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Characterization of a special case of hom-Lie superalgebra II

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Abstract. In this paper, we introduce the notion of sympathetic hom-Lie superalgebras. We prove some results on sympathetic multiplicative hom-Lie superalgebras with surjective α. In particular, we find some equivalence condition in which a sympathetic graded hom-ideal is direct factor of multiplicative hom-Lie superalgebra.

Keywords: hom-Lie superalgebra, sympathetic hom-Lie superalgebra, multiplicative hom-Lie superalgebra.

1. **Introduction**

Hom-Lie algebras and quasi-hom-Lie algebras were introduced first by Hartwig, Larsson, and Silvestrov in 2003 in [17] devoted to a general method for construction of deformations and discretizations of Lie algebras of vector fields and deformations of Witt and Virasoro type algebras based on general twisted derivations (σ-derivations) obeying twisted Leibniz rule, and motivated also by the examples of q-deformed Jacobi identities in q-deformations of Witt and Visaroro algebras and in related q-deformed algebras discovered in 1990'th in string theory, vertex models of conformal field theory, quantum field theory and quantum mechanics, and q-deformed differential calculi and q-deformed homological algebra [1, 20]. In 2005, Larsson and Silvestrov introduced quasi-Lie and quasi-Leibniz algebras in [18] and graded color quasi-Lie and graded color quasi-Leibniz algebras in [19] incorporating within the same framework the hom-Lie algebras

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and quasi-hom-Lie algebras, the color hom-Lie algebras and hom-Lie superalgebras, quasi-hom-Lie color algebras, quasi-hom-Lie superalgebras, quasi-Leibniz algebras and graded color quasi-Leibniz algebras. Investigation of color hom-Lie algebras and hom-Lie superalgebras and n-ary generalizations have been further expanded recently in [2, 4, 3, 5, 6, 7, 8, 9, 10, 11, 12, 13, 16, 22, 26, 27].

In [14, 15], the complete Lie superalgebras were introduced and studied. Recently the notion of compact hom-Lie superalgebra was introduced in [8]. In this article, we introduce the notion of sympathetic hom-Lie superalgebras. We prove some results on sympathetic multiplicative hom-Lie superalgebras with surjective α. In particular, we find some equivalence condition in which a sympathetic graded hom-ideal is direct factor of multiplicative hom-Lie superalgebra.

2. Preliminaries on hom-Lie superalgebras and their representation and derivations

Throughout this article, all linear spaces are assumed to be over a field $\mathbb{K}$ of characteristic different from 2. A linear space V is said to be a G-graded by an abelian group G if, there exists a family $\{V_g\}_{g\in G}$ of linear subspaces of V such that $V = \bigoplus_{g\in G} V_g$. The elements of V_g are said to be homogeneous of degree $g \in G$. The set of all homogeneous elements of V is denoted $\mathcal{H}(V) = \bigcup_{g\in G} V_g$. A linear mapping $f : V \to V'$ of two G-graded linear spaces $V = \bigoplus_{g\in G} V_g$ and $V' = \bigoplus_{g\in G} V'_g$ is called homogeneous of degree d if $f(V_g) \subseteq V'_{g+d}$, for all $g \in G$. Homogeneous linear maps of degree zero, $f(V_g) \subseteq V'_g$ for any $g \in G$, are also called even. In $\mathbb{Z}_2$-graded linear spaces $A = A_0 \oplus A_1$, the elements of A_j are homogeneous of degree (parity) $j \in \mathbb{Z}_2$, and the set of all homogeneous elements is $\mathcal{H}(A) = A_0 \cup A_1$. The parity of a homogeneous element $x \in \mathcal{H}(A)$ is denoted $|x|$.

