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LW^*W-PROPERTY AND TOPOLOGICAL CENTERS OF BANACH ALGEBRAS

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Abstract. In this paper we introduce the new concepts LW^*W-property and RW^*W-property for a Banach algebra A. Under certain conditions, we show that if A has LW^*W-property and RW^*W- property, then A is Arens regular. We also offer some applications of these new concepts in group algebras.

Key Words: Arens regularity, LW^*W- property, Topological center, Dual space.
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1. Introduction and Preliminaries

Through this paper, A is a Banach algebra and A^* and A^{**} are the first and second dual of A, respectively. Arens [1] has shown that for any given Banach algebra A, there exist two algebra multiplications on the second dual of A which extend multiplication on A. In the following, we introduce both multiplications which are given in [11]. Let $a, b \in A$, $f \in A^*$ and $F, G \in A^{**}$. Then the first Arens multiplication is defined by

$$\langle f.a, b\rangle = \langle f, ab\rangle,$$

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$$\langle F.f, a\rangle = \langle F, f.a\rangle,$$
$$\langle F.G, f\rangle = \langle F, G.f\rangle.$$

Clearly $F.f \in A^*$ and $F.G \in A^{**}$. We use the notion $(A^{**}, .)$ for A^{**} equipped with the first Arens multiplication. Also we use fa instead of $f.a$ for all $a \in A$ and $f \in A^*$. The second Arens product is defined as follows:

For $a, b \in A$, $f \in A^*$ and $F, G \in A^{**}$, the elements aof , foF of A^* and FoG of A^{**} are defined respectively by the equalities:

$$\langle a \circ f, b\rangle = \langle f, ba\rangle,$$
$$\langle f \circ F, a\rangle = \langle F, aof\rangle,$$
$$\langle F \circ G, f\rangle = \langle G, foF\rangle.$$

An element E of A^{**} is said to be a mixed unit if E is a right unit for the first Arens multiplication and a left unit for the second Arens multiplication. That is, E is a mixed unit if and only if, for each $F \in A^{**}$, $F.E = E \circ F = F$.

We say that a bounded net $(e_\alpha)_{\alpha\in I}$ in A is an approximate identity $(= BAI)$ if, for each $a \in A$, $ae_\alpha \to a$ and $e_\alpha a \to a$. For $a \in A$ and $f \in A^*$, we denote by $f.a$ and $a \circ f$ respectively, the functionals on A^* defined by $\langle f.a, b\rangle = \langle f, ab\rangle = f(ab)$ and $\langle a \circ f, b\rangle = \langle f, ba\rangle = f(ba)$. We say that A^{**} is unital with respect to the first Arens product if there exists an element $E \in A^{**}$ such that $F.E = E.F = F$ for all $F \in A^{**}$, and A^{**} is unital with respect to the second Arens product if there exists an element $E \in A^{**}$ such that $F \circ E = E \circ F = F$ for all $F \in A^{**}$. By [3, p.146], an element E of A^{**} is mixed unit if and only if it is a $weak^*$ cluster point of some BAI $(e_\alpha)_{\alpha\in I}$ in A.

Suppose that $F, G \in A^{**}$ and $F.G$ and $F \circ G$ are the first and second Arens multiplications in A^{**}, respectively. Then the mapping $F \to F.G$, for G fixed in A^{**}, is $w^* - w^*-$ continuous, while the mapping $F \to G.F$ for G fixed in A^{**} is not in general $w^* - w^*-$ continuous on A^{**} unless $G \in A$.

As an example for the algebras $L^1(G)^{**}$ and $M(G)^{**}$ whenever G is an infinite topological group the mapping $F \to G.F$, in general, is not $w^* - w^* -$ *continuous on* $L^1(G)^{**}$ and $M(G)^{**}$, see [10, 11]. The first topological center of A^{**} with respect to the first Arens product is defined as follows

$$Z_1 = \{G \in A^{**} :\ F \longrightarrow G.F \ is\ w^* - w^* - continuous\ on\ A^{**}\}.$$

For fixed G in A^{**}, the mapping $F \to GoF$ is $w^* - w^*-$ continuous on A^{**}, but the mapping $F \to FoG$ is not in general $w^* - w^*-$ continuous on A^{**} unless $G \in A$. The second topological center of A^{**} with respect to second Arens product is defined as follows

$$Z_2 = \{G \in A^{**} :\ F \longrightarrow F \circ G \ is\ w^* - w^* - continuous\ on\ A^{**}\}.$$

It is clear that $A \subseteq Z_1 \bigcap Z_2$ and $Z_1,\ Z_2$ are closed subalgebras of A^{**} endowed with the first and second Arens multiplication, respectively. If, for each $F, G \in A^{**}$, the equality $F.G = F \circ G$ holds, then the algebra A is said to be Arens regular, see [1, 2]. In this case $Z_1 = Z_2 = A^{**}$.

