Journal of Hyperstructures 1 *(2) (2012), 1-15.*
ISSN: 2251-8436 print/2322-1666 online

ON FUZZY SOFT RINGS

B. PAZAR VAROL*, A. AYGÜNOĞLU AND H. AYGÜN

ABSTRACT. In the present paper, we introduce the concept of fuzzy soft ring and study several algebraic properties. Thereafter, we define the notions of fuzzy soft ideal of a fuzzy soft ring, idealistic fuzzy soft ring and soft homomorphism between fuzzy soft rings and discuss some of their properties.

Key Words: Soft set, Fuzzy soft set, Fuzzy ring, Soft ring, Fuzzy soft ring.
2010 Mathematics Subject Classification: Primary: 08A72; Secondary: 20N25.

1. Introduction

Exact solutions to the mathematical models are needed in classical mathematics. If the model is so complicated that we can not set an exact solution, we can derive an approximate solution. In 1999, Russian researcher Molodtsov [11] initiated the concept of soft set theory and started to develop the basics of the corresponding theory as a new approach for modeling uncertainties. A soft set can be considered as an approximate description of an object. Soft set theory has a rich potential for applications in several directions. At present, works on soft set theory and its applications are progressing rapidly. Maji et al.[9] presented some new definitions on soft sets. Pei et al.[14] discussed the relationship between soft sets and information systems. In 2001, Maji et al.[8] combined the fuzzy set and soft set models and introduced the concept of fuzzy soft set. To continue the investigation on fuzzy soft sets, Ahmad and Kharal [2] presented some more properties of them.

Received: 11 September 2012, Accepted: 08 October 2012. Communicated by I. Cristea
*Address correspondence to Banu Pazar Varol; E-mail: banupazar@kocaeli.edu.tr

The algebraic structure of soft sets has been studied increasingly in recent years. Aktaş and Cağman [3] defined the notion of soft groups and derived some properties. They also compared soft sets to the related concepts of fuzzy sets and rough sets. In 2010, Acar et al.[1] introduced basic notions of soft rings, which are actually a parametrized family of subrings of a ring. By using $t-norm$ the concept of fuzzy soft groups was introduced by Aygünoğlu and Aygün [4]. Furthermore, Jun and Park [6] introduced and investigated the notion of soft BCK/BCI algebras and soft subalgebras and then derived their basic properties. Nazmul and Samanta [12] defined soft topological group, normal soft topological group and homomorphism.

The concept of fuzzy ring was introduced by Liu [7] in 1982. Subsequently, Martinez [10] and Dixit et al. [5] studied on fuzzy ring and obtained certain ring theorical analogues.

In this work we study the algebraic properties of fuzzy soft sets in ring theory. This paper is organized as follows: In the first section as preliminaries, we give the concepts of soft set and fuzzy soft set. In section 3, we introduce fuzzy soft rings and study their characteristic properties. In sections 4 and 5, we introduce the notions of fuzzy soft ideal and idealistic fuzzy soft rings and in the last section, we define soft homomorphism between fuzzy soft rings. We prove that image and pre-image of a fuzzy soft ring are also fuzzy soft rings.

2. Preliminaries

In this section as a preparation, we will present the basic definitions and notations as introduced by Maji et al. [8] and Ahmad et al.[2].

Throughout this paper, let X be an initial universe and E be a set of parameters for the universe X and let I be the closed unit interval, i.e., $I = [0, 1]$. Denote the power set of X by $P(X)$.

Definition 2.1. [11] A pair (F, E) is called a soft set over X, where $F : E \longrightarrow P(X)$ is set-valued function.

In what follows we denote a soft set (F, E) over X as a triple (F, E, X).

Definition 2.2. [8] Let I^X denote the set of all fuzzy sets on X and consider $A \subset E$.

A triple (f, A, X) is called a fuzzy soft set, where f is a mapping from A into I^X,i.e., $f : A \longrightarrow I^X$.

That is, for each $a \in A$, $f(a) = f_a : X \longrightarrow I$, is a fuzzy set on X.

Remark 2.3. Obviously, a classical soft set (F, E, X) can be seen as a fuzzy soft set (f, E, X) according to this manner, for $e \in E$, the image of e under f is defined as the characteristic function of the set $F(e)$, i.e.,
$$f_e(a) = \chi_{F(e)}(a) = \begin{cases} 1, & \text{if } a \in F(e); \\ 0, & \text{otherwise.} \end{cases}.$$

Definition 2.4. [8] Let (f, A, X) and (g, B, X) be fuzzy soft sets. Then (f, A, X) is called a fuzzy soft subset of (g, B, X) and write $(f, A, X) \sqsubseteq (g, B, X)$ if

(i) $A \subset B$ and

(ii) For each $a \in A$, $f_a \leqslant g_a$, that is, f_a is fuzzy subset of g_a.

Definition 2.5. [8] Let (f, A, X) and (g, B, X) be two fuzzy soft sets with $A \bigcap B \neq \emptyset$. The intersection of (f, A, X) and (g, B, X) is the fuzzy soft set (h, C, X) where $C = A \cap B$ and $h_c = f_c \wedge g_c$, $\forall c \in C$.

We write $(f, A, X) \sqcap (g, B, X) = (h, C, X)$.

Definition 2.6. [8] Let (f, A, X) and (g, B, X) be two fuzzy soft sets. The union of (f, A, X) and (g, B, X) is the fuzzy soft set (h, C, X) where $C = A \cup B$ and
$$h(c) = \begin{cases} f_c, & \text{if } c \in A - B \\ g_c, & \text{if } c \in B - A \\ f_c \vee g_c, & \text{if } c \in A \cap B \end{cases}, \forall c \in C.$$
We write $(f, A, X) \sqcup (g, B, X) = (h, C, X)$.

