Journal of Finsler Geometry and its Applications
Vol. 3, No. 1 (2022), pp 49-65
DOI: 10.22098/jfga.2022.10470.1062

Geometric Analysis of the Lie Algebra of Killing Vector Fields for a Significant Cosmological Model of Rotating Fluids

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Abstract. The investigation of rotating fluids in the context of general relativity received remarkable consideration principally after Godel proposed relativistic model of a rotating dust universe. In this paper, a comprehensive analysis regarding the structure of the Lie algebra of Killing vector fields for a specific solution of field equations describing the behaviour of rotating fluid models is presented. For this purpose, we specifically concentrate on detailed investigation of the Killing vector fields by reexpressing the analyzed cosmological solution in the orthogonal frame. Significantly, for the resulted Lie algebra of Killing vector fields, the associated basis for the original Lie algebra is determined in which the Lie algebra will be appropriately decomposed into an internal direct sum of subalgebras, where each summand is indecomposable. Ultimately, the preliminary group classification of the symmetry algebra of the killing vector fields is presented. This noteworthy objective is thoroughly fulfilled via constructing the adjoint representation group, which generically insinuates a conjugate relation in the set of all one-dimensional subalgebras. Consequently, the corresponding set of invariant solutions can be reckoned canonically as the mimimal list from which all the other invariant solutions of one-dimensional subalgebras are comprehensively designated unambiguously by virtue of transformations.

Keywords: Killing vector fields, Five dimensional spacetime, Rotating fluids, Orthogonal frame, Adjoint representation.

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AMS 2020 Mathematics Subject Classification: 53Z05, 83C20

1. **Introduction**

Killing vector fields can be regarded as one of the most significant types of symmetries and are considered as the smooth vector fields which preserve the metric tensor. These vector fields are extensively applied in various physical fields including in classical mechanics and are closely related to conservation laws. Specifically, remarkable applications of Killing vector fields in relativistic theories is undeniable. The noticeable fact is that the flow corresponding to a Killing vector field generates a symmetry in a way that if each point moves on an object at the same distance in the direction of the Killing vector field then distances on the object will not distorted at all. In particular, a vector field K is a Killing field if the Lie derivative with respect to K of the metric g vanishes. Moreover, the Lie bracket of two Killing vector fields is still a Killing field and the Killing fields on a manifold M thus form a Lie subalgebra of vector fields on M which can be considered as the isometry group of the manifold whenever M is complete [3, 4]. Taking into account the significant properties declared above, one naturally expects Killing vectors to be of substantial use in the study of geodesic motion. When one investigates the Lagrangian explaining the motion of a particle, one can realize that Killing vectors are the symmetries of the system and lead to conserved canonical momenta analogous to cyclic coordinates in classical mechanics. Furthermore, One can also try to obtain another conserved quantity related to the spacetime itself, if the background metric contains globally well defined Killing vectors [1, 9].

The Kaluza-Klein theory can be profoundly reckoned as a precursor to string theory and is uncontrovertibly the first illustration of a unification theory by adding an extra dimension. Concretely, Kaluza-klein theories depicts a physically feasible strategy for the unification of gravity with other interactions which is fundamentally derived from the hypothesis that the gauge symmetries are intrinsically geometrical. Predominantly, after Godel's relativistic scheme for rotating dust universe, inquest of rotating fluids in the context of general relativity has received remarkable consideration (refer to [2, 7] for extra details).

Considering the fact that stationary Kaluza-Klein perfect fluid models in standard Einstein theory are not available in literature, obtaining and analyzing such solutions is so constructive in order to investigate the effects of dimensionality on the different physical parameters.

In [8], R. Tikekar and L. K. Patel have formulated the Kaluza-Klein field equations for cylindrically symmetric rotating distributions of perfect fluid. They have reported a set of physically viable solutions which is believed to be the first such Kaluza-Klein solutions and it includes the Kaluza-Klein counterpart of Davidson's solution.