The other extreme situation is that $Z_1 = A$, in this case A is called left strongly Arens irregular, see [9, 10, 11]. We recall that the topological center of A^{**} is defined to be the set of all functionals $F \in A^{**}$ which satisfy $F.G = F \circ G$ for all $G \in A^{**}$, see [11]. In the other words, the topological centers of A^{**} with respect to the first and second Arens products can also be defined as the following sets

$$Z_1 = \{F \in A^{**} :\ F.G = F \circ G\ \forall G \in A^{**}\},$$

$$Z_2 = \{F \in A^{**} :\ G.F = G \circ F\ \forall G \in A^{**}\}.$$

respectively.

For a Banach algebra A the topological center of the algebra $(A^*A)^*$ is defined as follows, see [11].

$$\widetilde{Z} = \{\mu \in (A^*A)^* :\ \lambda \to\ \lambda.\mu \ is\ w^* - w^* - continuous\ on\ (A^*A)^*\}.$$

A functional f in A^* is said to be *wap* (weakly almost periodic) on A if the mapping $a \to f.a$ from A into A^* is weakly compact. Pym in [12] proved that this definition is equivalent to the following condition:

For any two nets $(a_\alpha)_\alpha$ and $(b_\beta)_\beta$ in $\{a \in A : \| a \| \leq 1\}$, we have

$$\lim_\alpha \lim_\beta \langle f, a_\alpha b_\beta \rangle = \lim_\beta \lim_\alpha \langle f, a_\alpha b_\beta \rangle,$$

whenever both iterated limits exist. The collection of all *wap* functionals on A is denoted by $wap(A)$. By [12], $f \in wap(A)$ if and only if $\langle F.G, f\rangle = \langle F \circ G, f\rangle$ for every $F, G \in A^{**}$.

In this paper, for a Banach algebra A, the notation WSC is used for weakly sequentially complete and WCC for weakly completely continuous, that is, A is said to be weakly sequentially complete, if every weakly Cauchy sequence in A has a weak limit, and A is said to be WCC, if for each $a \in A$, the multiplication operator $x \to ax$ is weakly compact, see [6].

2. Main results

In this section, we define new concepts LW^*W-property and RW^*W-property for a Banach algebra A and we study the relationships between these new concepts and Arens regularity of A

Definition 2.1. Let $(f_\alpha)_{\alpha \in I} \subseteq A^*$. If for $a \in A$, the convergence $a.f_\alpha \overset{w^*}{\to} 0$ implies $a.f_\alpha \overset{w}{\to} 0$, then we say that a has $Left{-}Weak^*{-}Weak$ property or LW^*W-property with respect to the first Arens product. The definition of RW^*W-property is similar.

We say that A has the LW^*W-property if for every $a \in A$, a has LW^*W-property.

Remark and Example 2.2. *It is clear that if A is reflexive, A has LW^*W and RW^*W-property, so that if A has a member which does not have the LW^*W-property or RW^*W-property, then A is not reflexive. If A is commutative, then it is clear that the LW^*W-property and RW^*W-property are equivalent. For Banach algebras $l^1(G)$ and $M(G)$ where G is an infinite group, the unit elements of them do not have*

LW^*W-property or RW^*W-property (We know that $l^\infty(G)^* \neq l^1(G)$ and $M(G)^{**} \neq M(G)$).

For a Banach algebra A, if $A^{**} = A^{**}c$ or $A^* = cA^*$ for some $c \in A$ and c has LW^*W-property, then it is easy to see that $A = A^{**}$, and also if $A^{**} = cA^{**}$ or $A^* = A^*c$ for some $c \in A$ and c has RW^*W-property, then $A = A^{**}$.

