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ACTION OF G AND M ON $Q(\sqrt{m})$ AND ALGORITHMIC IMPLEMENTATION FOR GROUP ACTIONS

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Abstract. This paper calls for a relationship among the actions of groups G and M on $Q(\sqrt{m})$. It characterized several significant elements of G in terms of generators of M and vice versa. In this way we cultivate a correlation between the rudiments of these two modular groups. This will help us in discovering various G and M-Subsets of $Q(\sqrt{m})$. We have also generated an algorithm by using Visual Basic for calculating the congruence classes of different Moduli and manipulating the group actions.

Key Words: Mobius Groups, Quadratic Fields, Generators, Group Action.

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1. Introduction

The study of groups through their actions on different sets and algebraic structures has grown into a very worthwhile technique. Mobius groups have always been of keen attention in finding group actions on quadratic fields. G Higman familiarized coset diagrams for presenting the action of Modular groups on number fields. Qaiser Mushtaq laid

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the foundation and established it further. The theory of congruence was introduced by Card Friedrich Gauss (1777-1855) one of the greatest Mathematicians of all times. Although, Pierre De Fermat (1601-1665) had earlier studied number theory. The *congruence* is nothing more than a statement of divisibility. However, it often helps to discover proofs and we realize that congruence advocates new ideas to resolve the problems that will lead us to further motivating philosophies. We have also used congruence classes to explore the action of Mobius groups on the real quadratic fields.
This paper is concerned with the investigation of real quadratic fields $Q(\sqrt{m})$ under the actions of two important Mobius groups G and M given below:

$$G = \langle \acute{x}, \acute{y}; \acute{x}^2 = \acute{y}^3 = 1 \rangle$$

where $\acute{x}(\alpha) = \frac{-1}{\alpha}$ and $\acute{y}(\alpha) = \frac{\alpha-1}{\alpha}$ are linear fractional transformations.

$$M = \langle x, y; x^2 = y^6 = 1 \rangle$$

where $x(\alpha) = \frac{-1}{3\alpha}$ and $y(\alpha) = \frac{-1}{3(1+\alpha)}$ are linear fractional transformations.

The actions of groups G and M on $Q(\sqrt{m})$ have yielded certain valuable and elegant results. G Higman and Mushtaq proved that $Q^*(\sqrt{n})$ is invariant under the action of G. Mushtaq and Aslam [12] studied an action of infinite Mobius group M on the projective real line over real quadratic field. M. Ashiq in [6] studied an action of two generator group on a real quadratic field. M. Aslam Malik and M. Riaz [11] explore proper G-Subsets of $Q^*(\sqrt{n})$.In [3]the classification of the real quadratic irrational numbers $(a+\sqrt{n})/c$ of $Q^*(\sqrt{n})$ with respect to $modulo 3^r$ is provided. M. Aslam Malik et al [9, 10] have studied various properties of groups G and M. Aslam et al [9] have proved that the subsets $Q'''(\sqrt{n})$

and $Q^{***}(\sqrt{n})$ of $Q(\sqrt{m})$ are M-Subsets of $Q(\sqrt{m})\bigcup\{\infty\}$.

Therefore it turn out to be motivating to discover the relationship between the action of different Mobius groups. So we started with the groups G and M acting on quadratic fields. Thus we established a correlation among the actions of both groups.

Secondly to produce the classes of different modes and to compute group actions by hand is very hectic, tedious and time consuming activity. This cumbersome activity can be resolved if computer is used for calculations. Therefore we make a computer program by using Visual Basic to resolve this problem. Main interaction manual for *Class Generation* and *Group Action* Calculations is given in Figure1. Now we have some important definitions which are required for our subsequent sections.

Definition 1.1. An algebraic integer of the form $a + b(\sqrt{m})$ where m is a square free, form a *Quadratic Field* and is denoted by $Q(\sqrt{m})$. If $m > 0$ then $Q(\sqrt{m})$ is called Real Quadratic Field. If $m < 0$ then it is called Imaginary Quadratic Field.

