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Research Paper

A NEW APPROACH TO SOLVING UNCERTAIN LINEAR SYSTEMS

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Abstract. Uncertain linear systems are defined and these linear systems using Log-normal and Zigzag variables based on their inverse distributions are investigated. A method for solving Log-normal and Zigzag linear uncertain devices has been designed and conditions for the existence of an uncertain linear has been provided. To show the effectiveness of the proposed method, two examples are given. Finally, a diet is presented to demonstrate the scientific importance of uncertain linear systems.

Key Words: Linear system, Uncertainty theory, Log-normal variable of uncertainty, Zigzag variable of uncertainty.

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1. Introduction

Linear systems have important roles in various areas such as mathematics, engineering, finance, physics and social sciences. Usually, uncertainty parameters in a linear system are described by fuzzy sets, if they are evaluated with experts' belief degrees.However, fuzzy sets theory as a mathematical system, has not evolved because of its inconsistency.

That is, belief degree cannot be considered as a probability subject or fuzzy concept. In other words, some contradictory results may occur

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[1]. To rationally deal with belief degrees, the uncertainty theory was introduced by Liu [2] and refined by Liu [3]. Currently, there have been many applications about uncertainty theory, such as uncertain statistics, uncertain programming, uncertain process, uncertain logic. Also, uncertain differential equation introduced by Liu [4] is a useful concept based on uncertain theory. It is employed to uncertain finance models by Liu [7] and uncertain optimal control by Zhu [6]. Linear systems is a useful concept in engineering sciences, because it models many engineering problems or it is used to approximate nonlinear systems. It is necessary to extend some mathematical methods and numerical formula for a linear system whose parameters are uncertain variables, to solving such as a linear system. In this paper we will create a model of uncertain linear system and propose a method to solving an uncertain linear systems.

In section 2, some Concepts of uncertainty theory will be reviewed. In section 3, a model of the uncertain linear system will be introduced and a concept of the solution in the distribution for model is defined. In section 4, the solutions of uncertain linear systems respectively, with Zigzag uncertain variables and Log-normal will be discussed. In Section 5, a healthy diet problem will be solved as an application of the results obtained in this paper.

2. Basic concepts uncertainty theory

In this section, we discuss and explore the concepts of uncertainty measure and uncertainty space and uncertainty processes that are needed in this article.

Definition 2.1. The most important concept is uncertainty measure, which is a set function that applies to the axioms of uncertainty theory. This measure is used to belief degree in the occurrence of uncertain event.

Let $(\Gamma, \mathcal{L})$ be a measurable space. Each element Λ in $\mathcal{L}$ is called a measurable set. We rename measurable set as event in uncertainty theory and define on uncertain measure $\mathcal{M}$ on the σ-algebra. That is, each event Λ is assigned by a number $\mathcal{M}(\Lambda)$ number to indicate the degree of belief that we believe this event will happen.

To deal rationally with belief degrees, Liu [2] proposed the following three axioms:

Axiom 1. (Normality Axiom) $\mathcal{M}\{\Gamma\} = 1$ for the universal set Γ.
Axiom 2. (Duality Axiom) $\mathcal{M}\{\Lambda\} + \mathcal{M}\{\Lambda^c\} = 1$ for any event Λ.
Axiom 3. (Subadditivity Axiom) For every countable sequence of events $\Lambda_1, \Lambda_2, ...$

$$\mathcal{M}\left\{\bigcup_{i=1}^{\infty} \Lambda_i\right\} \leq \sum_{i=1}^{\infty} \mathcal{M}\{\Lambda_i\}$$

Definition 2.2. Let Γ be a nonempty set, let $\mathcal{L}$ be a σ-algebra over Γ, and let $\mathcal{M}$ be an uncertain measure. Then the triplet $(\Gamma, \mathcal{L}, \mathcal{M})$ is called an uncertainty space [2].

Definition 2.3. An uncertain variable ξ is called Zigzag, if it has the folowing uncertainty distribution [3]

$$\Phi(x) = \begin{cases} 0, & x < a \\ \frac{x-a}{2(b-a)}, & a \leq x < b \\ \frac{x+c-2b}{2(c-b)}, & b \leq x < c \\ 1, & x \geq c \end{cases}$$

A Zigzag uncertain variable denoted by $\mathcal{Z}(a, b, c)$, where a, b, c are real number with $a < b < c$.