Now we give an example of a Banach algebra A such that it has LW^*W-property or RW^*W-property.

Let S be a compact semigroup and $A = C(S)$. Then every $f \in C(S)$ such that $f \geq \alpha$ where $\alpha \in (0,\infty)$ has LW^*W-property. Let $(\mu_\alpha)_\alpha \subseteq C(S)^* = M(S)$ and $\mu_\alpha.f \overset{w^*}{\to} 0$. Hence $\langle \mu_\alpha.f, g\rangle \to 0$ for all $g \in C(S)$. Consequently we have

$$\langle \mu_\alpha.f, g\rangle = \langle \mu_\alpha, f.g\rangle = \int_S fg d\mu_\alpha \to 0.$$

If we set $g = \frac{1}{f}$, then $\mu_\alpha \to 0$. Now let $F \in C(S)^{**} = M(S)^*$. Then we have

$$\langle F, \mu_\alpha.f\rangle = \langle F.\mu_\alpha, f\rangle = \int_S f dF\mu_\alpha \to 0.$$

It follows that $\mu_\alpha.f \overset{w}{\to} 0$.

In the following theorems, by using LW^*W-property or RW^*W-property of Banach algebra A, we study the Arens regularity of A.

Theorem 2.3. *For a Banach algebra A the following statements hold.*

(1) If $A^{**} = cA^{**}$ for some $c \in A$ and c has RW^*W-property, then A is Arens regular.

(2) If $A^{**} = A^{**}c$ for some $c \in A$ and c has LW^*W-property, then A is Arens regular.

(3) If $A^* = A^*c$ and c has LW^*W-property, then $A^{**}c \subseteq Z_1 \cap Z_2$ and $cA^* \subseteq wap(A) \subseteq A^*c$.

(4) If $A^* = cA^*$ and c has RW^*W-property, then $cA^{**} \subseteq Z_1 \cap Z_2$ and $A^*c \subseteq wap(A) \subseteq cA^*$.

Proof. (1) Let $G \in A^{**}$. We show that the mapping $F \to G.F$ is $w^* - w^* - continuous$. Let $f \in A^*$ and $(F_\alpha)_\alpha \subseteq A^{**}$ be such that $F_\alpha \overset{w^*}{\to} F$. Then for all $a \in A$ we have

$$\langle F_\alpha.f, a\rangle = \langle F_\alpha, f.a\rangle \to \langle F, f.a\rangle = \langle F.f, a\rangle.$$

Consequently, we have $F_\alpha.f \overset{w^*}{\to} F.f$ for all $f \in A^*$ which implies that $(F_\alpha.f)a \overset{w^*}{\to} (F.f)a$ for all $a \in A$. Since c has RW^*W-property, $(F_\alpha.f)c \overset{w}{\to} (F.f)c$. Thus for all $G \in A^{**}$, we have

$$\langle cG, F_\alpha.f\rangle = \langle G, (F_\alpha.f)c\rangle \to \langle G, (F.f)c\rangle = \langle cG, F.f\rangle.$$

Since $A^{**} = cA^{**}$, we can replace cG by G, and consequently we have $\langle G.F_\alpha, f\rangle \to \langle G.F, f\rangle$ for all $f \in A^*$. Therefore $G \in Z_1$ so that $Z_1 = A^{**}$.

(2)The proof is completely similar to that of (1).

(3) We show that the mapping $F \to Gc.F$ is $w^* - w^*$-continuous whenever $F, G \in A^{**}$. Let $(F_\alpha) \subseteq A^{**}$ be a net such that $F_\alpha \overset{w^*}{\to} F$. Then, we have $(F_\alpha.f) \overset{w^*}{\to} (F.f)$ for every $f \in A^*$, hence $(F_\alpha.f - F.f) \overset{w^*}{\to} 0$. Since $(F_\alpha.f - F.f) \in A^* = A^*c$, there is $(g_\alpha)_\alpha \subseteq A^*$ such that $(F_\alpha.f - F.f) = g_\alpha c$. Then we have $c(g_\alpha c) \overset{w^*}{\to} 0$, it follows that $c(g_\alpha c) \overset{w}{\to} 0$, since c has LW^*W-property. Consequently, for all $G \in A^{**}$, we have $\langle G, c(g_\alpha c)\rangle \to 0$ which implies $\langle Gc, (F_\alpha.f - F.f)\rangle \to 0$. Hence $Gc \in Z_1$ so $A^{**}c \subseteq Z_1$. Similarly $A^{**}c \subseteq Z_2$. Hence we have $A^{**}c \subseteq Z_1 \cap Z_2$. Now let $F, G \in A^{**}$. Since $Gc \in Z_2$, we have $Fo(Gc) = F.(Gc)$, consequently, for all $f \in A^*$, we have the following statements

$$\langle FoG, cf\rangle = \langle Fo(Gc), f\rangle = \langle F.(Gc), f\rangle = \langle F.G, cf\rangle.$$