In the following, according to [8], we will present a brief description of Kaluza-Klein field equations for stationary cylindrically symmetric fluid models in standard Einstein theory. For further complete information refer to [8].

A general stationary cylindrically symmetric five dimensional spacetime is denoted by the following metric:

$$ds^2 = D^2(dt + Hd\phi)^2 - A^2dr^2 - B^2dz^2 - r^2C^2d\phi^2 - E^2d\psi^2, \tag{1.1}$$

where t is the time coordinate, r, z and ϕ are cylindrical polar coordinates, ψ represents the coordinate corresponding to the extra spatial dimension and A, B, C, D and H are functions of the radial coordinate r only. By expressing with respect to pentad

$$\theta^1 = Adr, \quad \theta^2 = Bdz, \quad \theta^3 = rCd\phi, \quad \theta^4 = Ed\psi, \quad \theta^5 = D(dt + Hd\phi), \tag{1.2}$$

the metric (1.1) has the following form:

$$ds^2 = (\theta^5)^2 - (\theta^1)^2 - (\theta^2)^2 - (\theta^3)^2 - (\theta^4)^2. \tag{1.3}$$

If the metric (1.1) is to denote the spacetime of a stationary perfect fluid rotating about the regular axis $r = 0$, the metric coefficients will be related to the dynamical variables through the Einstein field equations which are in the pentad notation applying the system of units rendering $c = G = 1$, adopted in the form

$$\begin{cases} \mathbf{R}_{(11)} = \mathbf{R}_{(22)} = \mathbf{R}_{(33)} = \mathbf{R}_{(44)} = -\dfrac{8\pi}{3}(\rho - p), \\ \mathbf{R}_{(55)} = -\dfrac{16\pi}{3}(\rho + 2p), \\ \mathbf{R}_{(35)} = 0, \end{cases} \tag{1.4}$$

The field equations comprise a system of six equations relating the two physical parameters ρ and p of the fluid and the six metric coefficients A, B, C, D, E and H. Accordingly, the equation $\mathbf{R}_{(35)} = 0$ in (1.4) yields the following two significant identities:

$$H = \alpha r^2, \tag{1.5}$$

$$a + c = 2b + 3d, \tag{1.6}$$

where α is the arbitrary constant of integration.

In [8] certain specific cases for physical relevance which follow for certain particular choices of the free parameters, are discussed. In this paper, we will comprehensively analyze the structure of the Lie algebra of Killing vector fields for the following three specific solutions which is reported in [8].

When $a = -1/2$, $b = e = c = -d = 1/4$, $\alpha^2 = k^2$, the Kaluza-Klein equations are all satisfied and the spacetime of this class of solutions has the metric

$$\begin{aligned} ds^2 = (1 + k^2r^2)^{-1/2}(dt + kr^2d\phi)^2 - (1 + k^2r^2)^{-1}dr^2 \\ -(1 + k^2r^2)^{1/2}\big(dz^2 + r^2d\phi^2 + d\psi^2\big). \end{aligned} \tag{1.7}$$

which denotes a five dimensional spacetime of a cylindrically symmetric stationary fluid with constant density and pressure related by this equation of state: $\rho + p = 0$. By setting $\Lambda = -(3/2)k^2$, the metric above denotes a five dimensional solutions of the field equations: $\mathbf{R}_{ij} = \Lambda g_{ij}$, where Λ represents the cosmological constant.

This paper is organized as follows: In the next section, we have specifically concentrated on complete investigation of the problem of Killing vector fields for our analyzed physically viable five dimensional cosmological solution. First of all, by considering the Lagrangian which is determined directly from the metric, we will compute the corresponding symmetries which preserve the metric tensor. Additionally, the flow corresponding to a Killing vector field generates a symmetry in a way that if each point moves on an object at the same distance in the direction of the Killing vector field then distances on the object will not distorted at all. Therefore, Killing vector fields are inherently expected to be of significant application in the study of geodesic motion. When one investigates the Lagrangian explaining the motion of a particle, one can realize that Killing vectors are the symmetries of the system and lead to conserved canonical momenta analogous to cyclic coordinates in classical mechanics.

