K}{N}$. So $\frac{K}{N}$ is an extraordinary . By Theorem 3.14 $\frac{M}{N}$ is a top X-module.

In particular let $\Phi: M\to M'$ be an epimorphism and M be a top X-module. By above, $\frac{M}{ker(\Phi)}$ is a top X-module. Then $M'\simeq\frac{M}{ker(\Phi)}$ is a top X-module.

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