Definition 2.4. An uncertain variable ξ is called Log-normal if $\ln \xi$ is a normal uncertain variable $\mathcal{N}(e, \sigma)$. In other words, a Log-normal uncertain variable has uncertainty distribution

$$\Phi(x) = \left(1 + \exp\left(\frac{\pi(e - \ln x)}{\sqrt{3}\sigma}\right)\right)^{-1}, \qquad x \geq 0$$

denoted by $\mathcal{LOGN}(e, \sigma)$, where e and σ with $\sigma > 0$.

Example 2.5. The inverse uncertainty distribution of Log-normal uncertain variable $\mathcal{LOGN}(e, \sigma)$ is

$$\Phi^{-1}(\alpha) = \exp\left(e + \frac{\sigma\sqrt{3}}{\pi} \ln \frac{\alpha}{1-\alpha}\right)$$

Example 2.6. The inverse uncertainty distribution of Zigzag uncertain variable $\mathcal{Z}(a, b, c)$ is

$$\Phi^{-1}(\alpha) = \begin{cases} (1-2\alpha)a + 2\alpha b, & if\ \alpha < 0.5 \\ (2-2\alpha)b + (2\alpha - 1)c, & if\ \alpha \geq 0.5 \end{cases}$$

Definition 2.7. The uncertain variables $\xi_1, \xi_2, ..., \xi_n$ are said to be independent [7] if

$$\mathcal{M}\left\{\bigcap_{i=1}^{n}(\xi_i \in B_i)\right\} = \bigwedge_{i=1}^{n}\mathcal{M}\{\xi_i \in B_i\}$$

for any Borel sets $B_1, B_2, ..., B_n$ of real numbers.

Theorem 2.8. *Let $\xi_1, \xi_2, ..., \xi_n$ be independent uncertain variables with regular uncertainty distribution $\Phi_1, \Phi_2, ..., \Phi_n$, respectively. If f is a continuous and strictly increasing function, then $\xi = f(\xi_1, \xi_2, ..., \xi_n)$ has an inverse uncertainty distribution*

$$\Psi^{-1}(\alpha) = f\left(\Phi_1^{-1}(\alpha), \Phi_2^{-1}(\alpha), ..., \Phi_n^{-1}(\alpha)\right).$$

3. Solving uncertain linear equation system

Definition 3.1. An uncertain linear equations is presented by $Ax = \xi$, where A is a crisp matrix $n \times n$ from real number, x is an uncertain vector in the form $x = (x_1, x_2, ..., x_n)^t$ and ξ is a crisp uncertain vector $\xi = (\xi_1, \xi_2, ..., \xi_n)^t$ [8].

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + ... + a_{1n}x_n = \xi_1 \\ a_{21}x_1 + a_{22}x_2 + ... + a_{2n}x_n = \xi_2 \\ \vdots \\ a_{n1}x_1 + a_{n2}x_2 + ... + a_{nn}x_n = \xi_n \end{cases} \tag{3.1}$$

This is a system with n equations and n unknowns.
Let $\xi_1, \xi_2, ..., \xi_n$ be given uncertain variables with uncertainty distributions $\psi_1, \psi_2, ..., \psi_n$, respectively. Assume that $x_1, x_2, ..., x_n$ are independent uncertain variables with uncertainty distributions $\phi_1, \phi_2, ..., \phi_n$ respectively.We will define a method to find the solution of the uncertain linear system (3.1).

Definition 3.2. The uncertain vector $(x_1, x_2, ..., x_n)^t$ is a solution of the uncertain linear system (3.1) in distribution if for any $\alpha \in (0, 1)$, we have

$$\begin{cases} a_{11}\Phi_1^{-1}(\varepsilon_{11}) + a_{12}\Phi_2^{-1}(\varepsilon_{12}) + ... + a_{1n}\Phi_n^{-1}(\varepsilon_{1n}) = \Psi_1^{-1}(\alpha) \\ a_{21}\Phi_1^{-1}(\varepsilon_{21}) + a_{22}\Phi_2^{-1}(\varepsilon_{22}) + ... + a_{2n}\Phi_n^{-1}(\varepsilon_{2n}) = \Psi_2^{-1}(\alpha) \\ \vdots \\ a_{n1}\Phi_1^{-1}(\varepsilon_{n1}) + a_{n2}\Phi_2^{-1}(\varepsilon_{n2}) + ... + a_{nn}\Phi_n^{-1}(\varepsilon_{nn}) = \Psi_n^{-1}(\alpha), \end{cases} \tag{3.2}$$

where

$$\varepsilon_{ij} = \begin{cases} \alpha, & if\ a_{ij} \geq 0 \\ 1-\alpha, & if\ a_{ij} < 0. \end{cases} \tag{3.3}$$

4. Some special uncertain linear systems

The uncertainty distributions $\Phi_1, \Phi_2, ..., \Phi_n$ of $x_1, x_2, ..., x_n$ as a solution of the uncertain linear system (3.1) are related to the uncertainty distributions of $\xi_1, \xi_2, ..., \xi_n$. In this paper, we will limit the discussion to special uncertain linear systems, $\xi_1, \xi_2, ..., \xi_n$ are Log-normal or Zigzag uncertain variables.