Definition 2.1 ([17, 23]). *Hom-Lie algebras are triples* $(\mathfrak{g}, [.,.], \alpha)$, *where* $\mathfrak{g}$ *is a linear space,* $[.,.] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ *is a bilinear map and* $\alpha : \mathfrak{g} \to \mathfrak{g}$ *is a linear map satisfying for all* $x, y, z \in \mathfrak{g}$,

$$[x, y] = -[y, x], \qquad \textit{Skew-symmetry} \quad (2.1)$$

$$[\alpha(x), [y, z]] + [\alpha(y), [z, x]] + [\alpha(z), [x, y]] = 0, \textit{Hom-Lie Jacobi identity} \quad (2.2)$$

(1) *Hom-Lie algebra is called a multiplicative hom-Lie algebra if* α *is an algebra morphism,* $\alpha([.,.]) = ([\alpha(.), \alpha(.)])$, *meaning that* $\alpha([x, y]) = [\alpha(x), \alpha(y)]$ *for any* $x, y \in \mathfrak{g}$.
(2) *Multiplicative hom-Lie algebra is called regular, if* α *is an automorphism.*

From the point of view of Hom-algebras, Lie algebras are a special subclass of Hom-Lie algebras obtained when $\alpha = id$ in Definition 2.1.

Now, we recall the notion of hom-Lie superalgebras as generalization of Lie superalgebras that were considered in [24, 25].

Definition 2.2 ([2, 19])**.** *Hom-Lie superalgebras are tripples* $(\mathfrak{g}, [.,.], \alpha)$ *which consist of* $\mathbb{Z}_2$*-graded linear space* $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$*, an even bilinear map* $[.,.] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ *and an even linear map* $\alpha : \mathfrak{g} \to \mathfrak{g}$ *satisfying the super skew-symmetry and hom-Lie super Jacobi identities for homogeneous elements* $x, y, z \in \mathcal{H}(\mathfrak{g})$*,*

$$[x, y] = -(-1)^{|x||y|}[y, x], \qquad \textit{Super skew-symmetry} \tag{2.3}$$

$$(-1)^{|x||z|}[\alpha(x), [y, z]] + (-1)^{|y||x|}[\alpha(y), [z, x]] + (-1)^{|z||y|}[\alpha(z), [x, y]] = 0. \tag{2.4}$$

Super Hom-Jacobi identity

(1) *Hom-Lie superalgebra is called multiplicative Hom-Lie superalgebra, if* α *is an algebra morphism,* $\alpha([x, y]) = [\alpha(x), \alpha(y)]$ *for any* $x, y \in \mathfrak{g}$*.*
(2) *Multiplicative hom-Lie superalgebra is called regular, if* α *is an algebra automorphism.*

In skew-symmetric hom-superalgebras, the super hom-Jacobi identity can be presented equivalently in the form of super hom-Leibniz rule for the maps $ad_x = [x, .] : \mathfrak{g} \to \mathfrak{g}$,

$$[\alpha(x), [y, z]] = [[x, y], \alpha(z)] + (-1)^{|x||y|}[\alpha(y), [x, z]]. \tag{2.5}$$

Remark 2.3. *If skew-symmetry* (2.1) *does not hold, then* (2.4) *and* (2.5) *are not necessarily equivalent, defining different Hom-superalgebra structures. The Hom-superalgebras defined by just super algebras identity* (2.5) *without requiring super hom-skew-symmetry on homogeneous elements are Leibniz Hom-superalgebras, a special class of general* Γ*-graded quasi-Leibniz algebras (color quasi-Leibniz algebras) first introduced in* [18, 19]*.*

Remark 2.4. *In any hom-Lie superalgebra,* $(\mathfrak{g}_0, [.,.], \alpha)$ *is a hom-Lie algebra since* $[\mathfrak{g}_0, \mathfrak{g}_0] \in \mathfrak{g}_0$ *and* $\alpha(\mathfrak{g}_0) \in \mathfrak{g}_0$ *and* $(-1)^{|a||b|} = (-1)^0 = 1$ *for* $a, b \in \mathfrak{g}_0$*. Thus, hom-Lie algebras can be also seen as special class of hom-Lie superalgebras when* $\mathfrak{g}_1 = \{0\}$*.*

