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Research Paper

FIXED POINTS OF MULTI-VALUED SUZUKI NONEXPANSIVE MAPPINGS IN HYPERBOLIC SPACES

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Abstract. In this paper, we have proved fixed point results for multi-valued Suzuki nonexpansive mappings in complete hyperbolic spaces along with application.

Key Words: Hyperbolic space, fixed point, multi-valued Suzuki nonexpansive mappings, Fejer monotone.

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1. Introduction

The fixed point theory was revealed as a powerful tool in the study of nonlinear analysis. In fact the technique of fixed point also have been applied in different fields such as biology, physics, engineering, chemistry, game theory, economics, computer science etc.

Fixed point theory plays an important role not only in the field of analysis, but also used to find out solutions of different mathematical

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problems like integral equations, differential equations, convex minimization problems, image recovery, signal processing (refer [6, 7, 19]) etc.

There are lots of fixed point results in different spaces. One of the most important and fruitful result in metric space was given by Banach [4] called "Banach Contraction Principle". This principle was generalized and its several variants were studied by mathematicians in different spaces.

The relationship between the geometric properties of a space and xed point theory makes it possible to obtain eective and useful results. In particular, geometric properties of a space play an important role in metric xed point theory. It is well known in the literature that Banach spaces have been studied extensively, because Banach spaces always have convex structures. However, metric spaces do not have this structure. Therefore, there is a need to introduce and define convex structures.

Hyperbolic spaces are rich in geometrical structure and it is suitable to obtain new results in topology, graph theory, multi-valued analysis and metric fixed point theory. The study of fixed point theory for nonexpansive mappings in the framework of hyperbolic spaces was initiated by Takahashi [23]. The concept of hyperbolic space was introduced by Kohlenbach [13] in 2005. Leustean [15] showed that CAT(0) spaces are uniformly convex hyperbolic metric spaces.

Recall that a subset K of a metric space (X, d) is called proximal, if there exists an element $y \in K$ such that

$$d(x, y) = d(x, K) = \inf_{z \in K} d(x, z)$$

for all $x \in X$. Let $CB(K)$ and $P(K)$ be the collection of all non-empty closed bounded subsets and the collection of all non-empty proximal bounded closed subsets of K, respectively. A mapping $T : K \to X$ is called nonexpansive if

$$d(Tx, Ty) \leq d(x, y),$$

for all $x, y \in K$.

Example 1.1. Consider the mapping $T : X \to X$ defined by $Tx = x + z$ where $z \neq 0$. Then T is nonexpansive mapping.

In [17], for any metric space (X, d), the Hausdorff metric is defined by

$$d_H(A, B) = \max\{\sup d(x, B), \sup d(A, y)\},$$

where $A, B \in CB(X)$, and $d(x, A) = \inf_{a \in A} d(x, a)$.

Example 1.2. Let $X = \{1, 2, ..., 10\} \subset \mathbb{R}$, $A, B \subset \mathbb{R}$ such that $A = \{1, 5\}$, $B = \{2, 6\}$. Suppose that d is metric on X defined by $d(x, y) = |x - y|$. Now $d(A, x) = \inf_{a \in A} d(a, x) = 0$. Similarly $d(x, B) = \inf_{b \in B} d(x, b) = 0$. Therefore $d_H(A, B) = \max\{\sup d(x, B), \sup d(A, y)\} = 0$.
Now for $y \in A$, $d(x, y) = d(A, x) = 0$, therefore $y = 1$ is proximal point of A. Similarly $y = 2$ is proximal point of B.

In 2008, Suzuki [22] introduced a generalization of nonexpansive mappings, which he named condition (C), is as follows:
A mapping T defined on a subset K of a Banach space X is said to satisfy condition (C), if

$$\frac{1}{2}||x - Tx|| \leq ||x - y|| \Rightarrow ||Tx - Ty|| \leq ||x - y||,$$

for $x, y \in K$. Note that T is generalization of nonexpansive mapping in the sense of Suzuki. It is obvious that every nonexpansive mapping satisfies condition (C), but the converse is not true. Consider the following examples:

Example 1.3. Let $T : [0, 2] \to [0, 2]$ defined by

$$Tx = \begin{cases} 0, & x \neq 2, \\ 2, & x = 2. \end{cases} \tag{1.1}$$

It is clear that T is Suzuki nonexpansive mapping and also nonexpansive.

Example 1.4. Let $X = \mathbb{R}$ and $K = [0, \frac{5}{2}]$ is subset of X. Let $d : X \times X \to \mathbb{R}$ such that $d(x, y) = |x - y|$. Clearly (X, d) is metric space. Let T be a mapping defined on K such that

$$Tx = \begin{cases} 0, & x \in [0, 2], \\ 4x - 12, & x \in [0, \frac{5}{2}]. \end{cases} \tag{1.2}$$

Then T is Suzuki nonexpansive mapping. However it is not nonexpansive mapping.

A mapping $T : X \to CB(X)$ is called multi-valued nonexpansive mapping if

$$d_H(Tx, Ty) \leq d(x, y),$$

for $x, y \in X$ and it is called quasi-nonexpansive mapping if $F(T) \neq \emptyset$ and

$$d_H(Tx, y) \leq d(x, y),$$

for all $y \in F(T)$.

Example 1.5. [9] Let $X = \mathbb{R}$ with usual metric d. Define

$$Tx = \{x - \tan^{-1} x, \frac{x}{2} - \tan^{-1} x\}, \ x \in X.$$

Then T is multi-valued nonexpansive mapping.

A point $x \in K$ is called fixed point of multi-valued mapping T, if $x \in Tx$. Here we denote the set of fixed point of T by $F(T)$.

In 2010, Akbar and Islamian [2] introduced Suzuki condition for multi-valued mapping as follows:

A multi-valued mapping $T : X \to CB(X)$ is said to satisfy condition (C), provided that

$$\frac{1}{2}d(x, Tx) \leq d(x, y) \Rightarrow d_H(Tx, Ty) \leq d(x, y)$$

for $x, y \in X$, where d_H is Hausdorff metric derived from d.