Hence $cA^* \subseteq wap(A)$.

(4) The proof is similar to that of (3). □

Let G be an infinite group. Since $M(G)$ is not Arens regular, by Theorem 2.3, the unit element of $M(G)$ does not have LW^*W-property or RW^*W-property. Similar conclusions hold for the Banach algebra $l^1(G)$.

Theorem 2.4. *Let A and B be Banach algebras and let h be a bounded homomorphism from A onto B. If A has LW^*W-property (resp. RW^*W-property), then B has LW^*W-property (resp. RW^*W-property).*

Proof. Since h is continuous, the second adjoint of h, h^{**}, is $weak^* - weak$ continuous from A^{**} into B^{**}. Hence we conclude that $h^{**}(F.G) = h^{**}(F).h^{**}(G)$ for all $F, G \in A^{**}$.

Let $(g_\alpha)_\alpha \subseteq B^*$ be such that $b.g_\alpha \stackrel{w^*}{\to} 0$ where $b \in B$. We define $h^*(g_\alpha) = f_\alpha \in A^*$ and get $h(a) = b$ for some $a \in A$. Then for every $x \in A$, we have

$$\langle a.f_\alpha, x\rangle = \langle f_\alpha, x.a\rangle = \langle h^*(g_\alpha), x.a\rangle = \langle g_\alpha, h(x.a)\rangle = \langle g_\alpha, h(x).h(a)\rangle$$

$$\langle g_\alpha, h(x).b\rangle = \langle b.g_\alpha, h(x)\rangle \to 0.$$

Thus $a.f_\alpha \stackrel{w^*}{\to} 0$. Since a has LW^*W-property, $a.f_\alpha \stackrel{w}{\to} 0$. Now, let $G \in B^{**}$. Since h is surjective, by [12, Theorem 3.1.22] there is $F \in A^{**}$ such that $h^{**}(F) = G$. Then we have the following assertions

$$\langle G, b.g_\alpha\rangle = \langle G.b, g_\alpha\rangle = \langle h^{**}(F)h(a), g_\alpha\rangle = \langle h^{**}(F.a), g_\alpha\rangle$$

$$= \langle (F.a), h^*(g_\alpha)\rangle = \langle (F.a), f_\alpha\rangle = \langle F, a.f_\alpha\rangle \to 0$$

Thus we conclude that $b.g_\alpha \stackrel{w}{\to} 0$, therefore B has LW^*W-property. □

Theorem 2.5. *For a Banach algebra A the following assertions hold.*

(1)If $A^*A = A^*$ and A has RW^*W-property, then A is Arens regular.

(2) If $AA^* = A^*$ and A has LW^*W-property, then A is Arens regular.

Proof. (1) Let $(F_\alpha) \subseteq A^{**}$ be a net such that $F_\alpha \stackrel{w^*}{\to} F$. Then, we have $(F_\alpha.f).a \stackrel{w^*}{\to} (F.f).a$ for every $a \in A$ and $f \in A^*$, and so $F_\alpha.(f.a) \stackrel{w^*}{\to} F.(f.a)$. Since A has RW^*W-property, we conclude that $F_\alpha.(f.a) \stackrel{w}{\to} F.(f.a)$. Therefore for all $G \in A^{**}$, $f \in A^*$ and $a \in A$ we have

$$\langle G.F_\alpha, f.a\rangle = \langle G, F_\alpha(f.a)\rangle = \langle G, (F_\alpha.f).a\rangle \to \langle G, (F.f).a\rangle$$

$$= \langle G.F, f.a\rangle.$$

Since $A^*A = A^*$, the result follows.

