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PROOF OF CANTOR'S CONTINUUM HYPOTHESIS

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ABSTRACT. We shall attempt to prove *Cantor's General Continuum Hypothesis*, a special case of which is known as the *Continuum Hypothesis*, namely that the cardinality of the power set of an infinite set is the consecutive cardinality. An ordered field of cardinality $\aleph_{S\rho}$ with interval topology of weight $\aleph_{S\rho}$ is constructed, where $\aleph_{S\rho}$ is an uncountable isolated cardinal.

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1. Introduction

Let ω and Ω be the first countably infinite and the first uncountable ordinal of cardinalities $\aleph_0$ and $\aleph_1$, respectively.

Cardinality of the power set 2^ω of the set ω, called the continuum cardinality, is uncountable, as the power set always has greater cardinality. Therefore the cardinality of Ω is less than or equal to the continuum. Below we prove that they do coincide, that is, prove the *Continuum Hypothesis* by means of building an ordered field from the elements of Ω and applying the triadic set construction. More precisely, we will prove:

Theorem 1.1. $\aleph_1 = 2^{\aleph_0}$.

This special case is instructive and therefore included.

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The proof of Cantor's *General Hypothesis* in its entirety will be given in the last section in a completely analogous way. Let λ be an infinite initial ordinal of cardinality $\aleph_\rho$ and Λ the consecutive initial ordinal of cardinality $\aleph_{S\rho}$. We shall prove that cardinalities of the power set 2^λ and the set Λ coincide, i.e. prove

Theorem 1.2. $\aleph_{S\rho} = 2^{\aleph_\rho}$.

2. Proof of the *Continuum Hypothesis*

Denote by $\oplus$ and $*$ the usual addition and multiplication of ordinals, respectively.

Definition 2.1. By transfinite recursion define the countable ordinals τ_α for any countable ordinal α. Let $\tau_0 = 1$, and let

$$\tau_\alpha = \sup\{\tau_\beta * j \mid \beta < \alpha,\, j \in \omega\}.$$

These are uncountable in number, and we show that we can construct a number system "of base ω" for countable ordinals (the method can be extended to any ordinal).

Lemma 2.2. *Any nonzero countable ordinal γ can be written uniquely as $\gamma = \bigoplus_{j=1}^n \tau_{\alpha_j} * k_j$ with $\alpha_1 > \alpha_2 > \cdots > \alpha_n$, the $k_j \in \omega$ are all nonzero.*

Proof. As the set of all the τ_α has supremum Ω, and for limit ordinals α we have $\sup\{\tau_\beta \mid \beta < \alpha\} = \tau_\alpha$, we see that $\bigcup_{\alpha\in\Omega}[\tau_\alpha, \tau_{\alpha+1}) = \Omega \setminus \{0\}$ (with the usual interval notation). The statement is clear for any ordinal from the interval $[1, \tau_1]$, where $\tau_1 = \omega$. Let $\gamma \in [\tau_{\alpha_1}, \tau_{\alpha_1+1})$. Then $\tau_{\alpha_1+1} = \tau_{\alpha_1} * \omega$ and $[\tau_{\alpha_1}, \tau_{\alpha_1+1}) = \bigcup_{i\in\omega^+}[\tau_{\alpha_1} * i, \tau_{\alpha_1} * (i+1))$. Let $\gamma \in [\tau_{\alpha_1} * k_1, \tau_{\alpha_1} * (k_1+1))$. If γ equals $\tau_{\alpha_1} * k_1$ then the assertion is clear. By induction, ordinals of form $\bigoplus_{j=2}^n \tau_{\alpha_j} * k_j$ with $\tau_{\alpha_2} < \tau_{\alpha_1}$ exhaust the whole interval $(0, \tau_{\alpha_1})$, and every ordinal from the interval can be written in this form uniquely. As the interval $[\tau_{\alpha_1} * k_1, \tau_{\alpha_1} * (k_1+1))$ is of order type τ_{α_1}, γ is of form $\tau_{\alpha_1} * k_1 \oplus \theta$, where θ is a unique ordinal from $(0, \tau_{\alpha_1})$, the assertion has been proved. □

Lemma 2.3. *Let ${}_{\mathbf{Z}}\Omega$ be the free left module over the ring of integers $\mathbf{Z}$ with well-ordered basis $\{\tau_\mu \mid \mu \in \Omega\}$. Then the pair $({}_{\mathbf{Z}}\Omega, +)$ is an ordered free abelian group of cardinality $\aleph_1$.*