Definition 1.2. Every Real Quadratic Irrational number can be represented uniquely as $\frac{a+\sqrt{n}}{c}$ with a non-square positive integer n, where $a, \frac{a^2-n}{c}$ and c are relatively prime integers.

Definition 1.3. If $n = k^2 m$ and $k > 0$ be an integer then we have following definitions:

$$Q^*(\sqrt{n}) := \{\frac{a+\sqrt{n}}{c} : a, b := \frac{a^2-n}{c}, c \in Z\text{and}(a, \frac{a^2-n}{c}, c) = 1\};$$

$$Q'''(\sqrt{n}) = \{\alpha/t; \alpha \in Q^*(\sqrt{n}); t = 1, 3\};$$

and

$$Q^{***}(\sqrt{n}) = \{\frac{(a+\sqrt{n})}{c} \in Q^*(\sqrt{n}) : 3|c\}.$$

2. RELATIONS BETWEEN THE ELEMENTS OF GROUP G AND GROUP M

The following table gives images of elements of $Q^*(\sqrt{n})$ under the actions of group $G = \langle \acute{x}, \acute{y}; \acute{x}^2 = \acute{y}^3 = 1\rangle$

	α	a	b	c
$\acute{x}(\alpha)$	$(-1)/\alpha$	$-a$	c	b
$\acute{y}(\alpha)$	$(\alpha-1)/\alpha$	$-a+b$	$-2a+b+c$	a
$\acute{y}^2(\alpha)$	$1/(1-\alpha)$	$-a+c$	c	$-2a+b+c$
$\acute{x}\acute{y}(\alpha)$	$\alpha/(1-\alpha)$	$a-b$	b	$-2a+b+c$
$\acute{x}\acute{y}^2(\alpha)$	$\alpha-1$	$a-c$	$-2a+b+c$	c
$\acute{y}\acute{x}(\alpha)$	$\alpha+1$	$a+c$	$2a+b+c$	c
$\acute{y}^2\acute{x}(\alpha)$	$\alpha/(1+\alpha)$	$a+b$	b	$2a+b+c$

The table below provides images of elements of $Q^*(\sqrt{n})$ under the actions of group $M = \langle x, y; x^2 = y^6 = 1\rangle$.

	α	a	b	c
$x(\alpha)$	$(-1)/3\alpha$	$-a$	$c/3$	$3b$
$y(\alpha)$	$(-1)/3(\alpha+1)$	$-a$	$c/3$	$3(2a+b+c)$
$y^2(\alpha)$	$-(\alpha+1)/(3\alpha+2)$	$-5a-3b-2c$	$2a+b+c$	$12a+9b+4c$
$y^3(\alpha)$	$-(3\alpha+2)/(3(2\alpha+1)$	$-7a-6b-2c$	$(12a+9b+4c)/3$	$3(4a+4b+c)$
$y^4(\alpha)$	$-(2\alpha+1)/(3\alpha+1)$	$-5a-6b-c$	$4a+4b+c$	$(6a+9b+c)$
$y^5(\alpha)$	$-(3\alpha+1)/3\alpha$	$-a-3b$	$(6a+9b+c))/3$	$3b$
$xy(\alpha)$	$\alpha+1$	$a+c$	$2a+b+c$	c
$xy^2(\alpha)$	$(3\alpha+2)/(3(\alpha+1))$	$5a+3b+2c$	$(12a+9b+4c)/3$	$3(2a+b+c)$
$xy^3(\alpha)$	$(2\alpha+1)/(3\alpha+2)$	$7a+6b+2c$	$4a+4b+c$	$12a+9b+4c$
$xy^4(\alpha)$	$(3\alpha+1)/(3(2\alpha+1))$	$((6a+9b+c))/3$	$(6a+9b+c)/3$	$3(4a+4b+c)$
$xy^5(\alpha)$	$\alpha/(3\alpha+1)$	$a+3b$	b	$6a+9b+c$
$yx(\alpha)$	$\alpha/(1-3\alpha)$	$a-3b$	b	$-6a+9b+c$
$y^2x(\alpha)$	$(1-3\alpha)/3(-1+2\alpha)$	$5a-6b-c$	$(-6a+9b+c)/3$	$3(-4a+4b+c)$
$y^3x(\alpha)$	$(1-2\alpha)/((-2+3\alpha))$	$7a+6b+2c$	$-4a+4b+c$	$-12a+9b+4c$
$y^4x(\alpha)$	$(2-3\alpha)/3(-1+\alpha)$	$5a-3b-2c$	$(-12a+9b+4c)/3$	$3(-2a+b+c)$
$y^5x(\alpha)$	$\alpha-1$	$5a-3b-2c$	$-2a+b+c$	c