(2) The proof is similar to that of (1). □

Let the algebra $C_0 = (C_0, .)$ to be the collection of all sequences of scalars that converge to 0, with the same vector space operations and norm as in ℓ^∞. Then, C_0 is a Banach algebra which satisfies all conditions of Theorem 2.5.

By Theorem 2.4 and Theorem 2.5(2), if A and B are Banach algebras and h is a bounded homomorphism from A onto B whenever B factors on the right as Banach $A-bimodule$, then if A has LW^*W-property, B is Arens regular. Also from Theorem 2.5 and [11, Proposition 2.6] we conclude that when A is WSC and $AA^* = A^*$ (or A is unital) and A has RW^*W-property, then A is Arens regular.

Lemma 2.6. *For a Banach algebra A the following statements hold.*

(1) For all $f \in A^*$ and $F \in A^{**}$ there is a net $(x_\alpha)_\alpha \subseteq A$ such that $f.x_\alpha \overset{w^*}{\to} foF$.

(2) Let $F \in A^{**}$ and $(x_\alpha)_\alpha \subseteq A$ be such that $x_\alpha \overset{w^*}{\to} F$. Then A is Arens regular if and only if $f.x_\alpha \overset{w}{\to} foF$ for all $f \in A^*$.

Theorem 2.7. *For a Banach algebra A the following assertions hold.*

(1) Suppose that

$$\lim_\alpha \lim_\beta \langle f_\beta, a_\alpha \rangle = \lim_\beta \lim_\alpha \langle f_\beta, a_\alpha \rangle$$

for every $(a_\alpha)_\alpha \subseteq A$ and $(f_\beta)_\beta \subseteq A^*$. Then, if $A^*A = A^*$ or $AA^* = A^*$ it follows that A is Arens regular. Also if $A^{**} = cA^{**}$ or $A^{**} = A^{**}c$ for some $c \in A$, then A is reflexive.

(2) Let A be Arens regular and let $(a_\alpha)_\alpha \subseteq A$ and $(f_\beta)_\beta \subseteq A^*$ be such that they are convergence to some points in the weak* and weak topology in A^{**} and A^*, respectively. Then for all $F \in A^{**}$, we have

$$\lim_\alpha \lim_\beta \langle F, f_\beta . a_\alpha \rangle = \lim_\beta \lim_\alpha \langle F, f_\beta . a_\alpha \rangle.$$

(3) If for some $a \in A$, we have

$$\lim_\alpha \lim_\beta \langle f_\beta a, a_\alpha \rangle = \lim_\beta \lim_\alpha \langle f_\beta a, a_\alpha \rangle,$$

then a has RW^*W-property. Also if

$$\lim_\alpha \lim_\beta \langle af_\beta, a_\alpha\rangle = \lim_\beta \lim_\alpha \langle af_\beta, a_\alpha\rangle,$$

then a has LW^*W-property.

Proof. (1) If we show that A has both LW^*W-property and RW^*W-property then by Theorem 2.3 and Theorem 2.5, the proof is complete. Suppose that $(f_\beta)_\beta \subseteq A^*$ and $a \in A$ are such that $a.f_\beta \stackrel{w^*}{\to} 0$. Let $F \in A^{**}$. Since $w^* - closureA = A^{**}$, there is $(a_\alpha)_\alpha \subseteq A$ such that $a_\alpha \stackrel{w^*}{\to} F$. Consequently, we have the following equalities

$$\lim_\beta \langle F, a.f_\beta\rangle = \lim_\beta \langle F.a, f_\beta\rangle = \lim_\beta \lim_\alpha \langle a_\alpha.a, f_\beta\rangle = \lim_\beta \lim_\alpha \langle f_\beta, a_\alpha.a\rangle$$

$$= \lim_\alpha \lim_\beta \langle f_\beta, a_\alpha.a\rangle = \lim_\alpha \lim_\beta \langle a.f_\beta, a_\alpha\rangle = 0.$$

Therefore, we obtain $a.f_\beta \stackrel{w}{\to} 0$. Similarly A has RW^*W-property.