Taking into account the outstanding properties declared above, the next section of this paper is particularly devoted to detailed investigation of the Killing vector fields by reexpressing the metric (1.7) in the orthogonal frame. In addition, the Lie algebra of the resulted Killing vector fields are redemonstrated in a new basis which leads to decomposition of the original Lie symmetry algebra into an internal direct sum of subalgebras, where each summand is indecomposable. Significantly, in section three, the preliminary group classification of the symmetry algebra of the killing vector fields is presented. This noteworthy objective is thoroughly fulfilled via constructing the adjoint representation group, which generically insinuates a conjugate relation in the set of all one-dimensional subalgebras. Meanwhile, some concluding remarks are declared at the end of the paper.

2. Classification of Killing Vector Fields

Let (M, g) be an arbitrary Lorentzian manifold and $\Im$ be a smooth vector field on M. A curve $\gamma : \mathbb{R} \longrightarrow M$ whose tangent vector at every point $p \in \gamma$ is equal to $\Im$ is denoted by an integral curve of $\Im$. Significantly, for a given congruence a one-parameter family of diffeomorphisms from M onto itself can be associated which is described as follows: corresponding to each $s \in \mathbb{R}$, designate a map $\mathcal{F}_s : M \longrightarrow M$, where $\mathcal{F}_s(p)$ is the point parameter distance s from p along $\Im$, i.e. if $p = \gamma(\wp_0)$ then $\mathcal{F}_s(p) = \gamma(\wp_0 + s)$. Furthermore, from the algebraic point of view, considering the composition law $\mathcal{F}_s o \mathcal{F}_t = \mathcal{F}_{s+t}$, the identity $\mathcal{F}_0$ and the inverse $(\mathcal{F}_s)^{-1} = \mathcal{F}_{-s}$, these transformations construct an abelian group. Specifically to a metric tensor g on M the Lie derivative is defined by:

$$\left(\mathcal{L}_{\Im} g\right)_p = \lim_{\delta\wp \longrightarrow 0} \frac{g_p - (\mathcal{F}_{\delta\wp})^* g_{\mathcal{F}_{\delta\wp}}(p)}{\delta\wp} \tag{2.1}$$

The remarkable fact is that the Lie derivative of g entails the pull-back $\mathcal{F}_s^*$ which maps a covector at $\mathcal{F}_s(p)$ to a covector at p, mainly due to the fact that the components of g transform covariantly. It can be demonstrated that:

$$\left(\mathcal{L}_{\Im} g\right)_{\mu\nu} = \nabla_\mu \Im_\nu + \nabla_\nu \Im_\mu. \tag{2.2}$$

Meanwhile, if the metric does not change under the transformation $\mathcal{F}_s$, the transformation is called an isometry and that the metric possesses a symmetry. In this case $\mathcal{L}_{\Im} g = 0$, which leads to the following identity [4, 9]:

$$\nabla_\mu \Im_\nu + \nabla_\nu \Im_\mu = 0 \tag{2.3}$$

This relation is denoted by Killing's equation and a vector $\Im$ which satisfies (2.3) is called a Killing vector. It is noticeable that this identity contains the metric implicity which is hidden in ∇. In addition, the symmetries of a spacetime explicitly leads to determining the vectors which satisfy the Killing equation; this can be thoroughly fulfilled either by inspection or via integrating (2.3). An isometry is a distance preserving mapping among different spaces.
In this section, we apply an orthogonal frame to obtain the Killing vector fields for the metric (1.7). First of all, we set up a five dimensional spacetime with coordinates $\left[t, r, z, \phi, \psi\right]$ denoted by ϖ given by:

$$\begin{aligned} \varpi = \Bigg[& \frac{1}{\sqrt{k^2r^2+1}}\ dr, \left(k^2r^2+1\right)^{1/4}\ dz,\ \left(k^2r^2+1\right)^{1/4}\ dx, \\ & \left(k^2r^2+1\right)^{1/4}\ dy, \frac{1}{\left(k^2r^2+1\right)^{1/4}}\ dt + \frac{kr^2}{\left(k^2r^2+1\right)^{1/4}}\ dx \Bigg] \end{aligned} \tag{2.4}$$