Definition 2.7. [2] Let $(f_i, A_i, X)_{i \in J}$ be a family of fuzzy soft sets with $\bigcap_{i \in J} A_i \neq \emptyset$. The intersection of these fuzzy soft sets is a fuzzy soft set (h, C, X) where $C = \bigcap\limits_{i \in J} A_i$ and $h(c) = \bigwedge\limits_{i \in J} f_i(c)$, $\forall c \in C$.

We write $\sqcap_{i \in J}(f_i, A_i, X) = (h, C, X)$.

Definition 2.8. [2] Let $(f_i, A_i, X)_{i \in J}$ be a family of fuzzy soft sets. The union of these fuzzy soft sets is a fuzzy soft set (h, C, X), $C = \bigcup\limits_{i \in J} A_i$ and for all $c \in C$, $h(c) = \bigvee\limits_{i \in J(c)} f_i(c)$, where $J(c) = \{i \in J : c \in A_i\}$.

We write $\bigsqcup\limits_{i \in J}(f_i, A_i, X) = (h, C, X)$.

Definition 2.9. [8] If (f, A, X) and (g, B, X) are two fuzzy soft sets, then (f, A, X) **AND** (g, B, X) is denoted by $(f, A, X)\widetilde{\wedge}(g, B, X)$. $(f, A, X)\widetilde{\wedge}(g, B, X)$ is defined as $(h, A \times B)$ where $h(a, b) = h_{a,b} = f_a \wedge g_b$, for every $(a,b) \in A \times B$.

Definition 2.10. [8] If (f,A,X) and (g,B,X) are two fuzzy soft sets, then (f,A,X) **OR** (g,B,X) is denoted by $(f,A,X)\widetilde{\vee}(g,B,X)$. $(f,A,X)\widetilde{\vee}(g,B,X)$ is defined by $(h, A\times B)$ where $h(a,b) = h_{a,b} = f_a \vee g_b$, $\forall(a,b) \in A \times B$.

Definition 2.11. [1] Let (F,A,X) be a soft set. The set $Supp(F,A,X) = \{x \in A : F(x) \neq \emptyset\}$ is called support of the soft set (F,A,X).

A soft set is called non-null if its support is not equal to the empty set.

Definition 2.12. Let (f,A,X) be a fuzzy soft set. The set $Supp(f,A,X) = \{x \in A : f(x) = f_x \neq 0_X\}$ is called the support of the fuzzy soft set (f,A,X).

A fuzzy soft set is called non-null if its support is not equal to the empty set.

3. Fuzzy soft rings

Acar et al.[1] introduced the notion of soft rings. In this section we introduce the definition of fuzzy soft rings and give some fundamental properties of them.

From now on, R denotes a commutative ring and all fuzzy soft sets are considered over R.

Definition 3.1. [1] Let (F,A,R) be a non-null soft set. Then (F,A,R) is said to be a *soft ring* over R iff $F(a)$ is a subring of R for each $a \in A$.

Definition 3.2. Let (f,A,R) be a non-null fuzzy soft set. Then (f,A,R) is called a *fuzzy soft ring* over R iff $f(a) = f_a$ is a fuzzy subring of R for each $a \in A$, i.e.,

$f_a(x-y) \geqslant f_a(x) \wedge f_a(y)$

$f_a(x \cdot y) \geqslant f_a(x) \wedge f_a(y)$, $\forall x,y \in R$.

That is, for each $a \in A$, f_a is a fuzzy subring in Liu's sense [7].

Example 3.3. Since each soft set can be considered as a fuzzy soft set and since each characteristic function of a subring of a ring is a fuzzy subring, we can consider a soft ring as fuzzy soft ring.

Example 3.4. Let $\mathbb{N}$ be the set of all natural numbers and define $f : \mathbb{N} \longrightarrow I^{\mathbb{R}}$ by $f(n) = f_n : \mathbb{R} \longrightarrow I$, for each $n \in \mathbb{N}$, where

$f_n(x) = \begin{cases} \frac{1}{n}, & \text{if } x = k2^n,\ \exists k \in \mathbb{Z}; \\ 0, & \text{otherwise.} \end{cases}$, where $\mathbb{Z}$ is the set of all integers.

Then the triple $(f, \mathbb{N}, \mathbb{R})$ forms a fuzzy soft set, and the fuzzy soft set $(f, \mathbb{N}, \mathbb{R})$ is a fuzzy soft ring.

Theorem 3.5. *Let (f, A, R) and (g, B, R) be two fuzzy soft rings. If $(f, A, R)\widetilde{\wedge}(g, B, R)$ is non-null, then it is a fuzzy soft ring.*

Proof. Let $(f, A, R)\widetilde{\wedge}(g, B, R) = (h, A \times B, R)$, where $h_{a,b} = f_a \wedge g_b$ for all $(a, b) \in A \times B$. Since $(h, A \times B, R)$ is non-null, there exists the pair $(a, b) \in A \times B$ such that $h_{a,b} = f_a \wedge g_b \neq 0_R$.