3. SOME ELEMENTS OF GROUP M IN TERMS OF GENERATORS OF G

At first we represent the elements of group M in terms of words of elements in G. Every element of M is a word in the generators $x, y \in M$. Let $x, y \in M$ and $\acute{x}, \acute{y} \in G$ be the generator of G. Clearly we have:

(1) $x = \frac{1}{3(\acute{x})}$, and
(2) $y = \frac{1}{3(\acute{x}\acute{y})\acute{x}}$

Thus the generators of M are represented in the form of generators of G. In addition to that we have the following identities.

Theorem 3.1. *For $\acute{x}, \acute{y} \in G$ and $x, y \in M$ we have:*
(1) $y^2 = (\acute{x}\acute{y}^2)(\acute{x}\acute{y}^2)(\acute{x})$.
(2) $y^3 = 1/3(\acute{x}\acute{y}^2)(\acute{x}\acute{y})(\acute{x}\acute{y}^2)(\acute{x})$.
(3) $y^4 = [(\acute{x}\acute{y})^2(\acute{x}\acute{y}^2)(\acute{x})]^2$.
(4) $y^5 = 1/3(\acute{x}\acute{y}^2)(\acute{x}\acute{y})(\acute{x}\acute{y})(\acute{y}^2\acute{x})$.

Proof. 1) Let $\alpha \in Q^*(\sqrt{n}) := \{\frac{a+\sqrt{n}}{c} : a, b := \frac{a^2-n}{c}, c \in Z$ and $(a, \frac{a^2-n}{c}, c) = 1\}$ and $G = \langle \acute{x}, \acute{y}; \acute{x}^2 = \acute{y}^3 = 1\rangle$. Then $\acute{x}(\alpha) = -1/\alpha$ and $(\acute{x}\acute{y}^2)(\alpha) = \alpha - 1$ by Table 1. We have $(\acute{x}\acute{y}^2)(\acute{x})(\alpha) = -1/\alpha - 1$. Hence $\Rightarrow (\acute{x}\acute{y}^2)(\acute{x})(\alpha) = -(\alpha+1)/\alpha$. Since $(\acute{x}\acute{y})(\alpha) = \alpha/(\alpha-1)$, we conclude that $(\acute{x}\acute{y})(\acute{x})\acute{y}^2)(\acute{x})(\alpha) = -(\alpha+1)/(2\alpha+1)$. That is $(\acute{x}\acute{y})^2(\acute{x}\acute{y}^2)(\acute{x})(\alpha) = -(\alpha+1)/3(2\alpha+1)$. Therefore $(\acute{x}\acute{y})^2(\acute{x})\acute{y}^2)(\acute{x})(\alpha) = y^2(\alpha)$. Thus we have $y^2 = (\acute{x}\acute{y})^2(\acute{x}\acute{y}^2)(\acute{x})$.