Example 1.6. Let $X = [0, 3]$ and d is metric on X such that $d(x, y) = |x - y|$. Let $T : [0, 3] \to CB([0, 3])$ defined by

$$Tx = \begin{cases} [0, \frac{x}{3}] \ x \neq 3, \\ \{1\}, \ x = 3. \end{cases} \tag{1.3}$$

We claim that T is multi-valued Suzuki nonexpansive mapping. Consider the following cases:

Case I: when $x = 3$, we have

$$\begin{aligned} \frac{1}{2}d(x, Tx) &= \frac{1}{2}d(3, \{1\}) \\ &= 1. \end{aligned}$$

and $d(x, y) = d(3, y) = |3 - y| \leq 3 + |y|$. Now suppose that $\frac{1}{2}d(x, Tx) \leq d(x, y)$. Then $1 \leq 3 + |y| \Rightarrow -2 < 0 \leq |y|$ for any $y \in X$. Also $d_H(Tx, Ty) = 0 \leq d(x, y)$.

Case II: when $x \neq 3$. we have

$$\begin{aligned}\frac{1}{2}d(x,Tx) &= \frac{1}{2}d(x,[0,\frac{x}{3}]) \\ &= \frac{1}{2}|x-\frac{x}{3}| \\ &= \frac{|x|}{3} \\ &< |x| \\ &\leq |x|+|y| = |x-y| = d(x,y).\end{aligned}$$

Also $d_H(Tx,Ty) \leq d(x,y)$. Hence T is Suzuki nonexpansive mapping. However it is not nonexpansive mapping.

Fixed point theorems are developed for single-valued as well as multi-valued functions in different spaces. The study of the fixed points for multi-valued nonexpansive mappings is difficult rather than single-valued nonexpansive mappings. The multi-valued version of Banach contraction principle was given by Nadler [18] in 1969. Sastry and Babu [20] introduced multi-valued version of Mann [16] and Ishikawa [10] iteration and proved convergence theorems for nonexpansive mappings in Hilbert space. In 2016 Kim et al. [12] introduced multi-valued version of Thakur iteration [24] and proved convergence results in uniformly convex Banach space.

In recent years, lots of fixed point results were established by several researcher in hyperbolic spaces. In 2016, Alagoz et al. [1] proved strong convergence of a finite family of nonexpansive multi-valued mappings in hyperbolic spaces. They studied the convergence of following iteration scheme:
Let K be a non-empty convex subset of a hyperbolic space X. Let $\{T_i : i = 1,2,...,k\}$ be a family of multi-valued mappings such that $T_i : K \to P(K)$ and $P_{T_i}(x) = \{y \in T_i x : d(x,y) = d(x,Tx)\}$ is nonexpansive mapping. Suppose that $\alpha_{nk} \in [0,1]$ for all $n = 1,2,...$ and $i = 1,2,...,k$

for $x_0 \in K$ and let $\{x_k\}$ be the sequence generated by the following:

$$(1.4)\quad \begin{cases} x_{k+1} = W(u_{(n-1)k}, y_{(n-1)k}, \alpha_{nk}) \\ y_{(n-1)k} = W(u_{(n-2)k}, y_{(n-2)k}, \alpha_{(n-1)k}) \\ . \\ . \\ . \\ y_{2k} = W(u_{1k}, y_{1k}, \alpha_{2k}) \\ y_{1k} = W(u_{0k}, y_{0k}, \alpha_{1k}) \end{cases}$$

where $u_{ik} \in P_{T_{i+1}}(y_{ik})$ for $i = 0, 1, ...k-1$ and $y_{0k} = x_k$.

In 2017, Bello et al. [5] studied some fixed point results and established demiclosedness principle for mean nonexpansive mappings by using iteration scheme (1.4) in hyperbolic space.

Inspired by work of Bello et al. [5], in this paper we have established strong convergence and $\Delta-$ convergence of the sequence $\{x_k\}$ defined by (1.4) for multi-valued Suzuki nonexpansive mapping in complete hyperbolic space.

2. Preliminaries

This section starts with some basic concepts and also contains some useful results, which are required to get main results.

Definition 2.1. [13] A hyperbolic space (X, d, W) is a metric space (X, d) together with a convexity mapping $W : X \times X \times [0, 1] \to X$ such that for all $x, y, z \in X$ and $\alpha, \beta \in [0, 1]$, we have

(i) $d(u, W(x, y, \alpha)) \leq (1-\alpha)d(u, x) + \alpha d(u, y)$,
(ii) $d(W(x, y, \alpha), W(x, y, \beta)) = |\alpha - \beta| d(x, y)$,
(iii) $W(x, y, \alpha) = W(y, x, 1-\alpha)$
(iv) $d(W(x, z, \alpha), W(y, w, \alpha)) \leq (1-\alpha)d(x, y) + \alpha d(z, w)$.

Example 2.2. Let $X = \mathbb{R}$ be a Banach space. Let $d : X \times X \to [0, \infty)$ be a mapping defined by

$$d(x, y) = ||x - y||.$$

It is clear that d is metric on X. Let $K = [0, 1]$ be a subset of X. Further we define a mapping $W : X \times X \times [0, 1]$ by

$$W(x, y, \alpha) = \alpha x + (1-\alpha)y,$$

for all $x, y \in X$ and $\alpha \in [0, 1]$. Then (X, d, W) is hyperbolic space.

Definition 2.3. [8] A non-empty subset K of a hyperbolic space X is said to be convex, if $W(x, y, \alpha) \in K$ for all $x, y \in K$ and $\alpha \in [0, 1]$.

Leustean [14] introduced the concept of uniformly convex hyperbolic spaces. Later [21] defined uniformly convex hyperbolic spaces in the following way:

Definition 2.4. [21] A hyperbolic space X is said to be uniformly convex if for any $r > 0$ and $\varepsilon \in (0, 2]$, there exists a $\delta \in (0, 1]$ such that for all $x, y, z \in X$,

$$d(W(x, y, \frac{1}{2}), z) \leq (1 - \delta)r,$$

provided $d(x, z) \leq r$, $d(y, z) \leq r$ and $d(x, y) \geq \varepsilon r$.

Definition 2.5. [5] Let K be a non-empty subset of a metric space X and $\{x_k\}$ be any bounded sequence in K. For $x \in X$, there is a continuous functional $r(., \{x_k\}) : X \to [0, \infty)$ defined by

$$r(x, \{x_k\}) = \limsup_{k \to \infty} d(x_k, x).$$

The asymptotic radius $r(K, \{x_k\})$ of $\{x_k\}$ with respect to K is given by

$$r(K, \{x_k\}) = \inf\{r(x, \{x_k\}) : x \in K\}.$$

A point $x \in K$ is said to be an asymptotic center of the sequence $\{x_k\}$ with respect to K, if

$$r(x, \{x_k\}) = \inf\{r(y, \{x_k\}) : y \in K\}.$$

The set of all asymptotic centres of $\{x_k\}$ with respect to K is denoted by $A(K, \{x_k\})$.