4.1. **Uncertain linear system with Log-normal uncertain variables.**

Let $\xi \sim \mathcal{LOGN}(e_i, v),\ 1 \leq i \leq n$, where e_i are large than one and v_i are larger than zero.

Theorem 4.1. *Assume that $x_i \sim \mathcal{LOGN}(u_i, v),\ 1 \leq i \leq n$. Then (3.2) is equal to*

$$A_1 \exp(u) = \exp(b_e) \tag{4.1}$$

where

$$A_1 = (\mid a_{ij} \mid)_{n\times n},\ \ u = (u_1, u_2, ..., u_n)^t$$
$$b_e = (e_1, e_2, ..., e_n)^t,\ \ v = (v_1, v_2, ..., v_n)^t$$

Proof. Since $x_i \sim \mathcal{LOGN}(u_i, v),\ 1 \leq i \leq n$, (3.2) can be written as

(4.2)

$$\begin{cases} a_{11}\left[\exp\left(u_1 + \frac{v\sqrt{3}}{\pi}\ln\frac{\varepsilon_{11}}{1-\varepsilon_{11}}\right)\right] + a_{12}\left[\exp\left(u_2 + \frac{v_2\sqrt{3}}{\pi}\ln\frac{\varepsilon_{12}}{1-\varepsilon_{12}}\right)\right] + ... \\ +a_{1n}\left[\exp\left(u_n + \frac{v\sqrt{3}}{\pi}\ln\frac{\varepsilon_{1n}}{1-\varepsilon_{1n}}\right)\right] = \exp\left(e_1 + \frac{v\sqrt{3}}{\pi}\ln\frac{\alpha}{1-\alpha}\right) \\ a_{21}\left[\exp\left(u_1 + \frac{v\sqrt{3}}{\pi}\ln\frac{\varepsilon_{21}}{1-\varepsilon_{21}}\right)\right] + a_{22}\left[\exp\left(u_2 + \frac{v\sqrt{3}}{\pi}\ln\frac{\varepsilon_{22}}{1-\varepsilon_{22}}\right)\right] + ... \\ +a_{2n}\left[\exp\left(u_n + \frac{v\sqrt{3}}{\pi}\ln\frac{\varepsilon_{2n}}{1-\varepsilon_{2n}}\right)\right] = \exp\left(e_2 + \frac{v\sqrt{3}}{\pi}\ln\frac{\alpha}{1-\alpha}\right) \\ \vdots \\ a_{n1}\left[\exp\left(u_1 + \frac{v\sqrt{3}}{\pi}\ln\frac{\varepsilon_{n1}}{1-\varepsilon_{n1}}\right)\right] + a_{n2}\left[\exp\left(u_2 + \frac{v\sqrt{3}}{\pi}\ln\frac{\varepsilon_{n2}}{1-\varepsilon_{n2}}\right)\right] + ... \\ = +a_{nn}\left[\exp\left(u_n + \frac{v\sqrt{3}}{\pi}\ln\frac{\varepsilon_{nn}}{1-\varepsilon_{nn}}\right)\right] = \exp\left(e_n + \frac{v\sqrt{3}}{\pi}\ln\frac{\alpha}{1-\alpha}\right) \end{cases}$$

where ε_{ij}, $1 \leq i,j \leq n$, are definead as (3.3). We have (4.2) for each equation

$$\sum_{j=1}^{n} a_{ij} \exp(u_j) \exp\left(\frac{v\sqrt{3}}{\pi} \ln \frac{\varepsilon_{ij}}{1-\varepsilon_{ij}}\right) = \exp(e_i) \exp\left(\frac{v\sqrt{3}}{\pi} \ln \frac{\alpha}{1-\alpha}\right).$$