As for all hom-superalgebras, an even homomorphism $\phi : \mathfrak{g} \to \mathfrak{g}'$ of the hom-Lie superalgebras $(\mathfrak{g}, [.,.], \alpha)$ and $(\mathfrak{g}', [.,.]', \beta)$ is said to be a homomorphism of hom-Lie superalgebras, if $\phi[u, v] = [\phi(u), \phi(v))]'$ and $\phi \circ \alpha = \beta \circ \phi$. The hom-Lie superalgebras $(\mathfrak{g}, [.,.], \alpha)$ and $(\mathfrak{g}', [.,.]', \beta)$ are isomorphic, if there is a hom-Lie superalgebra homomorphism $\phi : \mathfrak{g} \to \mathfrak{g}'$ such that ϕ be bijective [22]. Hom-subalgebras of hom-Lie superalgebra $(\mathfrak{g}, [.,.], \alpha)$ are defined as $\mathbb{Z}_2$-graded linear subspaces $I = (I \cap \mathfrak{g}_0) \oplus (I \cap \mathfrak{g}_1) \subseteq \mathfrak{g}$ closed under both α and $[.,.]$, that is

$\alpha(I) \subseteq I$ and $[I, I] \subseteq I$. Hom-subalgebra I is called a hom-ideal of the hom-Lie superalgebra $\mathfrak{g}$, if $[I, \mathfrak{g}] \subseteq I$, and notation $I \triangleleft \mathfrak{g}$ is used in this case.

Hom-Lie subalgebra I of a hom-Lie superalgebra is called commutative if $[I, I] = 0$. If I is not abelian, then $[x, y] \neq 0$ for some non-zero elements $x, y \in I$.

Definition 2.5 ([21])**.** *The center of a hom-Lie superalgebra* $\mathfrak{g}$ *is defined as*

$$C(\mathfrak{g}) = \{x \in \mathfrak{g} : [x, \mathfrak{g}] = 0\}.$$

The centralizer of a hom-ideal I *in a hom-Lie superalgebra* $\mathfrak{g}$ *is defined as*

$$C_{\mathfrak{g}}(I) = \{x \in \mathfrak{g} : [x, I] = 0\}.$$

In any hom-Lie superalgebra $(\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1, [.,.], \alpha)$, the center is the centraliser of hom-ideal $\mathfrak{g}$ in $(\mathfrak{g}, [.,.], \alpha)$, that is $C(\mathfrak{g}) = C_{\mathfrak{g}}(\mathfrak{g})$.

Lemma 2.6. *Let* $(\mathfrak{g}, [.,.], \alpha)$ *be a hom-Lie superalgebra. If* $(\mathfrak{g}, [.,.], \alpha)$ *is a multiplicative hom-Lie superalgebra with surjective* α*, that is* $\alpha([.,.]) = [\alpha(.), \alpha(.)]$ *and* $\alpha(\mathfrak{g}) = \mathfrak{g}$*, then the center* $C(\mathfrak{g})$ *is a commutative hom-ideal in* $(\mathfrak{g}, [.,.], \alpha)$*.*

Proof. The hom-supersubspace $C(\mathfrak{g}) = (C(\mathfrak{g}) \cap \mathfrak{g}_0) \oplus (C(\mathfrak{g}) \cap \mathfrak{g}_1)$ of the hom-Lie superalgebra $(\mathfrak{g}, [.,.], \alpha)$ is closed under $[.,.]$ and α. Indeed, $\alpha(C(\mathfrak{g})) \subseteq C(\mathfrak{g})$, since the preimage set $\alpha^{-1}(y) \neq \emptyset$ of any $y \in \mathfrak{g}$ is non-empty by surjectivity of α, and

$$\forall\ x \in C(\mathfrak{g}), y \in \mathfrak{g} : [\alpha(x), y] = [\alpha(x), \alpha(\alpha^{-1}(y))] = \alpha([x, \alpha^{-1}(y)]) = \alpha(\{0\}) = \{0\}.$$