Proof. A nonzero element a of the module is of the unique form $a = \sum_{i=1}^n k_i \tau_{\mu_i}$ with n a positive integer, $k_i \in \mathbf{Z}\setminus\{0\}$ and $\mu_1 > \mu_2 > \cdots > \mu_n$

(unless otherwise stated, we shall use this presentation of elements of the module). k_1 will be called the leading coefficient of the element a. The set Ω of countable ordinals has a natural embedding $f : \Omega \to {}_{\mathbf{Z}}\Omega$, $0 \mapsto 0$, $\bigoplus_{i=1}^n \tau_{\mu_i} * k_i \mapsto \sum_{i=1}^n k_i \tau_{\mu_i}$. The well-order of the basis induces an order on the module ${}_{\mathbf{Z}}\Omega$. Let the positivity domain ${}_{\mathbf{Z}}\Omega^+$ be the set of all elements with positive leading coefficients (hence $\sum_{i=1}^n k_i \tau_{\mu_i} < \sum_{j=1}^m l_j \tau_{\nu_j}$ if and only if $\mu_1 = \nu_1$, $k_1 = l_1$ $\mu_2 = \nu_2$, $k_2 = l_2, \dots, \mu_{t-1} = \nu_{t-1}$, $k_{t-1} = l_{t-1}$, and $\mu_t < \nu_t$, or $\mu_t = \nu_t$ and $k_t < l_t$ for some t). It is easy to check that the pair $({}_{\mathbf{Z}}\Omega, +)$ is indeed an ordered free abelian group of cardinality $\aleph_1$. □

Lemma 2.4. *Define multiplication of basis elements of the module* ${}_{\mathbf{Z}}\Omega$ *by the rule* $\tau_\mu \tau_\nu = \tau_{f^{-1}(f(\mu)+f(\nu))}$, *where* f *is the natural embedding. Extend the multiplication by distributivity, that is,* $(\sum_{i=1}^n k_i \tau_{\mu_i})(\sum_{j=1}^m l_j \tau_{\nu_j}) = \sum_{i=1}^n \sum_{j=1}^m k_i l_j \tau_{\mu_i} \tau_{\nu_j}$. *The triple* $({}_{\mathbf{Z}}\Omega, +, \cdot)$ *becomes an ordered integral domain.*

Proof. Note that f is an order-preserving mapping, although $f(\mu \oplus \nu)$ and $f(\mu) + f(\nu)$ do not necessarily coincide. By a straightforward computation multiplication of basis elements is associative and commutative. Again by a routine check multiplication in the whole module is associative and commutative, 1 is unity, distributivity holds, and the product of positive elements is positive. The triple $({}_{\mathbf{Z}}\Omega, +, \cdot)$ is indeed an ordered integral domain. □

Lemma 2.5. *Let* Q_Ω *be the quotient field of the integral domain* ${}_{\mathbf{Z}}\Omega$, *consider the integral domain as a subring of the field, elements of which are the fractions. Then* Q_Ω *is a linearly ordered field of cardinality* $\aleph_1$.

Proof. Let the positivity domain Q_Ω^+ be the set of all fractions with numerator and denominator both either positive or negative. Hence if $a, c \in {}_{\mathbf{Z}}\Omega$ and $b, d \in {}_{\mathbf{Z}}\Omega^+$ then $\frac{a}{b} < \frac{c}{d}$ if and only if $ad < bc$. We see that Q_Ω is indeed a linearly ordered field of cardinality $\aleph_1$. □

We shall consider the field Q_Ω endowed with the interval topology.

Lemma 2.6. *A strictly increasing sequence* $\{u_i\}_{i \in \omega^+}$ *in the unit interval* $[0, 1]$ *of the field* Q_Ω *has an upper bound* h *so that* $h - \varepsilon$ *is not an upper bound for an arbitrarily small* $\varepsilon \in Q_\Omega^+$.