Then coefficient of $\frac{\sqrt{3}}{\pi} \ln \frac{\varepsilon_{ij}}{1-\varepsilon_{ij}}$ on the left hand side is equal to the coefficient of $\frac{\sqrt{3}}{\pi} \ln \frac{\alpha}{1-\alpha}$ on the right hand side of each equation in (4.2). Then we have

$$\frac{\sqrt{3}}{\pi} \ln \frac{\varepsilon_{ij}}{1-\varepsilon_{ij}} = \frac{\sqrt{3}}{\pi} \ln \frac{\alpha}{1-\alpha},$$

since every equation (4.2) has $\alpha \in (0,1)$ and

$$\ln \frac{\varepsilon_{ij}}{1-\varepsilon_{ij}} = \begin{cases} \ln \frac{\alpha}{1-\alpha}, & if\ a_{ij} \geq 0 \\ -\ln \frac{\alpha}{1-\alpha}, & if\ a_{ij} < 0. \end{cases}$$

Then we have

$$A_1 \exp(u) = \exp(b_e).$$

which completes the proof. □

Example 4.2. Consider the uncertain linear system

$$\begin{cases} 3x_1 - 0.8x_2 = \xi_1 \\ -0.6x_1 + 2.6x_2 = \xi_2 \end{cases}$$

where $\xi_1 \sim \mathcal{LOGN}(2.5, v)$ and $\xi_2 \sim \mathcal{LOGN}(3, v)$.
Assume that $x_1 \sim \mathcal{LOGN}(u_1, v)$ and $x_2 \sim \mathcal{LOGN}(u_2, v)$. According to Theorem (4.1) we have

Solution 4.3.

$$A = \begin{pmatrix} 3 & -0.8 \\ -0.6 & 2.6 \end{pmatrix},$$

$$A_1 = \begin{pmatrix} 3 & 0.8 \\ 0.6 & 2.6 \end{pmatrix}.$$

Using equation (4.1) we have

$$\begin{pmatrix} 3 & 0.8 \\ 0.6 & 2.6 \end{pmatrix} \begin{pmatrix} \exp(u_1) \\ \exp(u_2) \end{pmatrix} = \begin{pmatrix} \exp(2.5) \\ \exp(3) \end{pmatrix}$$

$$\begin{pmatrix} \exp(u_1) \\ \exp(u_2) \end{pmatrix} = \begin{pmatrix} 2.1319 \\ 7.2332 \end{pmatrix}$$

Thus,

$$u_1 = \ln(\exp(u_1)) = 0.7570,\ u_2 = \ln(\exp(u_2)) = 1.9786\ .$$

4.2. **Uncertain linear systems with Zigzag uncertain variables.** Let $\xi_i \sim \mathcal{Z}(a_i, b_i, c_i),\ 1 \le i \le n$, where $a_i < b_i < c_i$ and are real numbers.

Theorem 4.4. *Assume that* $x_i \sim \mathcal{Z}(e_i, f_i, g_i),\ 1 \le i \le n$. *Then* (3.2) *is equivalent to*

$$A_1 r = h, \tag{4.3}$$

and

$$Ae + 2A_2 r = a \tag{4.4}$$

and

$$A_1 t = m, \tag{4.5}$$

and

$$2A_2 t + 2Af - Ag = 2b - c, \tag{4.6}$$

where

$$A = (a_{ij})_{n\times n}\,,\ A_1 = (\mid a_{ij} \mid)_{n\times n}\,,\ A_2 = (a_{ij}\lambda_{ij})$$
$$a = (a_1, a_2, \dots, a_n)^t\,,\ b = (b_1, b_2, \dots, b_n)^t$$
$$c = (c_1, c_2, \dots, c_n)^t\,,\ e = (e_1, e_2, \dots, e_n)^t$$
$$f = (f_1, f_2, \dots, f_n)^t\,,\ g = (g_1, g_2, \dots, g_n)^t$$
$$h = (b_1 - a_1, b_2 - a_2, \dots, b_n - a_n)^t\,,\ m = (c_1 - b_1, c_2 - b_2, \dots, c_n - b_n)^t$$
$$r = (f_1 - e_1, f_2 - e_2, \dots, f_n - e_n)^t\,,\ t = (g_1 - f_1, g_2 - f_2, \dots, g_n - f_n)^t$$
$$\lambda_{ij} = \begin{cases} 0, & if\ a_{ij} \ge 0 \\ 1, & if\ a_{ij} < 0\ , \end{cases}$$