Moreover, $[C(\mathfrak{g}), C(\mathfrak{g})] = [C(\mathfrak{g}), \mathfrak{g}] = \{0\} \subseteq C(\mathfrak{g})$ by definition of the center. Hence, $C(\mathfrak{g})$ is commutative hom-ideal. □

Lemma 2.7. *Let* $(\mathfrak{g}, [.,.], \alpha)$ *be a multiplicative hom-Lie superalgebra,* $(\alpha([.,.]) = [\alpha(.), \alpha(.)])$*. If* I *is a hom-ideal* I *in* $(\mathfrak{g}, [.,.], \alpha)$ *such that* α *is surjective on* I*, that is* $\alpha(I) = I$*, then*

(1) $C_{\mathfrak{g}}(I)$ *is a hom-ideal in hom-Lie superalgebra* $(\mathfrak{g}, [.,.], \alpha)$*.*
(2) $C(I) = C_I(I)$ *is a commutative hom-ideal in the hom-Lie superalgebra* $(I, [.,.]_I, \alpha_I)$*, where* $[.,.]_I$ *and* α_I *are restrictions of* $[.,.]$ *and* α *to* I*.*
(3) *If* $(\mathfrak{g}, [.,.], \alpha)$ *is a multiplicative hom-Lie superalgebra with surjective* α*, that is* $\alpha([.,.]) = [\alpha(.), \alpha(.)]$ *and* $\alpha(\mathfrak{g}) = \mathfrak{g}$*, then the center* $C(\mathfrak{g})$ *is a commutative hom-ideal in* $(\mathfrak{g}, [.,.], \alpha)$*.*

Proof. For any hom-ideal I, the hom-supersubspace $C_{\mathfrak{g}}(I) = (C_{\mathfrak{g}}(I)) \cap \mathfrak{g}_0) \oplus (C_{\mathfrak{g}}(I)) \cap \mathfrak{g}_1)$ of the hom-Lie superalgebra $(\mathfrak{g}, [.,.], \alpha)$ is closed under $[.,.]$ if $\alpha(I) = I$, since by super hom-Jacobi identity (2.5), definition of the centralizer, and the condition $I = \alpha(I)$ of surjectivity of the restriction of α on I,

$$\forall\ x \in I \cap H(\mathfrak{g}),\ y, z \in C_{\mathfrak{g}}(I) \cap H(\mathfrak{g}) :$$
$$[x, y] = 0,\ [\alpha(y), [x, z]] = [\alpha(y), 0] = 0,$$

which yields

$$[\alpha(x),[y,z]] = [[x,y],\alpha(z)] + (-1)^{|x||y|}[\alpha(y),[x,z]] = 0.$$

Thus

$$[I,[C_{\mathfrak{g}}(I),C_{\mathfrak{g}}(I)]] \overset{\alpha(I)=I}{=} [\alpha(I),[C_{\mathfrak{g}}(I),C_{\mathfrak{g}}(I)]] = \{0\}$$

which give us

$$[C_{\mathfrak{g}}(I),C_{\mathfrak{g}}(I)] \subseteq C_{\mathfrak{g}}(I).$$

The hom-supersubspace $C_{\mathfrak{g}}(I) = (C_{\mathfrak{g}}(I))\cap\mathfrak{g}_0)\oplus(C_{\mathfrak{g}}(I))\cap\mathfrak{g}_1)$ is closed under α, since $\mathfrak{H}$ of the centraliser, surjectivity $\alpha(I) = I$ of α on I and multiplicativity of α yield