We know that $f_a, \forall a \in A$ and $g_b, \forall b \in B$ are fuzzy subrings of R. Since intersection of two fuzzy subrings of R is a fuzzy subring of R, then $h(a, b) = h_{a,b}$ is a fuzzy subring of R. Hence $(h, A \times B, R) = (f, A, R)\widetilde{\wedge}(g, B, R)$ is fuzzy soft ring. □

Theorem 3.6. *Let (f, A, R) and (g, B, R) be two fuzzy soft rings. If $(f, A, R) \sqcap (g, B, R)$ is non-null, then it is a fuzzy soft ring.*

Proof. Let $(f, A, R) \sqcap (g, B, R) = (h, C, R)$, where $C = A \cap B$ and $h_c = f_c \wedge g_c$ for all $c \in C$. Since (h, C, R) is non-null, there exists $c \in C$ such that $h_c(x) \neq 0$ for some $x \in R$. We have that $f_c \wedge g_c$ is a fuzzy subring of R, since $h_c \neq 0_R$ and f_c, g_c are fuzzy subrings of R. Consequently, $(h, C, R) = (f, A, R) \sqcap (g, B, R)$ is a fuzzy soft ring over R. □

Theorem 3.7. *Let $(f_i, A_i, R)_{i\in J}$ be a family of fuzzy soft rings. Then we have the followings:*

(i) *If $\widetilde{\bigwedge}_{i\in J}(f_i, A_i, R)$ is non-null, it is a fuzzy soft ring.*

(ii) *If $\sqcap_{i\in J}(f_i, A_i, R)$ is non-null, it is a fuzzy soft ring.*

(iii) *If $\{A_i : i \in J\}$ are pairwise disjoint and $\bigsqcup_{i\in J}(f_i, A_i, R)$ is non-null, then it is a fuzzy soft ring.*

Proof. (i) Let $\widetilde{\bigwedge}_{i\in J}(f_i, A_i, R) = (g, B, R)$, where $B = \prod_{i\in J} A_i$ and $g_b = \bigwedge_{i\in J} f_i(b_i),\ \forall b = (b_i)_{i\in J} \in B$. Suppose that the fuzzy soft set (g, B, R) is non-null. If $b = (b_i)_{i\in J} \in Supp(g, B, R)$, then $g_b = \bigwedge_{i\in J} f_i(b_i) \neq 0_R$. Since (f_i, A_i, R) is a fuzzy soft ring for all $i \in J$, then $f_i(b_i)$ is a fuzzy subring of R. Hence g_b is a fuzzy subring of R for all $b \in Supp(g, B, R)$.

Therefore $\widetilde{\bigwedge}_{i\in J}(f_i, A_i, R) = (g, B, R)$ is a fuzzy soft ring.

(ii) Let $\sqcap_{i\in J}(f_i, A_i, R) = (g, B, R)$ where $B = \bigcap_{i\in J} A_i$ and $g_b = \bigwedge_{i\in J} f_i(b_i)$ for all $b \in B$. Suppose that (g, B, R) is non-null. If $b \in Supp(g, B, R)$, then $g_b = \bigwedge_{i\in J} f_i(b_i) \neq 0_R$. Since (f_i, A_i, R) is a fuzzy soft ring, then $f_i(b)$ is a fuzzy subring of R for all $i \in J$. Hence g_b is a fuzzy subring of R for all $b \in Supp(g, B, R)$ and consequently, $\sqcap_{i\in J}(f_i, A_i, R) = (g, B, R)$ is a fuzzy soft ring.

(iii) Let $\bigsqcup_{i\in J}(f_i, A_i, R) = (g, B, R)$. Then $B = \bigcup_{i\in J} A_i$ and for all $a \in B$, $g_a = \bigvee_{i\in J(a)} f_i(a)$, where $J(a) = \{i \in J : a \in A_i\}$. Since $Supp(g, B, R) = \bigcup_{i\in J} Supp(f_i, A_i, R) \neq \emptyset$, (g, B, R) is non-null.

Let $a \in Supp(g, B, R)$. Then $g_a = \bigvee_{i\in J(a)} f_i(a) \neq 0_R$. So, we have $f_{i_o}(a) \neq 0_R$ for some $i_o \in J(a)$. From the hypothesis, we have that $(A_i)_{i\in J}$ are pairwise disjoint. Hence i_o is unique. Then g_a coincides with $f_{i_o}(a)$. Moreover, since (f_{i_o}, A_{i_o}) is a fuzzy soft ring over R, we infer that $f_{i_o}(a)$ is a fuzzy subring of R. It follows that $g_a = f_{i_o}(a)$ is a fuzzy subring of R for all $a \in Supp(g, B, R)$. Hence $\bigsqcup_{i\in J}(f_i, A_i, R) = (g, B, R)$ is a fuzzy soft ring. This completes the proof. □

Definition 3.8. [4] Let (f, A, R) be a fuzzy soft set. For each $\alpha \in (0, 1]$, the soft set $(f, A, R)^\alpha = (f^\alpha, A, R)$ is called an *α-level soft set* of (f, A, R), where $f^\alpha(a) = (f_a)^\alpha = \{x \in R \mid f_a(x) \geqslant \alpha\}$ for each $a \in A$.

Obviously, $(f, A, R)^\alpha$ is a soft set.

Theorem 3.9. *Let (f, A, R) be a fuzzy soft set. Then (f, A, R) is a fuzzy soft ring if and only if for all $a \in A$ and for arbitrary $\alpha \in (0, 1]$ with $(f_a)^\alpha \neq \emptyset$, the α-level soft set $(f, A, R)^\alpha$ is a soft ring in the classical case.*

Proof. Let (f, A, R) be a fuzzy soft ring. Then for each $a \in A$, f_a is a fuzzy subring of R. Let $x, y \in (f_a)^\alpha$ for arbitrary $\alpha \in (0, 1]$ with $(f_a)^\alpha \neq \emptyset$. Then $f_a(x) \geqslant \alpha$ and $f_a(y) \geqslant \alpha$. Therefore, $f_a(x - y) \geqslant f_a(x) \wedge f_a(y) \geqslant \alpha$. Hence, $x - y \in (f_a)^\alpha$.