Proof. Clearly, for some $\xi \in \Omega$, $\frac{1}{\tau_\xi} < \varepsilon$, and

$$[0,1) = \bigcup_{\gamma \in \Omega,\, \gamma < \tau_\xi} \left[\frac{\gamma}{\tau_\xi}, \frac{\gamma+1}{\tau_\xi}\right)$$

is a division of the interval $[0,1)$. If $a < b$, $a, b \in [0,1)$, $b - a > \frac{1}{\tau_\xi}$ then a and b are in distinct intervals in the division, hence there exists some division point $\frac{\gamma}{\tau_\xi}$ between a and b. Assume that the endpoint 1 is not an ε-least upper bound. Then $1 - \varepsilon$ is an upper bound and some division point is also an upper bound, hence the set

$$\{\beta \in \Omega \,|\, \beta < \tau_\xi,\ \frac{\beta}{\tau_\xi} \text{ is an upper bound of the sequence } \{u_i\}\}$$

of ordinals is nonempty and has a minimal element β_0. We see that inside the interval $(\frac{\beta_0}{\tau_\xi} - \varepsilon, \frac{\beta_0}{\tau_\xi})$ there exists a division point $\frac{\gamma}{\tau_\xi}$, which is not an upper bound by the construction of the ordinal β_0. Thus $\frac{\beta_0}{\tau_\xi} - \varepsilon$ is not an upper bound, and $h = \frac{\beta_0}{\tau_\xi}$ satisfies the requirements of the lemma. □

Definition 2.7. We shall call h an ε-least upper bound of the strictly increasing sequence $\{u_i\}_{i \in \omega^+}$ in the unit interval $[0,1]$ of the field Q_Ω.

Proof of Theorem 1.1. It is clear that a nondegenerate interval of the field Q_Ω contains uncountably many elements, and, as supremum of countably many countable ordinals is countable, it is obvious that a countable set of positive elements of the field Q_Ω has positive upper and lower bounds.

For $i \in \omega^+$ let $[u_i, v_i]$ be a strictly decreasing sequence of closed intervals with $u_i, v_i \in [0,1]$. Let $\varepsilon \in Q_\Omega^+$ be such that $\varepsilon < v_i - u_i$ for all $i \in \omega^+$. The sequence $\{u_i\}$ is strictly increasing, let h be an ε-least upper bound. Then for some i_0 we have $u_{i_0} > h - \varepsilon$ and $u_{i_0} < h < u_{i_0} + \varepsilon < v_{i_0}$, consequently for all $i > i_0$, $u_{i_0} < u_i < h < u_{i_0} + \varepsilon < u_i + \varepsilon < v_i$. We conclude

$$[h, u_{i_0} + \varepsilon] \subseteq \bigcap_{i \in \omega^+} [u_i, v_i]$$

and in the field Q_Ω the intersection of a strictly decreasing sequence of closed intervals within the unit interval contains a nondegenerate closed interval.

Apply the construction of Cantor's triadic set in the field Q_Ω. It follows that Q_Ω contains a subset of continuum cardinality, and itself is also of continuum cardinality. Consequently $\aleph_1$ is the cardinality of the power set of a set of cardinality $\aleph_0$. The proof of the *Continuum Hypothesis* is complete.

3. Proof of the *General Continuum Hypothesis*

Denote by $\oplus$ and $*$ the usual addition and multiplication of ordinals, respectively.

Definition 3.1. By transfinite recursion define the ordinals $\tau_\alpha \in \Lambda$ for any ordinal $\alpha \in \Lambda$. Let $\tau_0 = 1$, and let

$$\tau_\alpha = \sup\{\tau_\beta * j \mid \beta < \alpha,\ j \in \omega\}.$$

Construct a number system "of base ω" for ordinals in Λ.

Lemma 3.2. *Any nonzero ordinal $\gamma \in \Lambda$ can be written uniquely as $\gamma = \bigoplus_{j=1}^{n} \tau_{\alpha_j} * k_j$ with $\alpha_1 > \alpha_2 > \cdots > \alpha_n$, the $k_j \in \omega$ are all nonzero.*