$$\begin{aligned}&\forall\ x \in C_{\mathfrak{g}}(I)\cap H(\mathfrak{g}):\\&[\alpha(C_{\mathfrak{g}}(I)),I] = [\alpha(C_{\mathfrak{g}}(I)),\alpha(I)] = \alpha([C_{\mathfrak{g}}(I),I]) \in \alpha(\{0\}) = \{0\}\end{aligned}$$

which yields

$$\alpha(C_{\mathfrak{g}}(I)) \in C_{\mathfrak{g}}(I).$$

Thus, $C_{\mathfrak{g}}(I)$ is a hom-supersubalgebra in the hom-superalgebra $(\mathfrak{g},[.,.],\alpha)$. Moreover,

$$\begin{aligned}&\forall\ x \in I\cap H(\mathfrak{g}),\ y \in \mathfrak{g}\cap H(\mathfrak{g}), z \in C_{\mathfrak{g}}(I)\cap H(\mathfrak{g}):\\&[x,y] \in I,\ [\alpha(y),[x,z]] = [\alpha(y),0] = 0,\end{aligned}$$

which yields

$$[\alpha(x),[y,z]] = [[x,y],\alpha(z)] + (-1)^{|x||y|}[\alpha(y),[x,z]] \in I.$$

Thus

$$[I,[\mathfrak{g},C_{\mathfrak{g}}(I)]] \overset{\alpha(I)=I}{=} [\alpha(I),[\mathfrak{g},C_{\mathfrak{g}}(I)]] \in I$$

which give us

$$[\mathfrak{g},C_{\mathfrak{g}}(I)] \subseteq C_{\mathfrak{g}}(I).$$

Hence, $C_{\mathfrak{g}}(I)$ is a hom-ideal. □

Now, we need the following definition throughout the rest of the paper.

Definition 2.8 ([4, 19])**.** *A representation of the hom-Lie superalgebra* $(\mathfrak{g},[.,.],\alpha)$ *on a* $\mathbb{Z}_2$*-graded linear space* $V = V_{\bar{0}}\oplus V_{\bar{1}}$ *with respect to* $\beta \in gl(V)_{\bar{0}}$ *is an even linear map* $\rho : \mathfrak{g} \to gl(V)$*, such that for all homogeneous* $x, y \in \mathcal{H}(\mathfrak{g})$*,*

$$\begin{aligned}\rho(\alpha(x))\circ\beta &= \beta\circ\rho(x),\\ \rho([x,y])\circ\beta &= \rho(\alpha(x))\circ\rho(y) - (-1)^{|x||y|}\rho(\alpha(y))\circ\rho(x).\end{aligned}$$

A representation V *of* $\mathfrak{g}$ *is called irreducible or simple, if it has no nontrivial subrepresentations. Otherwise* V *is called reducible.*

For any linear transformation $T : X \mapsto X$ of a set X, and any nonnegative integer s, the s-times composition is $T^s = T \circ \cdots \circ T \quad (\text{s}-times)$, $\quad T^0 = Id$, $\quad T^1 = T$, and if T is invertible with inverse map $T^{-1}\mathfrak{g} \to \mathfrak{g}$, then $T^{-s} = T^{-1} \circ \cdots \circ T^{-1} \quad (s - times)$.

Next, we recall the notion of α^s-derivations.

Definition 2.9 ([4]). *Let $(\mathfrak{g}, [.,.]_{\mathfrak{g}}, \alpha)$ be a hom-Lie superalgebra. For any nonnegative integer s, we call $D \in (End(\mathfrak{g}))_i$, where $i \in \mathbb{Z}_2$, an α^s-derivation of the multiplicative hom-Lie superalgebra $(\mathfrak{g}, [.,.]_{\mathfrak{g}}, \alpha)$, if for all homogeneous $x, y \in \mathcal{H}(\mathfrak{g})$,*

$$D \circ \alpha = \alpha \circ D, \tag{2.6}$$

$$D([x, y]_{\mathfrak{g}}) = [D(x), \alpha^s(y)]_{\mathfrak{g}} + (-1)^{|D||x|}[\alpha^s(x), D(y)]_{\mathfrak{g}}. \tag{2.7}$$