Similarly, we have $f_a(x \cdot y) \geqslant f_a(x) \wedge f_a(y) \geqslant \alpha$. Hence, $x \cdot y \in (f_a)^\alpha$. So, we obtain that $(f_a)^\alpha$ is a subring of R. Consequently, $(f, A, R)^\alpha$ is a soft ring in classical case.

Conversely, let $(f, A, R)^\alpha$ be a soft ring for each $\alpha \in (0, 1]$. Let $\alpha = f_a(x) \wedge f_a(y)$ for each $a \in A$ and $x, y \in R$, then $x, y \in (f_a)^\alpha$. Since $(f_a)^\alpha$ is a subring of R, then $x - y \in (f_a)^\alpha$ and $x \cdot y \in (f_a)^\alpha$. This means that $f_a(x - y) \geqslant \alpha = f_a(x) \wedge f_a(y)$ and $f_a(x \cdot y) \geqslant \alpha = f_a(x) \wedge f_a(y)$, i.e., f_a is a fuzzy subring of R. According to Definition 3.2, (f, A, R) is a fuzzy soft ring. This completes the proof. □

Definition 3.10. Let (f, A, R) and (g, B, R) be two fuzzy soft rings. Then (g, B, R) is said to be a *fuzzy soft subring* of (f, A, R) if the followings are satisfied:

(i) $B \subset A$

(ii) g_c is a fuzzy subring of f_c, for all $c \in Supp(g, B, R)$.

Example 3.11. Let $\mathbb{N}$ be the set of all natural numbers and define $f_n(x)$ as in Example 3.2 and $g : 2\mathbb{N} \longrightarrow I^{\mathbb{R}}$ by $g(n) = g_n : \mathbb{R} \longrightarrow I$, for each $n \in 2\mathbb{N}$, where

$$g_n(x) = \begin{cases} \frac{1}{2n}, & \text{if } x = k2^n, \ \exists k \in \mathbb{Z}; \\ 0, & \text{otherwise.} \end{cases}$$, where $\mathbb{Z}$ is the set of all integers.

Then the triples $(f, \mathbb{N}, \mathbb{R})$ and $(g, 2\mathbb{N}, \mathbb{R})$ are two fuzzy soft rings and the fuzzy soft ring $(g, 2\mathbb{N}, \mathbb{R})$ is a fuzzy soft subring of $(f, \mathbb{N}, \mathbb{R})$.

Theorem 3.12. *Let (f, A, R) and (g, B, R) be two fuzzy soft rings. If $g_x \leqslant f_x$, for all $x \in B \subset A$, then (g, B, R) is a fuzzy soft subring of (f, A, R).*

Proof. Straightforward. □

Theorem 3.13. *Let (f, A, R) and (g, B, R) be two fuzzy soft rings. If $(f, A, R) \sqcap (g, B, R)$ is non-null, then it is a fuzzy soft subring of (f, A, R) and (g, B, R).*

Proof. Let $(f, A, R) \sqcap (g, B, R) = (h, C, R)$, where $C = A \cap B$ and $h_c = f_c \wedge g_c, \forall c \in C$.

Since $C = A \cap B \subset A$ and $h_c = f_c \wedge g_c$ is a fuzzy subring of f_c, then (h, C, R) is a fuzzy soft subring of (f, A, R). Similarly, we obtain that (h, C, R) is a fuzzy soft subring of (g, B, R). □

4. The fuzzy soft ideal of a fuzzy soft ring

If B is an ideal of a ring R, we write $B \lhd R$.

Definition 4.1. Let (f, A, R) be a fuzzy soft ring. A non-null fuzzy soft set (g, B, R) is called a *fuzzy soft ideal* of (f, A, R), denoted by $(g, B, R) \widetilde{\lhd} (f, A, R)$, if it satisfies the following:

(i) $B \subset A$
(ii) g_b is a fuzzy ideal of fuzzy ring f_b, for all $b \in Supp(g, B, R)$.
That is, for each $b \in Supp(g, B, R)$, g_b is a fuzzy ideal in Martinez' s sense [10], i.e.,
$g_b(x - y) \geqslant g_b(x) \wedge g_b(y)$
$g_b(x \cdot y) \geqslant g_b(x) \wedge g_b(y)$
$g_b(x) \leqslant f_b(x)$, $\forall x, y \in R$.

Example 4.2. Let $R = A = \mathbb{Z}_4 = \{0, 1, 2, 3\}$ and $B = \{0, 1, 2\}$. Let define the function $f : A \to I^R$ such that $f(0) = \{\frac{0}{1}, \frac{1}{1}, \frac{2}{1}, \frac{3}{1}\}$, $f(1) = \{\frac{0}{0.5}, \frac{1}{0}, \frac{2}{0}, \frac{3}{0}\}$, $f(2) = \{\frac{0}{0.8}, \frac{1}{0.8}, \frac{2}{0.8}, \frac{3}{0.8}\}$, $f(3) = \{\frac{0}{0.2}, \frac{1}{0}, \frac{2}{0.2}, \frac{3}{0}\}$.
All these sets are fuzzy subrings of R. Hence, (f, A, R) is a fuzzy soft ring. Now, consider the function $g : B \to I^R$ where

$$g_x(y) = \begin{cases} f_x(y), & \text{if } x \cdot y = 0; \\ 0, & \text{otherwise.} \end{cases}$$

Thus, $g(0) = \{\frac{0}{1}, \frac{1}{1}, \frac{2}{1}, \frac{3}{1}\}, g(1) = \{\frac{0}{0.5}, \frac{1}{0}, \frac{2}{0}, \frac{3}{0}\}, g(2) = \{\frac{0}{0.8}, \frac{1}{0}, \frac{2}{0.8}, \frac{3}{0}\}$. Hence, $g(0), g(1)$ and $g(2)$ are fuzzy ideals of fuzzy rings $f(0), f(1)$ and $f(2)$, respectively. Consequently, (g, B, R) is a fuzzy soft ideal of (f, A, R).