Remark 2.6. In uniformly convex Banach spaces and $CAT(0)$ spaces, bounded sequences have unique asymptotic center with respect to closed convex subset.

Definition 2.7. [5] A sequence $\{x_k\}$ in X is said to be $\Delta-$ converge to $x \in X$, if x is the unique asymptotic center of $\{x_{k_n}\}$ of $\{x_k\}$. In this case $\Delta - \lim_{k \to \infty} x_k = x$.

Definition 2.8. [21] Let X be a hyperbolic space. A map $\eta : (0, \infty) \times (0, 2] \to (0, 1]$ which provides such a $\delta = \eta(r, \varepsilon)$ for a given $r > 0$ and $\varepsilon \in (0, 2]$ is known as a modulus of uniform convexity of X. The mapping η is said to be monotone, if it decreases with r.

Lemma 2.9. [15] *Let X be a complete uniformly convex hyperbolic space with monotone modulus of uniform convexity η. Then every bounded sequences $\{x_k\}$ in X has a unique asymptotic center with respect to any non-empty closed convex subset K of X.*

Lemma 2.10. [8] *Let X be a complete uniformly convex hyperbolic space with monotone modulus of uniform convexity η and let $\{x_k\}$ be a bounded sequence in X with $A(\{x_k\}) = \{x\}$. Suppose that $\{x_{k_n}\}$ is any subsequence of $\{x_k\}$ with $A(\{x_{k_n}\}) = \{x_1\}$ and $\{d(x_k, x_1)\}$ converges, then $x = x_1$.*

Lemma 2.11. [11] *Let (X, d, W) be a complete uniformly convex hyperbolic space with monotone modulus of uniform convexity η. Let $x^* \in X$ and $\{t_k\}$ be a sequence in $[a, b]$ for some $a, b \in (0, 1)$. If $\{x_k\}$ and $\{y_k\}$ are sequences in X such that $\limsup_{k\to\infty} d(x_k, x^*) \leq c$, $\limsup_{k\to\infty} d(y_k, x^*) \leq c$, and $\lim_{k\to\infty} d(W(x_k, y_k, t_k,)x^*) \leq c$, for some $c > 0$. Then $\lim_{k\to\infty} d(x_k, y_k) = 0$.*

Lemma 2.12. [8] *Let (X, d, W) be a complete hyperbolic space, K be a non-empty closed convex subset of X. Let $T : K \to P(K)$ be a multi-valued mapping with $F(T) \neq \emptyset$. Let $P_T : K \to 2^K$ be a multi-valued mapping defined by*

$$P_T(x) = \{y \in Tx : d(x, y) = d(x, Tx)\}, x \in K.$$

Then the following conclusion holds:

(a) *P_T is multi-valued mapping from K to $P(K)$.*
(b) *$F(T) = F(P_T)$.*
(c) *$P_T(p) = \{p\}$, for each $p \in F(T)$.*
(d) *For each $x \in K$, $P_T(x)$ is a closed subset of Tx and so it is compact.*
(e) *$d(x, Tx) = d(x, P_T(x))$ for each $x \in K$.*

Definition 2.13. [5] Let K be a non-empty closed subset of a complete metric space X and $\{x_k\}$ be a sequence in K. Then $\{x_k\}$ is called Fejer monotone sequence with respect to K, if for all $x \in K$ and $k \in \mathbb{N}$,

$$d(x_{k+1}, x) \leq d(x_k, x).$$

Proposition 2.14. [5] *Let $\{x_k\}$ be a sequence in X and K be a non-empty subset of X. Suppose $T : K \to K$ is any nonlinear mapping and the sequence $\{x_k\}$ is Fejer monotone with respect of K, then we have the following:*

(i) *$\{x_k\}$ is bounded.*

(ii) *The sequence* $\{d(x_k, x^*)\}$ *is decreasing and converges for all* $x^* \in F(T)$.

(iii) $\lim_{k\to\infty} d(x_k, F(T))$ *exists.*

Lemma 2.15. [3] *Let* K *be a non-empty closed subset of a complete metric space* X *and* $\{x_k\}$ *be a Fejer monotone with respect of* K. *Then* $\{x_k\}$ *converges to some* $x^* \in K$ *if and only if* $\lim_{k\to\infty} d(x_k, K) = 0$.

Lemma 2.16. [17] *Let* (X, d) *be a complete* $\mathbb{R}$ *tree and* $A, B \in CB(X)$. *Then for any* $z \in X$, $d(x, y) \leq d_H(A, B)$, *where the points* x, y *are respectively the unique closet points to* z *in* A *and* B.

3. Main Results

3.1. Structure of fixed point set of multi-valued Suzuki nonexpansive mapping.

Lemma 3.1. *Let* K *be a non-empty closed convex subset of a complete hyperbolic space* X. *Let* $T_i : K \to CB(K)$ $(i = 1, 2, ..., k)$ *be a finite family of multi-valued mappings such that* $F(T) = \cap_{i=1}^{k} F(T_i) \neq \emptyset$ *and* $P_{T_i} : K \to 2^K$ *are multi-valued Suzuki nonexpansive mappings. Then* T *is quasi-nonexpansive mapping.*

Proof. Let $F(T) \neq \emptyset$ with $p \in F(T)$. Then from Lemma 2.12, we have $p \in F(P_T)$ and $P_T(p) = \{p\}$. Since P_T is Suzuki nonexpansive mapping, we have

$$\begin{aligned}\frac{1}{2}d(x, Tp) &\leq \frac{1}{2}[d(x, p) + d(p, Tp)]\\ &\leq \frac{1}{2}d(x, p)\\ &\Rightarrow d(x, Tp) \leq d(x, p).\end{aligned}$$

□

Lemma 3.2. *Let* K *be a non-empty closed convex subset of a complete hyperbolic space* X. *Let* $T_i : K \to CB(K)$ $(i = 1, 2, ..., k)$ *be a finite family of multi-valued mappings such that* $F(T) = \cap_{i=1}^{k} F(T_i) \neq \emptyset$ *and* $P_{T_i} : K \to 2^K$ *are multi-valued Suzuki nonexpansive mappings. Then* $F(T)$ *is closed and convex.*

Proof. Let $F(T) \neq \emptyset$ with $p \in F(T)$. First we show that $F(T)$ is closed. Let $\{x_k\}$ be a sequence in $F(T)$ such that $\{x_k\}$ converges to some $y \in K$.