Proof. Substituting the uncertainty distributions of $x_1, x_2, ..., x_n$ into (3.2), we have two separate systems for $\alpha < 0.5$ and $\alpha \ge 0.5$ if $\alpha < 0.5$

the relation (3.2) is equivalent to
(4.7)

$$\begin{cases} a_{11}\left[\left(1-2\varepsilon_{11}\right)e_1+2\varepsilon_{11}f_1\right]+a_{12}\left[\left(1-2\varepsilon_{12}\right)e_2+2\varepsilon_{12}f_2\right]+... \\ +a_{1n}\left[\left(1-2\varepsilon_{1n}\right)e_n+2\varepsilon_{1n}f_n\right]=\left(1-2\alpha\right)a_1+2\alpha b_1 \\ a_{21}\left[\left(1-2\varepsilon_{21}\right)e_1+2\varepsilon_{21}f_1\right]+a_{22}\left[\left(1-2\varepsilon_{22}\right)e_2+2\varepsilon_{22}f_2\right]+... \\ +a_{2n}\left[\left(1-2\varepsilon_{2n}\right)e_n+2\varepsilon_{2n}f_n\right]=\left(1-2\alpha\right)a_2+2\alpha b_2 \\ \vdots \\ a_{n1}\left[\left(1-2\varepsilon_{n1}\right)e_1+2\varepsilon_{n1}f_1\right]+a_{n2}\left[\left(1-2\varepsilon_{n2}\right)e_2+2\varepsilon_{n2}f_2\right]+... \\ +a_{nn}\left[\left(1-2\varepsilon_{nn}\right)e_n+2\varepsilon_{nn}f_n\right]=\left(1-2\alpha\right)a_n+2\alpha b\alpha b_n \end{cases}$$

According to (3.3) we know

$$a_{ij}\left[\left(1-2\varepsilon_{ij}\right)e_j+2\varepsilon_{ij}f_j\right]=\begin{cases} a_{ij}e_j+2a_{ij}\left(f_j-e_j\right)\alpha, & if\ a_{ij}\geq 0 \\ \\ 2a_{ij}f_j+2\mid a_{ij}\mid\left(f_j-e_j\right)\alpha-a_{ij}e_j, & if\ a_{ij}<0\ , \end{cases}$$

Since every equation of (3.3) holds for any $\alpha\in(0,1)$, the coefficient of on the left hand side is equal to the coefficient of α on the right hand side of (3.3). Then we can get

$$2\alpha\mid a_{ij}\mid\left(f_j-e_j\right)=2\alpha\left(b_i-a_i\right),\ 1\leq i,j\leq n$$

that is

$$A_1r=h\ ,$$

The rest terms of (4.7) becom

$$\begin{cases} a_{11}\theta_{11}+a_{12}\theta_{12}+...+a_{1n}\theta_{1n}=a_1 \\ a_{21}\theta_{21}+a_{22}\theta_{22}+...+a_{2n}\theta_{2n}=a_2 \\ \vdots \\ a_{n1}\theta_{n1}+a_{n2}\theta_{n2}+...+a_{nn}\theta_{nn}=a_n \end{cases}$$

where

$$\begin{aligned} \theta_{ij} &=\left(1-2\varepsilon_{ij}\right)e_j+2\varepsilon_{ij}f_j \\ &=e_j-2\varepsilon_{ij}e_j+2\varepsilon_{ij}f_j \\ &=e_j+2\varepsilon_{ij}e_j\left(f_j-e_j\right) \\ &=e_j+2\lambda_{ij}\left(f_j-e_j\right) \\ \longrightarrow a_{ij}\theta_j &=a_{ij}\left(e_j+2\lambda_{ij}\left(f_j-e_j\right)\right) \\ &=a_{ij}e_j+2a_{ij}\lambda_{ij}\left(f_j-e_j\right) \end{aligned}$$

That is

$$Ae + 2A_2 r = a$$

Now if $\alpha \geq 0.5$ the relation (3.2) is equivalent to
(4.8)

$$\begin{cases} a_{11}\left[\left(2-2\varepsilon_{11}\right)f_1+\left(2\varepsilon_{11}-1\right)g_1\right]+a_{12}\left[\left(2-2\varepsilon_{12}\right)f_2+\left(2\varepsilon_{12}-1\right)g_2\right]+... \\ +a_{1n}\left[\left(2-2\varepsilon_{1n}\right)f_n+\left(2\varepsilon_{1n}-1\right)g_n\right]=\left(2-2\alpha\right)b_1+\left(2\alpha-1\right)c_1 \\ a_{21}\left[\left(2-2\varepsilon_{21}\right)f_1+\left(2\varepsilon_{21}-1\right)g_1\right]+a_{22}\left[\left(2-2\varepsilon_{22}\right)f_2+\left(2\varepsilon_{22}-1\right)g_2\right]+... \\ +a_{2n}\left[\left(2-2\varepsilon_{2n}\right)f_n+\left(2\varepsilon_{2n}-1\right)g_n\right]=\left(2-2\alpha\right)b_2+\left(2\alpha-1\right)c_2 \\ \vdots \\ a_{n1}\left[\left(2-2\varepsilon_{n1}\right)f_1+\left(2\varepsilon_{n1}\right)g_1\right]+a_{n2}\left[\left(2-2\varepsilon_{n2}\right)f_2+\left(2\varepsilon_{2n}-1\right)g_2\right]+... \\ +a_{nn}\left[\left(2-2\varepsilon_{nn}\right)f_n+\left(2\varepsilon_{nn}-1\right)g_n\right]=\left(2-2\alpha\right)b_n+\left(2\alpha-1\right)c_n \end{cases}$$