(2) Suppose that $(a_\alpha)_\alpha \subseteq A$ is such that $a_\alpha \stackrel{w^*}{\to} G$ for some $G \in A^{**}$. Let $(f_\beta)_\beta \subseteq A^*$ be such that $f_\beta \stackrel{w}{\to} f$ where $f \in A^*$. Then by Lemma 2.6, for fixed β, we have $f_\beta.a_\alpha \stackrel{w}{\to} f_\beta oG$. So, for all $F \in A^{**}$ we have the following relations

$$\lim_\beta \lim_\alpha \langle F, f_\beta.a_\alpha\rangle = \lim_\beta \langle F, f_\beta.oG\rangle = \lim_\beta \langle GoF, f_\beta\rangle = \langle GoF, f\rangle$$

$$= \langle G.F, f\rangle = \langle G, F.f\rangle = \lim_\alpha \langle F.f, a_\alpha\rangle = \lim_\alpha \langle F, f.a_\alpha\rangle$$

$$= \lim_\alpha \langle a_\alpha.F, f\rangle = \lim_\alpha \lim_\beta \langle a_\alpha.F, f_\beta\rangle = \lim_\alpha \lim_\beta \langle F, f_\beta.a_\alpha\rangle.$$

(3) Suppose that $(f_\beta)_\beta \subseteq A^*$ and $f_\beta a \stackrel{w^*}{\to} 0$. Let $F \in A^{**}$ and $(a_\alpha)_\alpha \subseteq A$ such that $a_\alpha \stackrel{w^*}{\to} F$. Then, we have

$$\lim_\beta \langle F, f_\beta a\rangle = \lim_\beta \langle aF, f_\beta\rangle = \lim_\beta \lim_\alpha \langle f_\beta, aa_\alpha\rangle$$

$$= \lim_\alpha \lim_\beta \langle f_\beta, aa_\alpha\rangle = \lim_\alpha \lim_\beta \langle f_\beta a, a_\alpha\rangle = 0$$

Consequently, we have $f_\beta a \stackrel{w}{\to} 0$. □

Definition 2.8. Let A be a Banach algebra. We say that A^* strongly left (resp., right) factors, if for all $(f_\alpha)_\alpha \subseteq A^*$, there are $(a_\alpha)_\alpha \subseteq A$ and

$f \in A^*$ such that $f_\alpha = f.a_\alpha$ (resp., $f_\alpha = a_\alpha.f$) where $(a_\alpha)_\alpha \subseteq A$ has limit in the *weak** topology in A^{**}. If A^* strongly left and right factors, then we say that A^* strongly factors.

For a Banach algebra A with a BAI, it is clear that if A^* strongly left (resp., right) factors, then A^* factors on the left (resp., right).

Theorem 2.9. *Let $AA^* \subseteq wapA$. If A^* strongly factors on the left (resp., right), then A has LW^*W-property (resp., RW^*W-property).*

Proof. Suppose that $a \in A$ and $(f_\alpha)_\alpha \subseteq A^*$ are such that $a.f_\alpha \xrightarrow{w^*} 0$. Since A^* strongly factors on the left, there are $(a_\alpha)_\alpha \subseteq A$ and $f \in A^*$ such that $f_\alpha = f.a_\alpha$ where $(a_\alpha)_\alpha \subseteq A$ has limit in the *weak** topology on A^{**}. Let $F \in A^{**}$ and $(a_\beta)_\beta \subseteq A$ be such that $b_\beta \xrightarrow{w^*} F$. Then

$$\lim_\alpha \langle F, a.f_\alpha\rangle = \lim_\alpha \lim_\beta \langle af_\alpha, b_\beta\rangle = \lim_\alpha \lim_\beta \langle af.a_\alpha, b_\beta\rangle$$
$$= \lim_\alpha \lim_\beta \langle af, a_\alpha b_\beta\rangle = \lim_\beta \lim_\alpha \langle af, a_\alpha b_\beta\rangle = \lim_\beta \lim_\alpha \langle af.a_\alpha, b_\beta\rangle$$
$$= \lim_\beta \lim_\alpha \langle af_\alpha, b_\beta\rangle = 0.$$

It follows that $a \in A$ has LW^*W-property and so A has LW^*W-property. $\square$

Problems

(1) Suppose that A has LW^*W-property and A is WCC and WSC. Is A reflexive?

(2) For a non abelian Banach algebra A, when LW^*W-property and RW^*W-property concide?

(3) If A^* factors on the right and A has RW^*W-property and A has LW^*W-property, is A Arens regular?

(4) If A factors on the left and it is Arens regular, does A have LW^*W-property?

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