2) Since $(\acute{x}\acute{y}^2)(\alpha) = \alpha - 1$, and $\acute{x}(\alpha) = -1/\alpha$ we have:

$$(\acute{x}\acute{y}^2)(\acute{x})(\alpha) = -1/\alpha - 1$$
$$\Rightarrow (\acute{x}\acute{y}^2)(\acute{x})((\acute{x}\acute{y}^2)(\acute{x})(\alpha)) = -(\alpha+1)/\alpha$$
$$\Rightarrow (\acute{x}\acute{y})(\acute{x}\acute{y}^2)(\acute{x})(\alpha) = -(\alpha+1)/(2\alpha+1)\text{by (1)}$$
$$\Rightarrow (\acute{x}\acute{y})^2(\acute{x}\acute{y})(\acute{x}\acute{y}^2)(\acute{x})(\alpha) = -(\alpha+1)/(2\alpha+1) - 1$$
$$\Rightarrow (\acute{x}\acute{y})^2(\acute{x}\acute{y})(\acute{x}\acute{y}^2)(\acute{x})(\alpha) = -(3\alpha+2)/(2\alpha+1)$$

Thus $1/3(\acute{x}\acute{y})^2(\acute{x}\acute{y})(\acute{x}\acute{y}^2)(\acute{x})(\alpha) = -(3\alpha+2)/3(2\alpha+1)$. Now

$$y^3(\alpha) = 1/3(\acute{x}\acute{y})^2(\acute{x}\acute{y})(\acute{x}\acute{y}^2)(\acute{x})(\alpha)$$

$$\Rightarrow y^3 = 1/3(\acute{x}\acute{y})^2(\acute{x}\acute{y})(\acute{x}\acute{y}^2)(\acute{x}).$$

The proofs of (3) and (4) are similar. □

Following are immediate consequences of Theorem 3.1.

Corollary 3.2. *For the generators x, y of M and $\acute{x}, \acute{y}$ of G we have:*
(1) $y = \acute{y}\acute{x}$
(2) $xy^2 = 1/3(\acute{y}\acute{x})^2(\acute{y}^2\acute{x})$.
(3) $xy^3 = (\acute{y}^2\acute{x})(\acute{y}\acute{x})(\acute{y}^2\acute{x})$.
(4) $xy^4 = 1/3(\acute{y}\acute{x})(\acute{y}^2\acute{x})^2$.
(5) $xy^5 = (\acute{y}^2\acute{x})^3$.

Corollary 3.3. *For $x, y \in M$ and $\acute{x}, \acute{y} \in G$ we have:*
(1) $yx = 1/3(\acute{x}\acute{y})$.
(2) $y^2x = (\acute{x}\acute{y})^2(\acute{x}\acute{y}^2)$.
(3) $y^3x = 1/3(\acute{x}\acute{y}^2)(\acute{x}\acute{y})(\acute{x}\acute{y}^2)$.
(4) $y^4x = (\acute{x}\acute{y})^2(\acute{x}\acute{y}^2)(\acute{x})(\acute{x}\acute{y}^2)$.
(5) $y^5x = 1/3(\acute{y}^2\acute{x})^3$.

Therefore, using Theorem 3.1, Corollary 3.2 and Corollary 3.3 the action of the group M in terms of the elements of G can be acquired.

4. SOME ELEMENTS OF GROUP G IN TERMS OF GENERATORS OF M

The elements of the group G can also be written as the words of M. Since $\acute{x}, \acute{y} \in G$ are the generators of the group G so at first we will represent these in terms of generators x, y of M. Obviously $\acute{x} = 3x$ and $\acute{y} = (3x)(3y)(3x)$ and follows from Tables 1, 2. Thus by the above relation we have $\acute{y}^2 = [(3x)(3y)(3x)]^2$. Using the above identities, we can have following Theorems:

Theorem 4.1. *Let $\acute{x}, \acute{y} \in G$ and $x, y \in M$, then*
(1) $\acute{x}\acute{y} = 3(yx)(1/3)$.
(2) $\acute{x}\acute{y}^2 = y^5x$.

(3)$\acute{y}\acute{x} = xy$.
(4) $\acute{y}^2\acute{x} = 3(y^2x)(1/3)$.