Proof. As the set of all the τ_α has supremum Λ, and for limit ordinals α we have $\sup\{\tau_\beta \mid \beta < \alpha\} = \tau_\alpha$, we see that $\bigcup_{\alpha\in\Lambda}[\tau_\alpha, \tau_{\alpha+1}) = \Lambda \setminus \{0\}$. The statement is clear for any ordinal from the interval $[1, \tau_1]$, where $\tau_1 = \omega$. Let $\gamma \in [\tau_{\alpha_1}, \tau_{\alpha_1+1})$. Then $\tau_{\alpha_1+1} = \tau_{\alpha_1} * \omega$ and $[\tau_{\alpha_1}, \tau_{\alpha_1+1}) = \bigcup_{i\in\omega^+}[\tau_{\alpha_1} * i, \tau_{\alpha_1} * (i+1))$. Let $\gamma \in [\tau_{\alpha_1} * k_1, \tau_{\alpha_1} * (k_1+1))$. If γ equals $\tau_{\alpha_1} * k_1$ then the assertion is clear. By induction, ordinals of form $\bigoplus_{j=2}^{n} \tau_{\alpha_j} * k_j$ with $\tau_{\alpha_2} < \tau_{\alpha_1}$ exhaust the whole interval $(0, \tau_{\alpha_1})$, and every ordinal from the interval can be written in this form uniquely. As the interval $[\tau_{\alpha_1} * k_1, \tau_{\alpha_1} * (k_1+1))$ is of order type τ_{α_1}, γ is of form $\tau_{\alpha_1} * k_1 \oplus \theta$, where θ is a unique ordinal from $(0, \tau_{\alpha_1})$, the assertion has been proved. $\square$

Lemma 3.3. *Let ${}_{\mathbf{Z}}\Lambda$ be the free left $\mathbf{Z}$-module with well-ordered basis $\{\tau_\mu \mid \mu \in \Lambda\}$. Then the pair $({}_{\mathbf{Z}}\Lambda, +)$ is an ordered free abelian group of cardinality $\aleph_{S\rho}$.*

Proof. A nonzero element a of the module is of the unique form $a = \sum_{i=1}^{n} k_i \tau_{\mu_i}$ with n a positive integer, $k_i \in \mathbf{Z}\setminus\{0\}$ and $\mu_1 > \mu_2 > \cdots > \mu_n$ (unless otherwise stated, we shall use this presentation of elements of the module). k_1 will be called the leading coefficient of the element a. The set Λ has a natural embedding $f : \Lambda \to {}_{\mathbf{Z}}\Lambda$, $0 \mapsto 0$, $\bigoplus_{i=1}^{n} \tau_{\mu_i} * k_i \mapsto \sum_{i=1}^{n} k_i \tau_{\mu_i}$. The well-order of the basis induces an order on the module ${}_{\mathbf{Z}}\Lambda$. Let the positivity domain ${}_{\mathbf{Z}}\Lambda^+$ be the set of all elements with positive leading coefficients (hence $\sum_{i=1}^{n} k_i \tau_{\mu_i} < \sum_{j=1}^{m} l_j \tau_{\nu_j}$ if and only if $\mu_1 = \nu_1$, $k_1 = l_1$ $\mu_2 = \nu_2$, $k_2 = l_2, \ldots, \mu_{t-1} = \nu_{t-1}$, $k_{t-1} = l_{t-1}$, and

$\mu_t < \nu_t$, or $\mu_t = \nu_t$ and $k_t < l_t$ for some t). It is easy to check that the pair $(\mathbf{z}\Lambda, +)$ is indeed an ordered free abelian group of cardinality $\aleph_{S\rho}$. □

Lemma 3.4. *Define multiplication of basis elements of the module* $\mathbf{z}\Lambda$ *by the rule* $\tau_\mu \tau_\nu = \tau_{f^{-1}(f(\mu)+f(\nu))}$, *where* f *is the natural embedding. Extend the multiplication by distributivity, that is,* $(\sum_{i=1}^n k_i \tau_{\mu_i})(\sum_{j=1}^m l_j \tau_{\nu_j}) = \sum_{i=1}^n \sum_{j=1}^m k_i l_j \tau_{\mu_i} \tau_{\nu_j}$. *The triple* $(\mathbf{z}\Lambda, +, \cdot)$ *is an ordered integral domain.*

Proof. Note that f is an order-preserving mapping, although $f(mu \oplus \nu)$ and $f(\mu) + f(\nu)$ do not necessarily coincide. By a straightforward computation multiplication of basis elements is associative and commutative. Again by a routine check multiplication in the whole module is associative and commutative, 1 is unity, distributivity holds, and the product of positive elements is positive. The triple $(\mathbf{z}\Lambda, +, \cdot)$ is indeed an ordered integral domain. □

Lemma 3.5. *Let* Q_Λ *be the quotient field of the integral domain* $\mathbf{z}\Lambda$ *(we consider the integral domain as a subring of the field, elements of which are the fractions). Then* Q_Λ *is a linearly ordered field of cardinality* $\aleph_{S\rho}$.