Theorem 4.3. *Let (g_1, B_1, R) and (g_2, B_2, R) be fuzzy soft ideals of a fuzzy soft ring (f, A, R). Then $(g_1, B_1, R) \sqcap (g_2, B_2, R)$ is a fuzzy soft ideal of (f, A, R) if it is non-null.*

Proof. Let $(g_1, B_1, R) \widetilde{\lhd} (f, A, R)$ and $(g_2, B_2, R) \widetilde{\lhd} (f, A, R)$. By the Definition 2.4, we write $(g_1, B_1, R) \sqcap (g_2, B_2, R) = (g, B, R)$, where $B = B_1 \cap B_2$ and $g(b) = g_1(b) \wedge g_2(b)$ for all $b \in B$. Since $B_1 \subset A$ and $B_2 \subset A$, we have $B_1 \cap B_2 = B \subset A$.
Suppose that (g, B, R) is non-null. If $b \in Supp(g, B, R)$, then $g(b) = g_1(b) \wedge g_2(b) \neq 0_R$.
Since $(g_1, B_1, R) \widetilde{\lhd} (f, A, R)$ and $(g_2, B_2, R) \widetilde{\lhd} (f, A, R)$, we infer that $g_1(b)$ and $g_2(b)$ are both fuzzy ideals of $f(b)$. Hence, $g(b)$ is a fuzzy ideal of $f(b)$ for all $b \in Supp(g, B, R)$. Therefore, $(g_1, B_1, R) \sqcap (g_2, B_2, R) = (g, B, R)$ is a fuzzy soft ideal of (f, A, R). □

Theorem 4.4. *Let (g_1, B_1, R) and (g_2, B_2, R) be fuzzy soft ideals of a fuzzy soft ring (f, A, R). If B_1 and B_2 are disjoint, then $(g_1, B_1, R) \sqcup (g_2, B_2, R)$ is a fuzzy soft ideal of (f, A, R).*

Proof. Let $(g_1, B_1, R)\widetilde{\lhd}(f, A, R)$ and $(g_2, B_2, R)\widetilde{\lhd}(f, A, R)$. By the Definition 2.5, we write $(g_1, B_1, R) \sqcup (g_2, B_2, R) = (g, B, R)$, where $B = B_1 \cup B_2$ and for all $b \in B$

$$g_b = \begin{cases} g_1(b), & \text{if } b \in B_1 - B_2 \\ g_2(b), & \text{if } b \in B_2 - B_1 \\ g_1(b) \vee g_2(b), & \text{if } b \in B_1 \cap B_2 \end{cases}$$

Obviously, we have $B \subset A$. Since B_1 and B_2 are disjoint, $b \in B_1 - B_2$ or $b \in B_2 - B_1$ for all $b \in Supp(g, B, R)$.

Let $b \in B_1 - B_2$. Since $(g_1, B_1, R)\widetilde{\lhd}(f, A, R)$, then $g(b) = g_1(b) \neq 0_R$ is a fuzzy ideal of $f(b)$. Similarly, if $b \in B_2 - B_1$, then $g(b) = g_2(b) \neq 0_R$ is a fuzzy ideal of $f(b)$.

Thus, for all $b \in Supp(g, B, R)$, $g(b) \lhd f(b)$. Consequently, (g, B, R) is a fuzzy soft ideal of (f, A, R). □

Theorem 4.5. *Let $(g_i, B_i, R)_{i\in J}$ be a family of fuzzy soft ideals of fuzzy soft ring (f, A, R). Then we have the followings:*

(i) *$\widetilde{\bigwedge}_{i\in J}(g_i, B_i, R)$ is a fuzzy soft ideal of (f, A, R) if it is non-null.*

(ii) *$\sqcap_{i\in J}(g_i, B_i, R)$ is a fuzzy soft ideal of (f, A, R) if it is non-null.*

(iii) *If $\{B_i : i \in J\}$ are pairwise disjoint, then $\bigsqcup_{i\in J}(g_i, B_i, R)$ is a fuzzy soft ideal of (f, A, R) if it is non-null.*

Proof. (i) It is obvious from the fact that the intersection of an arbitrary nonempty family of fuzzy ideals of a fuzzy ring is a fuzzy ideal of it.

The proof of (ii) and (iii) are similar to those of the corresponding parts of Theorem 3.3. □

5. Idealistic fuzzy soft rings

Definition 5.1. Let (f, A, R) be a non-null fuzzy soft set. Then (f, A, R) is said to be an *idealistic fuzzy soft ring* if f_a is a fuzzy ideal of R for all $a \in Supp(f, A, R)$.

That is, for each $a \in Supp(f, A, R)$, f_a is a fuzzy ideal of R as defined in [5], i.e.,

$f_a(x - y) \geqslant f_a(x) \wedge f_a(y)$

$f_a(x \cdot y) \geqslant f_a(x) \vee f_a(y)$, $\forall x, y \in R$.