From Lemma 2.12, we have $p \in F(P_T)$ and $P_T(p) = \{p\}$. By using Lemma 2.16, we have

$$\begin{aligned} d(x_k, Ty) &\leq d(x_k, p) + d(p, Ty) \\ &\leq d(x_k, p) + d_H(P_T(p), P_T(y)) \\ &\leq d(x_k, p) + d_H(Tp, Ty) \\ &\leq d(x_k, p) + d(p, y) \\ &\leq d(x_k, y) \end{aligned}$$

Taking $\lim_{k\to\infty}$ on both sides, we have

$$\lim_{k\to\infty} d(x_k, Ty) = 0.$$

By uniqueness of limit, we have $y \in Ty$. Hence $F(T)$ is closed.

Next we will show that $F(T)$ is convex. Let $x, y \in F(T)$ and $\alpha \in [0, 1]$. By using Lemma 2.16, we have

$$\begin{aligned} d(x, T(W(x, y, \alpha))) &\leq d_H(P_T(x), P_T(W(x, y, \alpha))) \\ &\leq d(x, W(x, y, \alpha)). \end{aligned}$$

Hence

$$d(x, T(W(x, y, \alpha))) \leq d(x, W(x, y, \alpha)) \tag{3.1}$$

Using similar argument, we have

$$d(y, T(W(x, y, \alpha))) \leq d(y, W(x, y, \alpha)). \tag{3.2}$$

By using Lemma 2.16, (3.1) and (3.2), we have

$$\begin{aligned} d(x, y) &\leq d(x, T(W(x, y, \alpha))) + d(T(W(x, y, \alpha)), y) \\ &\leq d_H(P_T(x), P_T(W(x, y, \alpha))) + d_H(P_T(W(x, y, \alpha)), P_T(y))) \\ &\leq d(x, W(x, y, \alpha)) + d(y, W(x, y, \alpha)) \\ &= d(x, y). \end{aligned}$$

Therefore

$$d(x, y) \leq d(x, y). \tag{3.3}$$

Hence, we conclude that (3.1) and (3.2) are $d(x, T(W(x, y, \alpha))) = d(x, W(x, y, \alpha))$ and $d(y, T(W(x, y, \alpha))) = d(y, W(x, y, \alpha))$, because if we take strictly less than sign $<$, then from (3.3), we comes to the contradiction that $d(x, y) < d(x, y)$. Therefore

$$T(W(x, y, \alpha)) = W(x, y, \alpha),$$

for all $x, y \in F(T)$ and $\alpha \in [0,1]$. Thus $W(x, y, \alpha) \in F(T)$, which implies that $F(T)$ is convex. $\square$

Corollary 3.3. *Let K be a non-empty closed convex subset of a complete hyperbolic space X. Let $T_i : K \to CB(K)$ $(i = 1, 2, ..., k)$ be a finite family of multi-valued mappings such that $F(T) = \cap_{i=1}^{k} F(T_i) \neq \emptyset$ with $p \in F(T)$ and $P_{T_i} : K \to 2^K$ are multi-valued Suzuki nonexpansive mappings. Let $\{x_k\}$ be a bounded sequence in K such that $\lim_{k\to\infty} d(x_k, Tx_k) = 0$. Then $F(T)$ is closed and convex.*

Proof. Let $\{x_k\}$ be a bounded sequence in $F(T)$ such that $\{x_k\}$ converges to some $y \in K$. From Lemma 3.1, T is quasi-nonexpansive, we have

$$\begin{aligned} d(x_k, Ty) &\leq d(x_k, Tx_k) + d(Tx_k, p) + d(p, Ty) \\ &\leq d(x_k, Tx_k) + d_H(P_T(x_k), P_T(p)) + d_H(P_T(p), P_T(y)) \\ &\leq d(x_k, Tx_k) + d(x_k, p) + d(p, y) \\ &\leq d(x_k, Tx_k) + d(x_k, y). \end{aligned}$$

Taking $\lim_{k\to\infty}$ on both sides, we have

$$\lim_{k\to\infty} d(x_k, Ty) = 0.$$

Hence $F(T)$ is closed. Rest are the same as of the proof in Lemma 3.2. $\square$

Theorem 3.4. *Let K be a non-empty closed convex subset of a complete uniformly convex hyperbolic space X with monotone modulus of convexity η. Let $T_i : K \to CB(K)$ $(i = 1, 2, ..., k)$ be a finite family of multi-valued mappings such that $F(T) = \cap_{i=1}^{k} F(T_i) \neq \emptyset$ with $p \in F(T)$ and $P_{T_i} : K \to 2^K$ are multi-valued Suzuki nonexpansive mappings. Let $\{x_k\}$ be a bounded sequence in K such that $\lim_{k\to\infty} d(x_k, Tx_k) = 0$ and $\Delta - \lim_{k\to\infty} x_k = x^*$. Then $x^* \in F(T)$.*

Proof. Since $\{x_k\}$ is a bounded sequence in K, hence from Lemma 2.9, $\{x_k\}$ has a unique asymptotic center in K. Since $\Delta - \lim_{k\to\infty} x_k = x^*$, we have $A(\{x_k\}) = \{x^*\}$. Observe that

$$\begin{aligned} d(x_k, Tx^*) &\leq d(x_k, Tx_k) + d(Tx_k, Tx^*) \\ &\leq d(x_k, Tx_k) + d(Tx_k, p) + d(p, Tx^*) \\ &\leq d(x_k, Tx_k) + d_H(P_T(x_k), P_T(p)) + d_H(P_T(p), P_T(x^*)). \\ &\leq d(x_k, Tx_k) + d(x_k, p) + d(p, x^*). \end{aligned}$$

Taking $\lim_{k\to\infty}$ on both sides, we have

$$\lim_{k\to\infty} d(x_k, Tx^*) \leq \lim_{k\to\infty} d(x_k, x^*).$$

Since

$$\begin{aligned} r(Tx^*, \{x_k\}) &= \limsup_{k\to\infty} d(x_k, Tx^*) \\ &\leq \limsup_{k\to\infty} d(x_k, x^*) \\ &= r(x^*, \{x_k\}). \end{aligned}$$