Then we define a coframe and calculate the structure equations for this coframe as follows:

$$\begin{cases} d\Theta_1 = 0, \quad d\Theta_2 = \dfrac{k^2 r}{2\sqrt{k^2r^2+1}}\ \Theta_1 \wedge \Theta_2, \\ d\Theta_3 = \dfrac{3k^2r^2+2}{2r\sqrt{k^2r^2+1}}\ \Theta_1 \wedge \Theta_3, \\ d\Theta_4 = \dfrac{k^2 r}{2\sqrt{k^2r^2+1}}\ \Theta_1 \wedge \Theta_4, \\ d\Theta_5 = 2k\ \Theta_1 \wedge \Theta_3 - \dfrac{k^2 r}{2\sqrt{k^2r^2+1}}\ \Theta_1 \wedge \Theta_5. \end{cases} \tag{2.5}$$

Therefore, we can state the following theorem:

Theorem 2.1. *Taking into account the pentad (1.2), the metric (1.7) is expressed by (1.3) in the orthogonal frame. Subsequently, the following seven Killing vectors corresponding to the metric (1.7) are resulted in the adapted frame:*

$$
\begin{cases}
(\mathbf{1}) : \mathbf{K_1} = t\Big(k^2r^2+1\Big)^{1/4} \ \mathbf{E_2} + \Big(k^2r^2+1\Big)^{1/4} \ kzr \ \mathbf{E_3} \\
\qquad\qquad +z\Big(k^2r^2+1\Big)^{3/4} \ \mathbf{E_5}, \\
(\mathbf{2}) : \mathbf{K_2} = \Big(k^2r^2+1\Big)^{1/4} \ kyr \ \mathbf{E_3} + \Big(k^2r^2+1\Big)^{1/4} \ t \ \mathbf{E_4} \\
\qquad\qquad +y\Big(k^2r^2+1\Big)^{3/4} \ \mathbf{E_5}, \\
(\mathbf{3}) : \mathbf{K_3} = \dfrac{1}{\Big(k^2r^2+1\Big)^{1/4}} \ \mathbf{E_5}, \\
(\mathbf{4}) : \mathbf{K_4} = \dfrac{\Big(k^2r^2+1\Big)^{1/4} \ r}{k} \ \mathbf{E_3} + \dfrac{r^2}{\Big(k^2r^2+1\Big)^{1/4}} \ \mathbf{E_5}, \\
(\mathbf{5}) : \mathbf{K_5} = \Big(k^2r^2+1\Big)^{1/4} \ y \ \mathbf{E_2} - \Big(k^2r^2+1\Big)^{1/4} \ z \ \mathbf{E_4}, \\
(\mathbf{6}) : \mathbf{K_6} = -\Big(k^2r^2+1\Big)^{1/4} \ \mathbf{E_4}, \\
(\mathbf{7}) : \mathbf{K_7} = -\Big(k^2r^2+1\Big)^{1/4} \ \mathbf{E_2}.
\end{cases}
$$

Furthermore, here are the structure equations for the Lie algebra of Killing vectors denoted by $\mathcal{K}$:

$$[\mathbf{K_1}, \mathbf{K_2}] = -\mathbf{K_5}, \quad [\mathbf{K_1}, \mathbf{K_3}] = \mathbf{K_7}, \quad [\mathbf{K_1}, \mathbf{K_5}] = -\mathbf{K_2},$$

$$[\mathbf{K_1}, \mathbf{K_7}] = k^2 \ \mathbf{K_4} + \mathbf{K_3}, \quad [\mathbf{K_2}, \mathbf{K_5}] = \mathbf{K_1}, \quad [\mathbf{K_2}, \mathbf{K_6}] = k^2 \ \mathbf{K_4} + \mathbf{K_3},$$

$$[\mathbf{K_5}, \mathbf{K_6}] = -\mathbf{K_7}, \quad [\mathbf{K_5}, \mathbf{K_7}] = \mathbf{K_6}.$$