According to (3.3) ,(4.8) we have:

$$a_{ij}\left[\left(2-2\varepsilon_{ij}\right)f_j+\left(2\varepsilon_{ij}-1\right)g_j\right]=\begin{cases}\left(2-2\alpha\right)a_{ij}f_j+\left(2\alpha-1\right)a_{ij}g_j, & if\ \ a_{ij}\geq 0 \\ 2\alpha\mid a_{ij}\mid\left(g_j-f_j\right)+a_{ij}g_j, & if\ \ a_{ij}<0\end{cases}$$

since every equation of (4.8) holds for any $\alpha \in (0, 1)$, the coefficient of 2α on the left hand side is equal to the coefficient of 2α on the right hand side of (4.8). Then we can get

$$\sum_{j=1}^{n}\left(2\alpha\right)\mid a_{ij}\mid\left(g_j-f_j\right)=\left(2\alpha\right)\left(c_i-b_i\right),\ 1\leq i,j\leq n$$

That is

$$A_1 t = m$$

The rest terms of (4.8) become

$$\begin{cases} a_{11}\gamma_{11}+a_{12}\gamma_{12}+...+a_{1n}\gamma_{1n}=2b_1-c_1 \\ a_{21}\gamma_{21}+a_{22}\gamma_{22}+...+a_{2n}\gamma_{2n}=2b_2-c_2 \\ \vdots \\ a_{n1}\gamma_{n1}+a_{n2}\gamma_{n2}+...+a_{nn}\gamma_{nn}=2b_n-c_n \end{cases}$$

where

$$\begin{aligned}\gamma_{ij} &= (2-2\varepsilon_{ij})\,f_j + (2\varepsilon_{ij}-1)\,g_j\\ &= 2f_j - 2\varepsilon_{ij}f_j + 2\varepsilon_{ij}g_j - g_j\\ &= 2\varepsilon_{ij}\,(g_j - f_j) + 2f_j - g_j\\ &\longrightarrow a_{ij}\gamma_{ij} = 2a_{ij}\varepsilon_{ij}\,(g_j - f_j) + 2a_{ij}f_j - a_{ij}g_j\\ &= 2b_i - c_i,\ 1 \le i,j \le n\end{aligned}$$

That is

$$2A_2t + 2Af - Ag = 2b - c$$

Which completes the proof.

□

Theorem 4.5. *Assume that* $x_i \sim \mathcal{Z}(a_i, b_i, c_i),\ 1 \le i \le n$. *If* A *and* A_1 *are nonsingular matrices, that we have*

$$r = A_1^{-1}h \tag{4.9}$$

$$e = A^{-1}a - 2A^{-1}A_1^{-1}A_2h \tag{4.10}$$

$$f = A^{-1}a - 2A^{-1}A_1^{-1}A_2h + A_1^{-1}h \tag{4.11}$$

$$g = A^{-1}a - 2A^{-1}A_1^{-1}A_2h + A_1^{-1}h + A_1^{-1}m \tag{4.12}$$

where

$$\begin{aligned}&A = (a_{ij})_{n\times n},\ A_1 = (|\ a_{ij}\ |)_{n\times n}, A_2 = (a_{ij}\lambda_{ij})\\ &a = (a_1, a_2, \ldots, a_n)^t,\ b = (b_1, b_2, \ldots, b_n)^t\\ &c = (c_1, c_2, \ldots, c_n)^t,\ e = (e_1, e_2, \ldots, e_n)^t\\ &f = (f_1, f_2, \ldots, f_n)^t,\ g = (g_1, g_2, \ldots, g_n)^t\\ &h = (b_1 - a_1, b_2 - a_2, \ldots, b_n - a_n)^t,\ m = (c_1 - b_1, c_2 - b_2, \ldots, c_n - b_n)^t\\ &r = (f_1 - e_1, f_2 - e_2, \ldots, f_n - e_n)^t,\ t = (g_1 - f_1, g_2 - f_2, \ldots, g_n - f_n)^t\\ &\lambda_{ij} = \begin{cases} 0, & if\ a_{ij} \ge 0\\ 1, & if\ a_{ij} < 0\ ,\end{cases}\end{aligned}$$