Example 5.2. Let $R = A = \mathbb{Z}_3 = \{0, 1, 2\}$. Let define the function $f : A \to I^R$ such that

$f(0) = \{\frac{0}{1}, \frac{1}{1}, \frac{2}{1}\}, f(1) = \{\frac{0}{0.5}, \frac{1}{0}, \frac{2}{0}\}, f(2) = \{\frac{0}{0.8}, \frac{1}{0.8}, \frac{2}{0.8}\}$.

(f, A, R) is an idealistic fuzzy soft ring since $f(a)$ is a fuzzy ideal of R for all $a \in A$.

Remark 5.3. Every idealistic fuzzy soft ring is a fuzzy soft ring but the inverse of this statement is not true in general. It is shown by the following example:

Example 5.4. Let $R = A$ and define $f : A \to I^R$ by $f(a) = f_a = R \to I$, for each $a \in A$, where

$$f_a(x) = \begin{cases} 1, & \text{if } ax = xa; \\ 0, & \text{otherwise.} \end{cases}$$

Then the triple (f, A, R) is a fuzzy soft ring but not an idealistic fuzzy soft ring.

Proposition 5.5. *Let (f, A, R) be a fuzzy soft set and $B \subset A$. If (f, A, R) is an idealistic fuzzy soft ring, then (f, B, R) is an idealistic fuzzy soft ring whenever it is non-null.*

Proof. Straightforward. □

Theorem 5.6. *Let (f, A, R) and (g, B, R) be two idealistic fuzzy soft rings. Then $(f, A, R) \sqcap (g, B, R)$ is an idealistic fuzzy soft ring if it is non-null.*

Proof. Let $(f, A, R) \sqcap (g, B, R) = (h, C, R)$ where $C = A \cap B$ and $h_c = f_c \wedge g_c$ for all $c \in C$.

Suppose that (h, C, R) is non-null fuzzy soft set. If $c \in Supp(h, C, R)$, then $h_c = f_c \wedge g_c \neq 0_R$. Thus, f_c and g_c are both fuzzy ideals of R. It follows that h_c is a fuzzy ideal of R for all $c \in Supp(h, C, R)$. Hence, $(h, C, R) = (f, A, R) \sqcap (g, B, R)$ is an idealistic fuzzy soft ring over R. □

Theorem 5.7. *Let (f, A, R) and (g, B, R) be two idealistic fuzzy soft rings. If A and B are disjoint, then $(f, A, R) \sqcup (g, B, R)$ is an idealistic fuzzy soft ring.*

Proof. Let $(f, A, R) \sqcup (g, B, R) = (h, C, R)$ where $C = A \cup B$ and for all $c \in C$,

$$h_c = \begin{cases} f_c, & \text{if } c \in A - B \\ g_c, & \text{if } c \in B - A \\ f_c \vee g_c, & \text{if } c \in A \cap B \end{cases}$$

Suppose that $A \cap B = \emptyset$. Then for all $c \in Supp(h, C, R)$ we have that either $c \in A - B$ or $c \in B - A$.

If $c \in A - B$, then $h_c = f_c$ is a fuzzy ideal of R since (f, A, R) is an idealistic fuzzy soft ring.

If $c \in B - A$, then $h_c = g_c$ is a fuzzy ideal of R since (g, B, R) is an idealistic fuzzy soft ring.

Thus, for all $c \in Supp(h, C, R)$, h_c is a fuzzy ideal of R. Consequently, $(h, C, R) = (f, A, R) \sqcup (g, B, R)$ is an idealistic fuzzy soft ring. □

If A and B are not disjoint in Theorem 5.7, then the theorem is not true in general, because the union of two different fuzzy ideals of a ring R may not be a fuzzy ideal of R.

Theorem 5.8. *Let (f, A, R) and (g, B, R) be two idealistic fuzzy soft rings. Then $(f, A, R)\widetilde{\wedge}(g, B, R)$ is an idealistic fuzzy soft ring if it is non-null.*

Proof. Let $(f, A, R)\widetilde{\wedge}(g, B, R) = (h, A \times B, R)$ where $h(a, b) = h_{a,b} = f_a \wedge g_b$ for all $(a, b) \in A \times B$. Suppose that $(h, A \times B, R)$ is non-null fuzzy soft set. If $(a, b) \in Supp(h, A \times B, R)$, then $h_{a,b} = f_a \wedge g_b \neq 0_R$. Since (f, A, R) and (g, B, R) are idealistic fuzzy soft rings, we infer that f_a and g_b are both fuzzy ideals of R. Hence, $h_{a,b}$ is a fuzzy ideal of R for all $(a, b) \in Supp(h, A \times B, R)$. Thus, $(h, A \times B, R) = (f, A, R)\widetilde{\wedge}(g, B, R)$ is an idealistic fuzzy soft ring. □

6. Homomorphism of fuzzy soft rings

In this section we show that the homomorphic image and pre-image of a fuzzy soft ring (fuzzy soft ideal of a fuzzy soft ring) are also fuzzy soft ring (fuzzy soft ideal of a fuzzy soft ring).

Definition 6.1. [4] Let $\varphi : X \longrightarrow Y$ and $\psi : A \longrightarrow B$ be two functions, where A and B are parameter sets for the crisp sets X and Y, respectively. Then the pair (φ, ψ) is called a fuzzy soft function from X to Y.

Definition 6.2. [4] Let (f, A, X) and (g, B, Y) be two fuzzy soft sets and let (φ, ψ) be a fuzzy soft function from X to Y.