Proof. 1) Let $\alpha \in Q^*(\sqrt{n})$. Since $(yx)(\alpha) = \frac{\alpha}{1-3(\alpha)}$, by Table 2, we have $(yx)(\alpha/3) = (\alpha/3/(1-\alpha))$. So $(yx)(\alpha/3) = (\frac{1}{3})(\frac{\alpha}{1-\alpha})$. Therefore:

$$3(yx)(\alpha/3) = (\alpha/(1-\alpha))$$
$$\Rightarrow 3(yx)(\alpha/3) = \acute{x}\acute{y}(\alpha)$$
$$\Rightarrow 3(yx)(1/3)(\alpha) = \acute{x}\acute{y}(\alpha)$$
$$\Rightarrow \acute{x}\acute{y} = 3(yx)\frac{1}{3}.$$

2) $\acute{x}\acute{y}^2(\alpha) = y^5x(\alpha)$ by Tables 1,2.

The proofs of (3) and (4) are similar. □

Thus by knowing the rudiments of one of the groups, we can find the actions of other group. By this way, finding actions have become very tranquil. So one can implement this relationship for finding more G-subsets and it will help in discovering proper M-Subsets of $Q(\sqrt{m})$.

5. COMPUTER PROGRAM FOR CLASS GENERATION AND GROUP ACTION CALCULATIONS

Generation of congruence classes of different moduli and to compute Group Actions by hand is very hectic, tedious and cumbersome activity. This cumbersome activity can be resolved and lessened if computer is used for calculations. Programming language Visual Basic has a fundamental role to surmount this task, after initial algorithm study and according to the problem requirements Visual Basic is selected to code the problem. The selection of Visual Basic is obvious due to its ease of Graphical User Interface capabilities and easy coding. The main interaction panel of the code is show below in figure 1.

This code has two major frames. One for *Class Generation* and other is for *Group Action*. In the class generation frame, Congruence classes of different modes can be generated; classes of mode 3, 9, 27 and 81 can