By uniqueness of asymptotic center of $\{x_k\}$, we have $Tx^* = x^*$. Hence $x^* \in F(T)$. □

3.2. **Strong convergence and $\Delta-$ convergence of a sequence in hyperbolic space.**

Lemma 3.5. *Let K be a non-empty closed convex subset of a complete uniformly convex hyperbolic space X. Let $T_i : K \to CB(K)$ $(i = 1, 2, ..., k)$ be a finite family of multi-valued mappings such that $F(T) = \cap_{i=1}^{k} F(T_i) \neq \emptyset$ with $p \in F(T)$ and $P_{T_i} : K \to 2^K$ are multi-valued Suzuki nonexpansive mappings. Let $\{x_k\}$ be a sequence in K defined by (1.4) and let $y_{0k} = x_k$, then*

(i) *$d(y_{ik}, p) \leq d(x_k, p)$, for $i = 1, 2, ..., k-1$,*
(ii) *$\lim_{k\to\infty} d(x_k, p)$ exists for all $p \in F(T)$.*
(iii) *$\lim_{k\to\infty} d(x_k, F(T))$ exists.*

Proof. (i) We proceed by induction on i.

$$\begin{aligned} d(y_{1k}, p) &= d(W(u_{0k}, y_{0k}, \alpha_{1k}), p) \\ &\leq (1 - \alpha_{1k}) d(u_{0k}, p) + \alpha_{1k} d(y_{0k}, p) \\ &\leq (1 - \alpha_{1k}) d_H(P_{T_1}(y_{0k}), P_{T_1}(p)) + \alpha_{1k} d(y_{0k}, p) \\ &\leq (1 - \alpha_{1k}) d(y_{0k}, p) + \alpha_{1k} d(y_{0k}, p) \\ &= d(y_{0k}, p) \\ &= d(x_k, p). \end{aligned}$$

Hence, we have $d(y_{1k},p) \leq d(x_k,p)$. Assuming that $d(y_{ik},p) \leq d(x_k,p)$ holds for some $1 \leq i \leq k-2$. Now

$$\begin{aligned} d(y_{(i+1)k},p) &= d(W(u_{ik},y_{ik},\alpha_{(i+1)k}),p) \\ &\leq (1-\alpha_{(i+1)k})d(u_{ik},p) + \alpha_{(i+1)k}d(y_{ik},p) \\ &\leq (1-\alpha_{(i+1)k})d_H(P_{T_{(i+1)}}(y_{ik}),P_{T_{(i+1)}}(p)) + \alpha_{(i+1)k}d(y_{ik},p) \\ &\leq d(x_k,p). \end{aligned}$$

We now show that $d(y_{ik},p) \leq d(x_k,p)$, for $i = 1,2,...,k-1$.

$$\begin{aligned} d(y_{(k-1)k},p) &= d(W(u_{(k-2)k},y_{(k-2)k},\alpha_{(k-1)k}),p) \\ &\leq (1-\alpha_{(k-1)k})d(u_{(k-2)k},p) + \alpha_{(k-1)k}d(y_{(k-2)k},p) \\ &\leq (1-\alpha_{(k-1)k})d_H(P_{T_{(k-1)}}(y_{(k-2)k}),P_{T_{(k-1)k}}(p)) \\ &+ \alpha_{(k-1)k}d(y_{(k-2)k},p) \\ &\leq (1-\alpha_{(k-1)k})(d(y_{(k-2)k},p) + \alpha_{(k-1)k}d(y_{(k-2)k},p) \\ &\leq d(x_k,p). \end{aligned}$$

Thus by induction $d(y_{ik},p) \leq d(x_k,p)$, for $i = 1,2,...,k-1$.

(ii)

$$\begin{aligned} d(x_{k+1},p) &= d(W(u_{(n-1)k},y_{(n-1)k},\alpha_{nk}),p) \\ &\leq (1-\alpha_{nk})d(u_{(n-1)k},p) + \alpha_{nk}d(y_{(n-1)k},p) \\ &\leq (1-\alpha_{nk})d_H(P_{T_n}(y_{(n-1)k}),P_{T_n}(p)) + \alpha_{nk}d(y_{(n-1)k},p) \\ &\leq (1-\alpha_{nk})(d(y_{(n-1)k},p) + \alpha_{nk}d(y_{(n-1)k},p) \\ &\leq d(x_k,p). \end{aligned}$$

This implies that $\{x_k\}$ is Fejer monotone with respect to $F(T)$, so by Proposition 2.14, $\lim_{k\to\infty} d(x_k,p)$ exists.

(iii) By Proposition 2.14, $\lim_{k\to\infty} d(x_k,F(T))$ exists.

□

Theorem 3.6. *Let K be a non-empty closed convex subset of complete uniformly convex hyperbolic space X with monotone modulus of convexity η. Let $T_i : K \to CB(K)$ $(i-1,2,...,k)$ be a finite family of multi-valued mappings such that $F(T) = \cap_{i=1}^k F(T_i) \neq \emptyset$ with $p \in F(T)$ and $P_{T_i} : K \to 2^K$ are multi-valued Suzuki nonexpansive mappings. Let $\{x_k\}$ be a sequence in K defined by (1.4), then $\lim_{k\to\infty} d(x_k,T_ix_k) = 0$, for $i = 1,2,...,k$.*

Proof. From Lemma 3.5, we have $\lim_{k\to\infty} d(x_k, p)$ exists for all $p \in F(T)$. So suppose that $\lim_{k\to\infty} d(x_k, p) = w$, where $w \geq 0$. If $w = 0$, then the proof is obvious. Let $w > 0$. Since

$$\lim_{k\to\infty} d(x_k, p) = w \Rightarrow \limsup_{k\to\infty} d(x_k, p) \leq w.$$

Also from Lemma 3.5,

$$d(y_{ik}, p) \leq d(x_k, p),$$

we have

$$\limsup_{k\to\infty} d(y_{ik}, p) \leq w, \ \ for\ i = 1, 2, ..., k-1. \tag{3.4}$$

Note that for $i = 1, 2, ..., k$

$$\begin{aligned} d(u_{(i-1)k}, p) &\leq d_H(P_{T_i}(y_{(i-1)k}), P_{T_i}(p)) \\ &\leq d(y_{(i-1)k}, p). \end{aligned}$$