For any $x \in \mathfrak{g}$ satisfying $\alpha(x) = x$, the mapping $ad_s(x) : \mathfrak{g} \to \mathfrak{g}$ defined for all $y \in \mathfrak{g}$ by $ad_s(x)(y) = [x, \alpha^s(y)]_{\mathfrak{g}}$, is a α^{s+1}-derivation, called an inner α^{s+1}-derivation [4], and the set $Inn_{\alpha^{s+1}}(\mathfrak{g}) = \{[x, \alpha^s(.)]_{\mathfrak{g}} \mid \ x \in \mathfrak{g}, \ \alpha(x) = x\}$ is a linear space in $Der_{\alpha^{s+1}}(\mathfrak{g})$.

Now, by using the above defenitions and lemmas, we generalized some results from hom-Lie superalgebras to sympathetic hom-Lie superalgebras which are expressed in the next section.

3. Sympathetic hom-Lie superalgebras

In this section we introduce the notion of a sympathetic hom-Lie superalgebra and we state some results about it.

Definition 3.1. *A hom-Lie superalgebra $\mathfrak{g}$ is called sympathetic hom-Lie superalgebra if $\mathfrak{g}$ satisfies the following two conditions.*

- $C(\mathfrak{g}) = 0$
- $[\mathfrak{g}, \mathfrak{g}] = \mathfrak{g}$
- $Der_{\alpha^{t+1}}(\mathfrak{g}) = ad_t(\mathfrak{g})$,

for any nonnegative integer t.

Now, we define the notion of direct factor and characteristic hom-ideal as below.

Definition 3.2. *Let $\mathfrak{g}$ be a hom-Lie superalgebra, A be a graded hom-ideal of $\mathfrak{g}$. Then A is said to be a direct factor if there exists a graded hom-ideal B of $\mathfrak{g}$ such that $\mathfrak{g} = A \oplus B$.*

Definition 3.3. *Let $\mathfrak{g}$ be a hom-Lie superalgebra, A be a graded subspace of $\mathfrak{g}$. Then A is called characteristic hom-ideal if for every $D \in Der_{\alpha^{t+1}}(\mathfrak{g}), D(A) \subseteq A$.*

Lemma 3.4. *Let $(\mathfrak{g}, [.,.], \alpha)$ be a multiplicative hom-Lie superalgebra with surjective α and A be a graded hom-ideal of $\mathfrak{g}$. If A is perfect, then A is a characteristic hom-ideal of $\mathfrak{g}$.*

Proof. Let $D \in Der_{\alpha^{t+1}}(\mathfrak{g})$ and $x, y \in A$. Then by defenition of α^t-derivation we have

$$D([x,y]) = [D(x), \alpha^t(y)] + (1)|D||x|[\alpha^s(x), D(y)] \in A.$$

Since A is perfect, then we have $D(A) \subseteq A$. Thus A is a characteristic hom-ideal. □

By above notation, we have the following proposition.

Proposition 3.5. *Let $(\mathfrak{g}, [.,.], \alpha)$ be a perfect multiplicative hom-Lie superalgebra with surjective α , A be a graded hom-ideal of $\mathfrak{g}$. If A is a direct factor of $\mathfrak{g}$, then A is perfect.*

Proof. Since A is a direct factor of $\mathfrak{g}$, then there exists a graded hom-ideal B of $\mathfrak{g}$ such that $\mathfrak{g} = A \oplus B$, in particular, $[A, B] = \{0\}$. It follows that $[\mathfrak{g}, \mathfrak{g}] = [A, A] \oplus [B, B]$. So both A and B are perfect. □

By above defenitions and propositions, one can easily check the following proposition.