(1) The image of (f, A, X) under the soft function (φ, ψ), denoted by $(\varphi, \psi)(f, A, X)$, is the fuzzy soft set over Y defined by $(\varphi, \psi)(f, A, X) = (\varphi(f), \psi(A), Y)$, where

$$\varphi(f)_k(y) = \begin{cases} \bigvee_{\varphi(x)=y} \bigvee_{\psi(a)=k} f_a(x), & \text{if } x \in \varphi^{-1}(y); \\ 0, & \text{otherwise.} \end{cases}, \forall k \in \psi(A), \forall y \in Y.$$

(2) The preimage of (g, B, Y) under the fuzzy soft function (φ, ψ), denoted by $(\varphi, \psi)^{-1}(g, B, Y)$, is the fuzzy soft set over X defined by $(\varphi, \psi)^{-1}(g, B, Y) = (\varphi^{-1}(g), \psi^{-1}(B), X)$, where

$\varphi^{-1}(g)_a(x) = g_{\psi(a)}(\varphi(x))$, $\forall a \in \psi^{-1}(B)$, $\forall x \in X$.

If φ and ψ is injective (surjective), then (φ, ψ) is said to be injective (surjective).

Definition 6.3. Let R and S be two classical rings and (φ, ψ) be a fuzzy soft function from R to S. If φ is a homomorphism from R to S then (φ, ψ) is said to be fuzzy soft homomorphism. If φ is a isomorphism from X to Y and ψ is one-to-one mapping from A onto B then (φ, ψ) is said to be fuzzy soft isomorphism.

Theorem 6.4. *Let (f, A, R) be a fuzzy soft ring and (φ, ψ) be a fuzzy soft homomorphism from R to S. Then $(\varphi, \psi)(f, A, R)$ is a fuzzy soft ring over S.*

Proof. Let $k \in \psi(A)$ and $y_1, y_2 \in S$. If $\varphi^{-1}(y_1) = \emptyset$ or $\varphi^{-1}(y_2) = \emptyset$ the proof is straightforward. Let assume that there exist $x_1, x_2 \in R$ such that $\varphi(x_1) = y_1, \varphi(x_2) = y_2$.

$$\begin{aligned}
\varphi(f)_k(y_1 - y_2) &= \bigvee_{\varphi(t)=y_1-y_2} \bigvee_{\psi(a)=k} f_a(t) \\
&\geqslant \bigvee_{\psi(a)=k} f_a(x_1 - x_2) \\
&\geqslant \bigvee_{\psi(a)=k} (f_a(x_1) \wedge f_a(x_2)) \\
&= \bigvee_{\psi(a)=k} f_a(x_1) \wedge \bigvee_{\psi(a)=k} f_a(x_2)
\end{aligned}$$

This inequality is satisfied for each $x_1, x_2 \in R$, which satisfy $\varphi(x_1) = y_1, \varphi(x_2) = y_2$. Then we have

$$\varphi(f)_k(y_1 - y_2) \geqslant \left(\bigvee_{\varphi(t_1)=y_1} \bigvee_{\psi(a)=k} f_a(t_1) \right) \wedge \left(\bigvee_{\varphi(t_2)=y_2} \bigvee_{\psi(a)=k} f_a(t_2) \right) =$$

$\varphi(f)_k(y_1) \wedge \varphi(f)_k(y_2)$.

and similarly, we have $\varphi(f)_k(y_1 \cdot y_2) \geqslant \varphi(f)_k(y_1) \wedge \varphi(f)_k(y_2)$. Thus, we conclude that $(\varphi, \psi)(f, A, R)$ is a fuzzy soft ring over S. □

Theorem 6.5. *Let (g, B, S) be a fuzzy soft ring and (φ, ψ) be a fuzzy soft homomorphism from R to S. Then $(\varphi, \psi)^{-1}(g, B, S)$ is a fuzzy soft ring over R.*

Proof. Let $a \in \psi^{-1}(B)$ and $x_1, x_2 \in R$.

$$\begin{aligned}
\varphi^{-1}(g)_a(x_1 \cdot x_2) &= g_{\psi(a)}(\varphi(x_1 \cdot x_2)) \\
&= g_{\psi(a)}(\varphi(x_1) \cdot \varphi(x_2)) \\
&\geqslant g_{\psi(a)}(\varphi(x_1)) \wedge g_{\psi(a)}(\varphi(x_2)) \\
&= \varphi^{-1}(g)_a(x_1) \wedge \varphi^{-1}(g)_a(x_2).
\end{aligned}$$

and similarly, we have $\varphi^{-1}(g)_a(x_1 - x_2) \geqslant \varphi^{-1}(g)_a(x_1) \wedge \varphi^{-1}(g)_a(x_2)$. So, $(\varphi, \psi)^{-1}(g, B, S)$ is a fuzzy soft ring over R. $\square$

Theorem 6.6. *Let (f, A, R) be a fuzzy soft ring, $(g, B, R)\widetilde{\lhd}(f, A, R)$ and (φ, ψ) be a fuzzy soft homomorphism from R to S. Then $(\varphi, \psi)(g, B, R)\widetilde{\lhd}$ $(\varphi, \psi)(f, A, R)$.*

Proof. By Theorem 6.1, we know that $(\varphi, \psi)(f, A, R)$ is a fuzzy ring over S. Since (g, B, R) is a fuzzy soft ideal of (f, A, R), we have $B \subset A$ and hence, $\psi(B) \subset \psi(A)$.