Which implies that

$$\limsup_{k\to\infty} d(u_{(i-1)k}, p) \leq w. \tag{3.5}$$

Since $\lim_{k\to\infty} d(x_{k+1}, p) = w$, we have

$$\lim_{k\to\infty} d(W(u_{(n-1)k}, y_{(n-1)k}, \alpha_{nk}), p) = w. \tag{3.6}$$

From Lemma 2.11, (3.4), (3.5) and (3.6), we have

$$\lim_{k\to\infty} d(y_{(k-1)k}, u_{(k-1)k}) = 0.$$

Note that for $i = 1, 2, ..., k-1$, we have

$$d(x_{k+1}, p) \leq d(y_{ik}, p),$$

therefore

$$w \leq \liminf_{k\to\infty} d(y_{ik}, p).$$

Also

$$d(W(u_{(i-2)k}, y_{(i-2)k}, \alpha_{(i-1)k}), p) = d(y_{(i-1)k}, p),$$

therefore

$$\lim_{k\to\infty} d(W(u_{(i-2)k}, y_{(i-2)k}, \alpha_{(i-1)k}), p) = w.$$

Thus by induction, we have

$$\lim_{k\to\infty} d(y_{(i-1)k}, u_{(i-1)k}) = 0, \ \ for\ i = 1, 2, ..., k. \tag{3.7}$$

Also we have

$$\begin{aligned} d(y_{ik}, y_{(i-1)k}) &= d(W(u_{(i-1)k}, y_{(i-1)k}, \alpha_{ik}), y_{(i-1)k}) \\ &\leq (1-\alpha_{ik}) d(u_{(i-1)k}, y_{(i-1)k}) + \alpha_{ik} d(y_{(i-1)k}, y_{(i-1)k}) \\ &\Rightarrow \lim_{k\to\infty} d(y_{ik}, y_{(i-1)k}) = 0. \end{aligned}$$

$$\begin{aligned} d(x_k, y_{1k}) &= d(x_k, W((u_{0k}, y_{0k}, \alpha_{1k}))) \\ &\leq (1-\alpha_{1k}) d(x_k, u_{0k}) + \alpha_{1k} d(x_k, y_{0k}) \\ &= (1-\alpha_{1k}) d(x_k, u_{0k}) + \alpha_{1k} d(x_k, x_k) \\ &\Rightarrow \lim_{k\to\infty} d(x_k, y_{1k}) = 0. \end{aligned}$$

Since

$$d(x_k, y_{ik}) \leq d(x_k, y_{1k}) + d(y_{1k}, y_{12}) + ... + d(y_{(i-1)k}, y_{ik}),$$

we have

$$\lim_{k\to\infty} d(x_k, y_{ik}) = 0, \; for \; i = 1, 2, ..., k-1. \tag{3.8}$$

Now from (3.7) and (3.8), we have

$$\begin{aligned} d(x_k, T_i x_k) &\leq d(x_k, y_{(i-1)k}) + d(y_{(i-1)k}), (u_{(i-1)k}) + d(u_{(i-1)k}, T_i x_k) \\ &\leq d(x_k, y_{(i-1)k}) + d(y_{(i-1)k}), (u_{(i-1)k}) \\ &+ d_H(P_{T_i}(y_{(i-1)k}), P_{T_i}(x_k)) \\ &\leq d(x_k, y_{(i-1)k}) + d(y_{(i-1)k}), (u_{(i-1)k}) + d(y_{(i-1)k}, x_k) \\ &\Rightarrow \lim_{k\to\infty} d(x_k, T_i x_k) = 0. \end{aligned}$$

□

Theorem 3.7. *Let K be a non-empty closed convex subset of complete uniformly convex hyperbolic space X with monotone modulus of convexity η. Let $T_i : K \to CB(K)$ $(i-1,2,...,k)$ be a finite family of multi-valued mappings such that $F(T) = \cap_{i=1}^{k} F(T_i) \neq \emptyset$ with $p \in F(T)$ and $P_{T_i} : K \to 2^K$ are multi-valued Suzuki nonexpansive mappings. Let $\{x_k\}$ be a sequence in K defined by (1.4), then $\{x_k\}$ converges strongly to $p \in F(T)$ if and only if $\lim_{k\to\infty} d(x_k, F(T)) = 0$, where $d(x_k, F(T)) = \inf\{d(x_k, p) : p \in F(T)\}$.*

Proof. If $\{x_k\}$ converges strongly to $p \in F(T)$, then $\lim_{k\to\infty} d(x_k, p) = 0$. Since $0 \leq d(x_k, F(T)) = \inf\{d(x_k, p) : p \in F(T)\}$, we have

$$\lim_{k\to\infty} d(x_k, F(T)) = 0.$$

Conversely, suppose that $\lim_{k\to\infty} d(x_k, F(T)) = 0$. From Lemma 3.5, we have

$$d(x_{k+1}, p) \leq d(x_k, p),$$

which implies that

$$d(x_{k+1}, F(T)) \leq d(x_k, F(T)).$$

This implies that $\lim_{k\to\infty} d(x_k, F(T))$ exists. Therefore by our assumption, $\lim_{k\to\infty} d(x_k, F(T)) = 0$. Next we will show that $\{x_k\}$ is Cauchy sequence in K. For $k > n$,

$$\begin{aligned} d(x_k, x_n) &\leq d(x_k, p) + d(p, x_n) \\ &\leq 2d(x_k, p). \end{aligned}$$

Taking inf on right hand side, we have

$$d(x_k, x_n) \leq 2d(x_k, F(T)).$$

Hence, we have $d(x_k, x_n) \to 0$ as $k, n \to \infty$. Hence $\{x_k\}$ is Cauchy sequence in K, therefore it converges to some $q \in K$. Next we show that $q \in F(T_1))$. Since $d(x_k, F(T_1)) = \inf_{y\in F(T_1)} d(x_k, y)$. So for each $\varepsilon^* > 0$, there exists $p_k \in F(T_1)$ such that

$$d(x_k, p_k) < d(x_k, F(T_1)) + \frac{\varepsilon^*}{2}.$$

Since $d(p_k, q) \leq d(x_k, p_k) + d(x_k, q) \Rightarrow \lim_{k\to\infty} d(p_k, q) \leq \frac{\varepsilon^*}{2}$. Hence, we obtain that

$$\begin{aligned} d(T_1q, q) &\leq d(T_1q, p_k) + d(p_k, q) \\ &\leq d_H(P_{T_1}(p_k), P_{T_1}(q)) + d(p_k, q) \\ &\leq d(p_k, q) + d(p_k, q) \\ &\leq 2d(p_k, q) \end{aligned}$$