Proposition 3.6. *Let $(\mathfrak{g}, [.,.], \alpha)$ be a multiplicative hom-Lie superalgebra with surjective α and trivial center. Let A be a direct factor of $\mathfrak{g}$. Then $C(A) = \{0\}$.*

Now, we consider the sympathetic hom-Lie superalgebra to state some results.

Proposition 3.7. *Let $(\mathfrak{g}, [.,.], \alpha)$ be a multiplicative hom-Lie superalgebra with surjective α and A be a sympathetic graded hom-ideal of $\mathfrak{g}$. Then there exists a graded hom-ideal B such that $\mathfrak{g} = A \oplus B$.*

Proof. Let $B = C_{\mathfrak{g}}(A)$. Since α is surjective then $C_{\mathfrak{g}}(A)$ is a graded hom-ideal of $\mathfrak{g}$. We know that $A \lhd \mathfrak{g}$, then For any $x \in \mathfrak{g}$, $ad_t(x) \in Der_{\alpha^{t+1}}(A)$. By using $Der_{\alpha^{t+1}}(A) = ad_t(A)$, there exists a derivation D in $Der_{\alpha^{t+1}}(A)$ such that $ad_t(x) = D$. So there exists $y \in A$ such that

$$D(z) = [x, \alpha(z)] = [y, \alpha(z)],$$

for all $z \in A$. Then $[x - y, \alpha(z)] = 0$ and then $x - y \in C_{\mathfrak{g}}(A) = B$. Hence $x = b + y$ for some b $\in B$. $A \cap B = A \cap C_{\mathfrak{g}}(A) = C_A(A) = \{0\}$, since A is sympathetic. Therefore $\mathfrak{g} = A \oplus B$. □

Proposition 3.8. *Let $(\mathfrak{g}, [.,.], \alpha)$ be a sympathetic multiplicative hom-Lie superalgebra with surjective α and A be a graded hom-ideal of $\mathfrak{g}$. Then A is a direct factor of $\mathfrak{g}$ if and only if A is sympathetic.*

Proof. Let A is a direct factor of $\mathfrak{g}$, then there exists a graded hom-ideal B of $\mathfrak{g}$ such that $\mathfrak{g} = A \oplus B$. Therefore $\mathfrak{g} = [\mathfrak{g}, \mathfrak{g}] = [A, A] \oplus [B, B]$ and $[A, A] = A$. Let $D \in Der_{\alpha^{t+1}}(A)$ and $d : \mathfrak{g} \to \mathfrak{g}$ is defined as a linear map by $d|_A = D$, $d|B = 0$. So, D belongs to $Der_{\alpha^{t+1}}(\mathfrak{g})$. Then there exists $g \in \mathfrak{g}$ such that $d = ad_t(\mathfrak{g})$. Since $\mathfrak{g} = A \oplus B$ and $g = g_i + g_j$, for $g_i \in A$ and $g_j \in B$, therefore $D = ad_t(g_i)$. Thus we have $Der_{\alpha^{t+1}}(A) = ad_t(A)$. finally by Proposition 3.6 we have $C(A) = \{0\}$. Therefore A is sympathetic.
Conversely, it is a direct results of the Proposition 3.7. □

By using above Proposition, we have the following consequences immediately.

Corollary 3.9. *Let $(\mathfrak{g}, [.,.], \alpha)$ be a multiplicative hom-Lie superalgebra with surjective α and A be a sympathetic graded hom-ideal of $\mathfrak{g}$. Then A is a direct factor of $\mathfrak{g}$.*

Corollary 3.10. *Let $(\mathfrak{g}, [.,.], \alpha)$ be a multiplicative hom-Lie superalgebra with surjective α and A be a sympathetic graded hom-ideal of $\mathfrak{g}$. If $Der_{\alpha^{t+1}}(\mathfrak{g}) = ad_t(\mathfrak{g})$, then $Der_{\alpha^{t+1}}(A) = ad_t(A)$.*

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