Let $k \in Supp(\varphi, \psi)(g, B, R)$ and $y_1, y_2 \in S$. If $\varphi^{-1}(y_1) = \emptyset$ or $\varphi^{-1}(y_2) = \emptyset$, the proof is straightforward. Let assume that there exist $x_1, x_2 \in R$ such that $\varphi(x_1) = y_1, \varphi(x_2) = y_2$.

$$\begin{aligned}
\varphi(g)_k(y_1 - y_2) &= \bigvee_{\varphi(t)=y_1-y_2} \bigvee_{\psi(b)=k} g_b(t) \\
&\geqslant \bigvee_{\psi(b)=k} g_b(x_1 - x_2) \\
&\geqslant \bigvee_{\psi(b)=k} (g_b(x_1) \wedge g_b(x_2)) \\
&= \bigvee_{\psi(b)=k} g_b(x_1) \wedge \bigvee_{\psi(b)=k} g_b(x_2)
\end{aligned}$$

This inequality is satisfied for each $x_1, x_2 \in R$, which satisfy $\varphi(x_1) = y_1, \varphi(x_2) = y_2$. Then we have

$$\varphi(g)_k(y_1 - y_2) \geqslant \left(\bigvee_{\varphi(t_1)=y_1} \bigvee_{\psi(b)=k} g_b(t_1) \right) \wedge \left(\bigvee_{\varphi(t_2)=y_2} \bigvee_{\psi(b)=k} g_b(t_2) \right) =$$

$\varphi(g)_k(y_1) \wedge \varphi(g)_k(y_2)$.

Similarly, we have $\varphi(g)_k(y_1 \cdot y_2) \geqslant \varphi(g)_k(y_1) \wedge \varphi(g)_k(y_2)$.

Let $y \in S$.

$$\begin{aligned}
\varphi(g)_k(y) &= \bigvee_{\varphi(t)=y} \bigvee_{\psi(b)=k} g_b(t) \\
&\leqslant \bigvee_{\varphi(t)=y} \bigvee_{\psi(b)=k} f_b(t) \\
&= \varphi(f)_k(y)
\end{aligned}$$

This completes the proof. $\square$

Theorem 6.7. *Let (f, A, S) be a fuzzy soft ring, $(g, B, S)\widetilde{\lhd}(f, A, S)$ and (φ, ψ) be a fuzzy soft homomorphism from R to S. Then $(\varphi, \psi)^{-1}(g, B, S)$ $\widetilde{\lhd}(\varphi, \psi)(f, A, S)$*

Proof. Straightforward. $\square$

Conclusion. In this paper, the concept of fuzzy soft ring is introduced and its characteristic properties are studied. Further, the notions of fuzzy soft ideal of a fuzzy soft ring and idealistic fuzzy soft ring are introduced. The soft homomorphism between fuzzy soft rings are defined and they are proved that image and preimage of fuzzy soft rings are fuzzy soft rings. To extend this study, one could study the properties of fuzzy soft sets in other algebraic structures.

Acknowledgments

The authors are highly grateful to the editor in chief and referees for their valuable suggestions.

References

[1] U. Acar, F. Koyuncu and B. Tanay, *Soft sets and soft rings*, Comput. Math. Appl., **59** (2010), 3458–3463.

[2] B. Ahmat and A. Kharal, *On fuzzy soft sets*, Adv. Fuzzy Syst. (2009), Article ID 586507, 6 pages

[3] H. Aktaş and N. Çağman, *Soft sets and soft group*, Inform. Sci., **177** (2007) 2726–2735.

[4] A. Aygünoğlu and H. Aygün, *Introduction to fuzzy soft groups*, Comput. Math. Appl. **58** (2009) 1279–1286.

[5] V.N. Dixit, R. Kumar and N. Ajmal, On fuzzy rings, *Fuzzy Sets and systems*, **49** (1992) 205–213.

[6] Y. B. Jun and C.H. Park, *Applications of soft sets in ideal theory of BCK/BCI-algebras*, Inform. Sci. **178** (2008) 2466–2475.

[7] W. Liu, *Fuzzy invariant subgroups and fuzzy ideals*, Fuzzy Sets and Systems **8** (1982) 133–139.

[8] P.K. Maji, R. Biswas and A.R. Roy, *Fuzzy soft sets*, J. Fuzzy Math. **9 (3)** (2001) 589–602.

[9] P.K. Maji, R. Biswas and A.R. Roy, *Soft set theory*, Comput. Math. Appl. **45** (2003) 555–562.

[10] L. Martinez, *Prime and primary L-fuzzy ideals of L-fuzzy rings*, Fuzzy Sets and Systems **101** (1999) 489–494.

[11] D. Molodsov, *Soft set theory-First results*, Comput. Math. Appl. **37 (4/5)** (1999) 19–31.

[12] S. Nazmul and S. K. Samanta, *Soft topological groups*, Kochi J. Math. **5** (2010) 151–161.

[13] Z. Pawlak, *Rough sets*, Internat. J. Comput. Inform. Sci., **11** (1982) 341–356.

[14] D. Pei and D. Miao, *From soft sets to information systems*, Granular Computing, 2005 IEEE International Conference on (2) (2005) 617–621.

[15] A. Rosenfeld, *Fuzzy groups*, J. Math. Anal. Appl. **35** (1971) 512–517.

[16] L.A. Zadeh, *Fuzzy sets*, Information and Control **8**, (1965) 338– 353.

B. Pazar Varol
Department of Mathematics, Kocaeli University, P.O.Box 41380, Kocaeli, Turkey
Email: banupazar@kocaeli.edu.tr

A. Aygünoğlu
Department of Mathematics, Kocaeli University, P.O.Box 41380, Kocaeli, Turkey
Email: abdulkadir.aygunoglu@kocaeli.edu.tr

H. Aygün
Department of Mathematics, Kocaeli University, P.O.Box 41380, Kocaeli, Turkey
Email: halis@kocaeli.edu.tr