Which implies that $d(T_1q, q) \leq \varepsilon^*$. Hence $d(T_1q, q) = 0$. Similarly $d(T_iq, q) = 0$ for $i = 1, 2, ..., k$. Since $F(T)$ is closed, we have $q \in F(T)$. □

Theorem 3.8. *Let K be a non-empty closed convex subset of complete uniformly convex hyperbolic space X with monotone modulus of convexity η. Let $T_i : K \to CB(K)$ $(i - 1, 2, ..., k)$ be a finite family of multi-valued mappings such that $F(T) = \cap_{i=1}^{k} F(T_i) \neq \emptyset$ with $p \in F(T)$ and $P_{T_i} : K \to 2^K$ are multi-valued Suzuki nonexpansive mappings. Let $\{x_k\}$ be a sequence in K defined by (1.4), then $\{x_k\}$ $\Delta-$ converges to a common point $p \in F(T)$.*

Proof. Let $p \in F(T)$, then $p \in F(T_i)$, for $i = 1, 2, ..., k$. Also the sequence $\{x_k\}$ has unique asymptotic center, so suppose that $A(\{x_k\}) = \{x\}$. From Lemma 3.5, sequence $\{x_k\}$ is bounded and $\lim_{k\to\infty} d(x_k, p)$ exists, so we can find a subsequence $\{w_k\}$ of the sequence $\{x_k\}$ such that $A(\{w_k\}) = \{x^*\}$ for some $x^* \in K$.
From the Theorem 3.6, $\lim_{k\to\infty} d(w_k, T_i w_k) = 0, i = 1, 2, ..., k$.
We claim that x^* is a fixed point of T_1. For this, let $\{v_k\}$ be another sequence in $T_1 x^*$. Then

$$\begin{aligned} r(v_k, \{w_k\}) &= \limsup_{k\to\infty} d(v_k, w_k) \\ &\leq \limsup_{k\to\infty} (d(v_k, T_1 w_k) + d(T_1 w_k, w_k)) \\ &\leq \limsup_{k\to\infty} (d_H(P_{T_1}(x^*), P_{T_1}(w_k)) + d(T_1 w_k, w_k)) \\ &\leq \limsup_{k\to\infty} ((x^*, w_k) + d(T_1 w_k, w_k)) \\ &\leq \limsup_{k\to\infty} d(x^*, w_k) \\ &= r(x^*, \{w_k\}). \end{aligned}$$

Hence we have $|r(v_k, \{w_k\}) - r(x^*, \{w_k\})| \to 0$ as $k \to \infty$. From Lemma 2.10, we have $\lim_{k\to\infty} v_k = x^*$. Hence either $T_1 x^*$ is closed or bounded. Therefore $\lim_{k\to\infty} v_k = x^* \in T_1 x^*$. Similarly $x^* \in T_i x^*$, for $i = 1, 2, ..., k$, i.e., $x^* \in F(T)$. From Lemma 2.10, we have $p = x^*$. This implies that $\{x_k\}$ $\Delta-$ converges to $p \in F(T)$. □

4. Numerical Examples

To justify Theorem 3.4, let $X = \mathbb{R}$ with metric $d(x, y) = |x - y|$ and $K = [0, 2]$. Define $W : X \times X \times [0, 1] \to X$ by

$$W(x, y, \zeta) = \zeta x + (1 - \zeta)y,$$

for $x, y \in X$, $\zeta \in [0, 1]$. Then (X, d, W) is a complete uniformly hyperbolic space with monotone modulus of uniform convexity and K is non-empty compact convex subset of X. Now, we define a mapping $T : [0, 2] \to CB([0, 2])$ by

$$Tx = \begin{cases} [0, \frac{x}{2}] \ x \neq 2, \\ \{0\}, \ x = 2. \end{cases}$$

It is easy to proved that T is multi-valued Suzuki nonexpansive mapping (refer Example 1.6). Let us choose a sequence $\{x_k\} = \{\frac{1}{k}\}$ in K.

Clearly $\{x_k\}$ is bounded sequence in $K = [0, 2]$ and

$$\begin{aligned} d(x_k, Tx_k) &= |\frac{1}{k} - \frac{1}{2k}| \\ &= \frac{1}{2k} \to \ 0 \ as \ \lim_{k\to\infty} \end{aligned}$$

Since $\{x_k\} = \{\frac{1}{k}\}$ is convergent sequence in K and $\Delta - \lim_{k\to\infty} d(x_k, Tx_k) = 0$ and $0 \in F(T)$.

Now, in support of the Theorem 3.6, 3.7 and 3.8, we consider the following example.

Example 4.1. Let $X = \mathbb{R}$ with metric $d(x, y) = |x - y|$ and $K = [0, 1]$. Define $W : X \times X \times [0, 1] \to X$ by

$$W(x, y, \zeta) = \zeta x + (1 - \zeta)y,$$

for $x, y \in X$, $\zeta \in [0, 1]$. Then $(X, d, W,)$ is a complete uniformly hyperbolic space with monotone modulus of uniform convexity and K is non-empty compact convex subset of X. Define a mapping $T : [0, 1] \to CB([0, 1])$ by

$$Tx = \begin{cases} [0, 1 - x], \ x \in [0, \frac{1}{6}), \\ [0, \frac{x+5}{6}], \ x \in [\frac{1}{6}, 1]. \end{cases}$$

First, we prove that T is multi-valued Suzuki nonexpansive mapping. For this, we consider following cases:
Case I: when $x \in [0, \frac{1}{6})$. Then

$$\begin{aligned} \frac{1}{2}d(x, Tx) &= \frac{1}{2}d(x, [0, 1 - x]) \\ &= \frac{1}{2}|x - 1 + x| \\ &\leq \frac{1}{2} + |x|. \end{aligned}$$

Suppose that

$$\begin{aligned} &\frac{1}{2}d(x, Tx) \leq d(x, y) \\ &\Rightarrow \frac{1}{2} + |x| \leq |x| + |y| \\ &\Rightarrow |y| \geq \frac{1}{2}. \end{aligned}$$

Hence $y \in [\frac{1}{2}, 1] \subset [0,1]$. Now for $y \in [\frac{1}{2}, 1]$,

$$\begin{aligned} d_H(Tx, Ty) &= |Tx - Ty| \\ &= |\frac{y+5}{6} - (1-x)| \\ &= |\frac{y+6x-1}{6}| \\ &\leq \frac{1}{6}, \\ d(x,y) &= |x-y| \\ &\leq \frac{1}{2}. \end{aligned}$$

Clearly $\frac{1}{2}d(x,Tx) \leq d(x,y) \Rightarrow d_H(Tx,Ty) \leq d(x,y)$.