Proof. Since A_1 is a nonsingular matrix, from Theorem (4.4), we have

$$r = A_1^{-1} h$$

From (4.4) , we have

$$\begin{aligned} &Ae + 2A_2 r = a \\ &\longrightarrow e = A^{-1} a - 2A^{-1} A_1^{-1} (A_2 h) \end{aligned}$$

To get (4.10), we put (4.11) in (4.9) and we have

$$f = e + A_1^{-1} h$$

and we have (4.12) out of (4.5) to get

$$\begin{aligned} &A_1 t = m \\ &\longrightarrow g = f + A_1^{-1} m \ , \end{aligned}$$

□

Example 4.6. Consider the uncertain linear systems

$$\begin{cases} 1.5x_1 + 0.4x_2 = \xi_1 \\ 0.3x_1 + 1.3x_2 = \xi_2 \\ 0.2x_2 + 1.5x_3 = \xi_3 \end{cases}$$

where $\xi_1 \sim \mathcal{Z}(50, 85, 100)$, $\xi_2 \sim \mathcal{Z}(30, 40, 70)$ and $\xi_3 \sim \mathcal{Z}(70, 85, 95)$. Assume that $x_1 \sim \mathcal{Z}(e_1, f_1, g_1)$, $x_2 \sim \mathcal{Z}(e_2, f_2, g_2)$ and $x_3 \sim \mathcal{Z}(e_3, f_3, g_3)$.

Solution 4.7. According to Theorem (4.4), we have

$$A = A_1 = \begin{pmatrix} 1.5 & 0.4 & 0 \\ 0.3 & 1.3 & 0 \\ 0 & 0.2 & 1.5 \end{pmatrix}$$

$$A_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$A^{-1} = A_1^{-1} = \begin{pmatrix} 0.7104 & -0.2186 & 0 \\ -0.1639 & 0.8197 & 0 \\ 0.0219 & -0.1093 & 0.6667 \end{pmatrix} ,$$

From Theorem (4.5), we can get

$$\begin{pmatrix} f_1-e_1\\ f_2-e_2\\ f_3-e_3 \end{pmatrix} = \begin{pmatrix} 0.7104 & -0.2186 & 0\\ -0.1639 & 0.8197 & 0\\ 0.0219 & -0.1093 & 0.6667\,. \end{pmatrix} \begin{pmatrix} 35\\ 10\\ 15 \end{pmatrix} = \begin{pmatrix} 22.6776\\ 2.4590\\ 9.6721 \end{pmatrix}$$

Then we have (4.10) using

$$\begin{pmatrix} e_1\\ e_2\\ e_3 \end{pmatrix} = \begin{pmatrix} 28.9617\\ 16.3934\\ 44.4809 \end{pmatrix},$$

with an introduction e in (4.9) can be obtained

$$\begin{pmatrix} f_1\\ f_2\\ f_3 \end{pmatrix} = \begin{pmatrix} 51.6393\\ 18.8524\\ 54.1530 \end{pmatrix},$$

from (4.12) we can get

$$\begin{pmatrix} g_1\\ g_2\\ g_3 \end{pmatrix} = \begin{pmatrix} 55.7377\\ 40.9835\\ 57.8688 \end{pmatrix}.$$

Therefore the solution is $x_1 \sim \mathcal{Z}(28.9617, 51.6393, 55.7377)$, $x_2 \sim \mathcal{Z}(16.3934, 18.8524, 40.9835)$ and $x_3 \sim \mathcal{Z}(44.4809, 54.1530, 57.8688)$.

Lemma 4.8. *The inverse of a nonnegative matrix A is nonnegative if and only A is a generalized permutation matrix* [5].

Theorem 4.9. *Assume that $A = (a_{ij})_{n\times n}$ and $A_1 = (\mid a_{ij} \mid)_{n\times n}$ are nonsingular matrices. If A_1 is generalized permutation matrix, then the uncertain linear system (3.1) with Log-normal or Zigzag uncertain variables has a unique solution in distribution.*

Proof. If A_1 is generalized permutation matrix, if follows from Lemma (4.8) that A_1^{-1} is nonnegative. For the uncertain linear systems (3.1) with Log-normal and Zigzag uncertain variables has a unique solution in distribution. □

5. Application of uncertain linear equation system

A healthy diet is an important part of healthy life in society. A proper diet not only allows people to consume enough nutrients but also prevents them from eating extra nutrients. Typically, the consumption of different nutrients for people is not very accurate and may be recommended by experts as about 100 grams or about 100 milliliter.