Proof. Let the positivity domain Q_Λ^+ be the set of all fractions with numerator and denominator both either positive or negative. Hence if $a, c \in \mathbf{z}\Lambda$ and $b, d \in \mathbf{z}\Lambda^+$ then $\frac{a}{b} < \frac{c}{d}$ if and only if $ad < bc$. We see that Q_Λ is indeed a linearly ordered field of cardinality $\aleph_{S\rho}$. □

We shall consider the field Q_Λ endowed with the interval topology.

Lemma 3.6. *Let* κ *be a nonzero limit ordinal not exceeding* λ *and* $\{u_i\}_{i \in \kappa^+}$ *be a strictly increasing net in the unit interval* $[0, 1]$ *of the field* Q_Λ. *The net* $\{u_i\}$ *has an upper bound* h *so that* $h - \varepsilon$ *is not an upper bound for an arbitrarily small* $\varepsilon \in Q_\Lambda^+$.

Proof. Clearly, for some $\xi \in \Lambda$, $\frac{1}{\tau_\xi} < \varepsilon$, and

$$[0, 1) = \bigcup_{\gamma \in \Lambda,\, \gamma < \tau_\xi} \left[\frac{\gamma}{\tau_\xi}, \frac{\gamma + 1}{\tau_\xi}\right)$$

is a division of the interval $[0, 1)$. If $a < b$, $a, b \in [0, 1)$, $b - a > \frac{1}{\tau_\xi}$ then a and b are in distinct intervals in the division, hence there exists some division point $\frac{\gamma}{\tau_\xi}$ between a and b. Assume that the endpoint 1 is not an

ε-least upper bound. Then $1-\varepsilon$ is an upper bound and some division point is also an upper bound, hence the set

$$\{\beta \in \Lambda \,|\, \beta < \tau_\xi,\ \frac{\beta}{\tau_\xi} \text{ is an upper bound of the net } \{u_i\}\}$$

of ordinals is nonempty and has a minimal element β_0. We see that inside the interval $(\frac{\beta_0}{\tau_\xi} - \varepsilon, \frac{\beta_0}{\tau_\xi})$ there exists a division point $\frac{\gamma}{\tau_\xi}$, which is not an upper bound by the construction of the ordinal β_0. Thus $\frac{\beta_0}{\tau_\xi} - \varepsilon$ is not an upper bound, and $h = \frac{\beta_0}{\tau_\xi}$ satisfies the requirements of the lemma. □

Definition 3.7. We shall call h an ε-least upper bound of the strictly increasing net $\{u_i\}_{i\in\kappa^+}$ in the unit interval $[0,1]$ of the field Q_Λ.

Proof of Theorem 1.2. It is clear that a nondegenerate interval of the field Q_Λ is of cardinality $\aleph_{S\rho}$, and, as supremum of a set of cardinality not exceeding $\aleph_\rho$ of ordinals from Λ is also in Λ, it is obvious that a set of cardinality not exceeding $\aleph_\rho$ of positive elements of the field Q_Λ has positive upper and lower bounds.

For $i \in \kappa^+$, where κ is a nonzero limit ordinal not exceeding λ, let $[u_i, v_i]$ be a strictly decreasing chain of closed intervals with $u_i, v_i \in [0,1]$. Let $\varepsilon \in Q_\Lambda^+$ be such that $\varepsilon < v_i - u_i$ for all $i \in \kappa^+$. The net $\{u_i\}$ is strictly increasing, let h be an ε-least upper bound. Then for some i_0 we have $u_{i_0} > h - \varepsilon$ and $u_{i_0} < h < u_{i_0} + \varepsilon < v_{i_0}$, consequently for all $i > i_0$, $u_{i_0} < u_i < h < u_{i_0} + \varepsilon < u_i + \varepsilon < v_i$. We conclude

$$[h, u_{i_0} + \varepsilon] \subseteq \bigcap_{i\in\kappa^+} [u_i, v_i]$$

and in the field Q_Ω the intersection of a strictly decreasing (well-ordered) chain of closed intervals of length not exceeding λ within the unit interval contains a nondegenerate closed interval.

This fact yields that by transfinite induction the iterated construction of Cantor's triadic set can be applied in the field Q_Λ to obtain a set (a tree) of strictly decreasing (well-ordered) chains of closed intervals of length λ, and the cardinality of this set equals the cardinality of the power set of λ. As the intersection of each chain is nonempty, the set Q_Λ contains a subset of cardinality that of the power set of λ. It follows that the power set of λ is of cardinality $\aleph_{S\rho}$.

The proof of the *General Hypothesis* is complete.

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