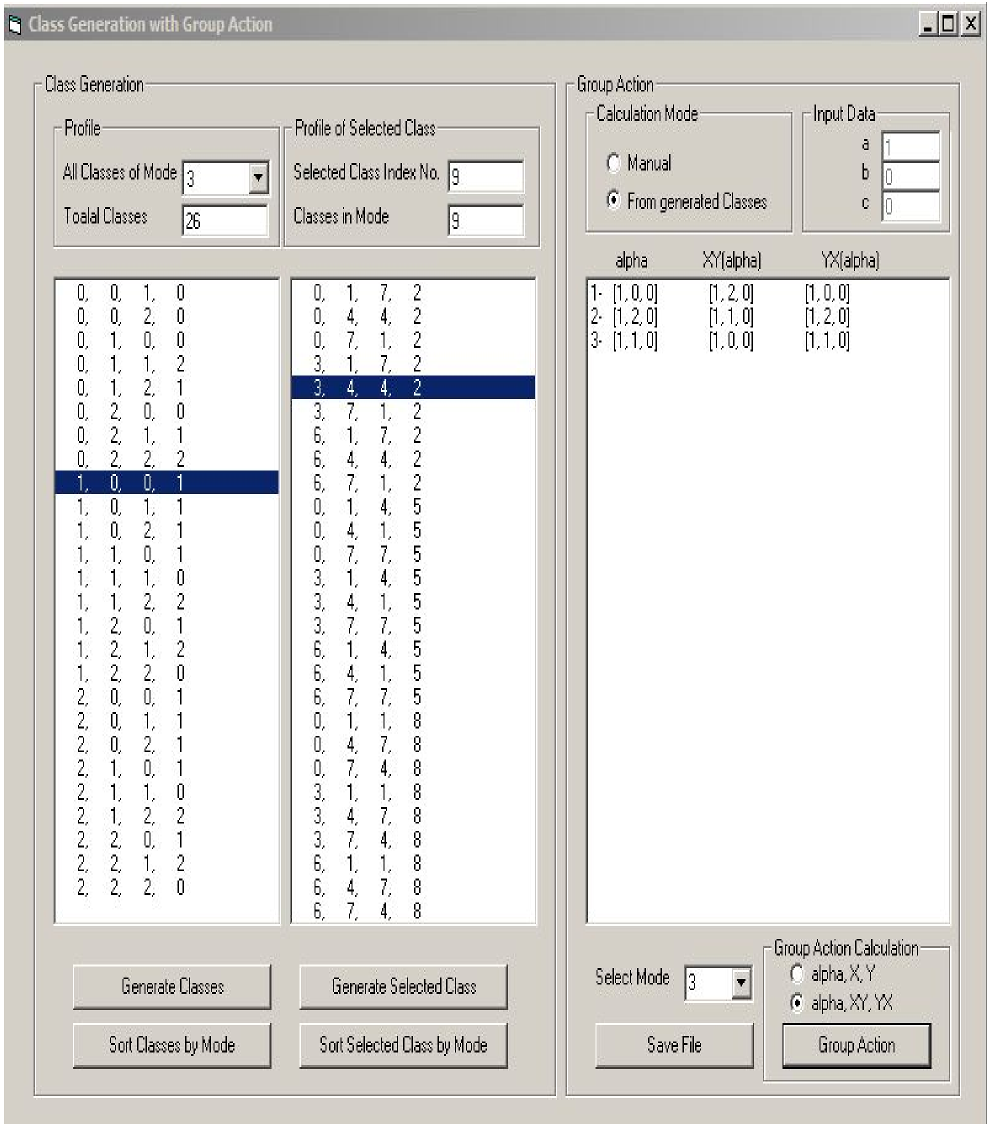

FIGURE 1. The Manual for Class Generation and Group Action Calculations

be generated using this code. In the second Group Action frame, action of classes is calculated. Two methods are available for calculating the Group Action, first one is *manual* and the second one is from *generated classes*. The result of Group Action is available as alpha, X, Y and alpha, XY, YX.

Example 5.1. To generate the classes of Mode 3, select 3 in all classes of Mode *pull down list* and then press the *generate classes push button*, all the classes of mode 3 will be displayed in the list 1. As show in figure above all the classes of mode 3 are displayed. To display classes of mode 9 for each generated classes of mode 3. Put cursor on any parent generated class and press the *generate selected classes push button*, then all next level mode classes will be generated and displayed in List 2. In the figure [1, 0, 0] class of mode 3 is selected and to generated the classes in mode 9 press generate selected class *push button*, all the classes of mode 9 will be displayed in List 2. Both the parent classes and child classes for example classes of mode 3(the parent class) and mode 9 (the child class) can be sorted by pressing the *sort classes by mode* and *sort selected classes by mode push buttons* respectively. Similarly all the classes of mode 9, 27 and 81 are generated as parent classes and their child classes can be generated and displayed in List 1 and List 2 respectively.

Example 5.2. If we want to calculate the Group Action of class [1, 0, 0] manually, then select the *manual radio button* and put the values of class [1, 0, 0] in the text boxes a, b and c. [1, 0, 0] is a class of mode 3, we can have the action of this mode 3 in mode 3, 9, 27 and 81, these modes can be selected from the *pull down list* button select mode. The Action of Group can be calculated as alpha, X, Y or as alpha, XY, YX, which can be selected from the *Group Action calculation radio button.* In the Group Action calculated shown above in the figure alpha, X, Y radio button is selected. To calculate the Group Action presses the *Group Action push button.* The action of class [1, 0, 0] over mode 3 is shown in the list 3. We can also store this group action result in a file for further analysis in the file system.

Conclusion

We conclude the paper with the remarks that after having the relationship among elements of two Mobius groups G and M one can find their actions in a tranquil way and more G-subsets and M-subsets can be explored with the help of this relationship. Secondly computing group actions and finding congruence classes of different moduli has become relaxed by using the algorithm presented.

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