These quantities may be considered as uncertain variables and may be represented by different uncertainty distributions. Here we use the Log-normal uncertainty distribution to describe these quantities.
Nutritionists suggest that at the right time, a person's daily intake of calcium, vitamin C and phosphor is ln(1000), ln(850) and ln(700) mg, respectively. These elements may be due to the diet of goat's milk, jujube and beef. The relationships of elements (calcium, vitamin C and phosphor) and nutrients (goat's milk, jujube and beef) are shown in Table 1.

Table 1. Nutrition diet relations

Nutrients	Nutrient content per gram of food(mg)			Demands of nutrients (mg)
	Goat's milk	Jujube	Beef	
Calcium	1.5	0.4		ξ_1
Vitamin C	0.3	1.3		ξ_2
Phosphor		0.2	1.5	ξ_3

Let ξ_i be uncertainty orders for the ith nutrient elements, $i = 1, 2, 3$ respectively, where $\xi_1 \sim \mathcal{LOGN}(\ln(1000), v)$, $\xi_2 \sim \mathcal{LOGN}(\ln(850), v)$ and $\xi_3 \sim \mathcal{LOGN}(\ln(700), v)$. We need to find the amount of goat's milk, jujube and beef in a man's diet so that the needs of calcium, Vitamin C and phosphor are satiesfied.
Let x_1, x_2 and x_3 be the diet amount of Goat's milk, Jujube and Beef, respectively. So according to the table 1, we have the following uncertain linear systems

$$\begin{cases} 1.5x_1 + 0.4x_2 = \xi_1 \\ 0.3x_1 + 1.3x_2 = \xi_2 \\ 0.2x_2 + 1.5x_3 = \xi_3 \end{cases}, \tag{5.1}$$

Assume that $x_1 \sim \mathcal{LOGN}(u_1, v)$, $x_2 \sim \mathcal{LOGN}(u_2, v)$ and $x_3 \sim \mathcal{N}(u_3, v)$. We have

$$A_1 = A = \begin{pmatrix} 1.5 & 0.4 & 0 \\ 0.3 & 1.3 & 0 \\ 0 & 0.2 & 1.5 \end{pmatrix},$$

Therefore, by using Theorem (4.9) we know that the uncertain linear system (5.1) has a unique solution. Now we want to get the solution of uncertain (5.1) linear system when $\xi_1 \sim \mathcal{LOGN}(6.9077, v)$, $\xi_2 \sim \mathcal{LOGN}(6.7452, v)$ and $\xi_3 \sim \mathcal{LOGN}(6.5510, v)$, $x_1 \sim \mathcal{LOGN}(u_1, v)$,

$x_2 \sim \mathcal{LOGN}(u_2, v)$ and $x_3 \sim \mathcal{LOGN}(u_3, v)$. We have

$$A_1 = A = \begin{pmatrix} 1.5 & 0.4 & 0 \\ 0.3 & 1.3 & 0 \\ 0 & 0.2 & 1.5 \end{pmatrix}, \ \exp(b_e) = \begin{pmatrix} 1000 \\ 850 \\ 700 \end{pmatrix},$$

According to Theorem (4.1)

$$A_1 \exp(u) = \exp(b_e)$$

$$\rightarrow \begin{pmatrix} \exp(u_1) \\ \exp(u_2) \\ \exp(u_3) \end{pmatrix} = \begin{pmatrix} 524.3712 \\ 53207754 \\ 395.5985 \end{pmatrix},$$

Therefore, by using Theoremc (4.1), we can get the solution $x_1 \sim \mathcal{LOGN}(6.2622, v)$, $x_2 \sim \mathcal{LOGN}(6.2781, v)$ and $x_3 \sim \mathcal{LOGN}(5.9804, v)$.

Conclusions

In this paper, uncertain linear systems in the form of $Ax = \xi$, were presented and investigated, where A is a crisp matrix and ξ is an uncertain vector. It has been shown that uncertain linear systems can be solved with Log-normal and Zigzag uncertain variables and examples are provided to show the effectiveness and practicality of this method for solving some uncertain linear systems.

Finally, a healthy diet problem was modeled by uncertain linear system with Log-normal variables.

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