Case II: when $x \in [\frac{1}{6}, 1]$. Then

$$\begin{aligned} \frac{1}{2}d(x,Tx) &= \frac{1}{2}d(x,[0,\frac{x+5}{6}]) \\ &= \frac{1}{2}|\frac{5-5x}{6}| \\ &= \frac{5}{12}|1-x|. \end{aligned}$$

Again, we have two possibilities. If $x < y$, then by assuming that

$$\begin{aligned} &\frac{1}{2}d(x,Tx) \leq d(x,y) \\ \Rightarrow &\frac{5}{12}(1-x) \leq y - x \\ &\Rightarrow y \geq \frac{5+7x}{12}. \end{aligned}$$

Hence $y \in [\frac{37}{72}, 1] \subset [0,1]$.
If $x > y$, then

$$\begin{aligned} &\frac{1}{2}d(x,Tx) \leq d(x,y) \\ \Rightarrow &\frac{5}{12}(1-x) \leq x - y \\ &\Rightarrow y \leq \frac{17x-5}{12}. \end{aligned}$$

Hence $y \in [\frac{-13}{72}, 1] \subset [\frac{1}{6}, 1] \subset [0,1]$. Therefore for $y \in [0,1]$, we have

$$y \leq \frac{17x-5}{12}$$
$$\Rightarrow x \geq \frac{12y+5}{17}.$$

Hence $x \in [\frac{5}{17}, 1] \subset [0,1]$. So, the case is $x \in [\frac{5}{17}, 1]$, $y \in [0,1]$. Now, choose $x \in [\frac{5}{17}, 1]$, $y \in [0, \frac{1}{6})$, then

$$\begin{aligned} d_H(Tx, Ty) &= |\frac{x+5}{6} - (1-y)| \\ &= |\frac{x+6y-1}{6}| \\ &\leq \frac{1}{6}, \\ d(x,y) &\leq 1. \end{aligned}$$

Hence, $\frac{1}{2}d(x, Tx) \leq d(x,y) \Rightarrow d_H(Tx, Ty) \leq d(x,y)$.

Next, we assume that $\{x_k\}$ is a sequence in K defined by (1.4). Start with initial value $x_1 = y_0 = \frac{1}{2}$ and $\alpha = \frac{2}{3}$. Then $F(T) = [0, \frac{x+5}{6}]$. From Lemma 2.12, we have $F(T) = F(P_T) = [0, \frac{x+5}{6}]$.
Let $u_0 \in P_T(x_1 = y_0) \Rightarrow u_0 \in P_T(\frac{x_1+5}{6})$, i.e., $u_0 = \frac{11}{12}$. Now

$$\begin{aligned} y_1 &= (1-\alpha)u_0 + \alpha y_0 \\ &= \frac{23}{36}. \end{aligned}$$

Now choose $x_2 = y_1 = \frac{23}{36}$, and let $u_1 \in P_T(x_2) \Rightarrow u_1 \in [\frac{x_2+5}{6}]$, i.e., $u_1 = \frac{203}{216}$. Then

$$\begin{aligned} y_2 &= (1-\alpha)u_1 + \alpha y_1 \\ &= \frac{479}{648}. \end{aligned}$$

Hence, $x_3 = y_2 = \frac{479}{648}$. Continuing this process, we get $x_1 < 1$, $x_2 < 1$, $x_3 < 1$,...,$x_k < 1$, and so on. Hence, sequence $\{x_k\}$ converges strongly to a common point of $F(T)$ and this point will be $\Delta-$ limit of the sequence. Hence, from Theorem 3.7, we have $\lim_{k\to\infty} d(x_k, F(T)) = 0$ and immediately Theorem 3.6 follows.

5. Application

Let $X = \mathbb{R}^n$ and $d : X \times X \to \mathbb{R}$ defined by

$$d(x, y) = \max |x_j - y_j| \tag{5.1}$$

Let $T : X \to X$ defined by

$$Tx = Cx + b, \tag{5.2}$$

where $C = [c_{jk}]$ be a $n \times n$ matrix, b is the fixed vector of X. Equation (5.2) can be written in component form as

$$Tx_j = \sum_{k=1}^{n} c_{jk} x_{jk} + \beta_j, \tag{5.3}$$

$b = (\beta_j)$, $j = 1, 2, ..., n$. Finding solution of system of equation (5.3) is equivalent to finding fixed points of T.

Theorem 5.1. *Let $X = \mathbb{R}^n$ and $d_H : X \times X \to \mathbb{R}$ defined by*

$$d_H(x, y) = \max |x_j - y_j| \tag{5.4}$$

and $T : X \to CB(X)$ defined by (5.2) with the assumption that $|C| \leq 1$. Let $\{x_k\}$ be a sequence in X defined by (1.4) such that it $\Delta-$ converges to a common point $p \in F(T)$ which is solution of the system of equation (5.3).

Proof. By using Lemma 2.16, we have

$$\begin{aligned} d_H(Tx, Tz) &= \max |Tx_j - Tz_j| \\ &= \max |\sum_{k=1}^{n} (c_{jk} x_{jk} - c_{jk} z_{jk})| \\ &= |\sum_{k=1}^{n} c_{jk}| \max |x_{jk} - z_{jk}| \\ &= |C| d_H(x, z) \\ &\leq d(x, z). \end{aligned}$$

It conclude that T is multivalued nonexpansive mapping. Obviously T satisfy condition (C), i.e., T is Suzuki nonexpansive mapping. Also by Lemma 3.1, T is quasi-nonexpansive mapping. Therefore $F(T) \neq \emptyset$. Let $p \in F(T)$. So by assumption there is a sequence $\{x_k\}$ in X such that it $\Delta-$ converges to a common point $p \in F(T)$ which is solution of the system of equation (5.3). □

6. Conflict of Interest

The author's declare that they have no conflict of interest.

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