Significantly, due to above theorem we have:

Corollary 2.2. *By considering the following basis for the original Lie algebra of Killing vector fields $\mathcal{K}$, it will decompose into an internal direct sum of subalgebras, where each summand is indecomposable.*

$$\Big\{\mathbf{F_1}, \mathbf{F_2}, \mathbf{F_3}, \mathbf{F_4}, \mathbf{F_5}, \mathbf{F_6}, \mathbf{F_7}\Big\} := \Big\{\mathbf{K_1}, \mathbf{K_2}, \mathbf{K_3} + k^2 \ \mathbf{K_4}, \mathbf{K_5}, \mathbf{K_6}, \mathbf{K_7}, \mathbf{K_4}\Big\}$$

The expression of $\mathcal{K}$ in this new basis described above, will be denoted by $\tilde{\mathcal{K}}$. Meanwhile, $\mathbb{A}$ is a matrix which defines a Lie algebra isomorphism from $\mathcal{K}$ to $\tilde{\mathcal{K}}$ (the Lie algebra defined by the direct sum of indecomposable Lie subalgebras) given by:

$$\mathbb{A} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & -k^2 & 1 & 0 & 0 & 0 \end{pmatrix}$$

The commutator table of $\tilde{\mathcal{K}}$ is illustrated in Table 1, where the entry in the i^{th} row and j^{th} column is defined as $[\mathbf{F_i}, \mathbf{F_j}] = \mathbf{F_i}\mathbf{F_j} - \mathbf{F_j}\mathbf{F_i}$, $i, j = 1, ..., 7$.

TABLE 1. Commutation relations satisfied by infinitesimal generators for the Lie algebra $\tilde{\mathcal{K}}$

$[\ ,\ ]$	$\mathbf{F_1}$	$\mathbf{F_2}$	$\mathbf{F_3}$	$\mathbf{F_4}$	$\mathbf{F_5}$	$\mathbf{F_6}$	$\mathbf{F_7}$
$\mathbf{F_1}$	0	$-\mathbf{F_4}$	$\mathbf{F_6}$	$-\mathbf{F_2}$	0	$\mathbf{F_3}$	0
$\mathbf{F_2}$	$\mathbf{F_4}$	0	$\mathbf{F_5}$	$\mathbf{F_1}$	$\mathbf{F_3}$	0	0
$\mathbf{F_3}$	$-\mathbf{F_6}$	$-\mathbf{F_5}$	0	0	0	0	0
$\mathbf{F_4}$	$\mathbf{F_2}$	$-\mathbf{F_1}$	0	0	$-\mathbf{F_6}$	$\mathbf{F_5}$	0
$\mathbf{F_5}$	0	$-\mathbf{F_3}$	0	$\mathbf{F_6}$	0	0	0
$\mathbf{F_6}$	$-\mathbf{F_3}$	0	0	$-\mathbf{F_5}$	0	0	0
$\mathbf{F_7}$	0	0	0	0	0	0	0

3. Preliminary Group Classification of Killing Vector Fields via the Adjoint Representation

3.1. General Framework of the Preliminary Group Classification Method.

It is worth noticing that in general to each one parameter subgroups of the full symmetry group of a system there will associate a family of solutions denoted by invariant solutions. Reckoning on the point that any linear combination of infinitesimal generators is also an infinitesimal generator, there are always infinitely many distinct symmetry subgroups for a differential equation. As a consequence, in order to complete understanding the invariant solutions, a mean of characterizing which subgroups give different types of solutions is indispensable.

Let $\mathcal{G}$ be the symmetry group associated to an arbitrary system of differential equations Ω. Then for each $g \in \mathcal{G}$ the group conjugation $\mathcal{Q}_g(h) \equiv ghg^{-1},\ h \in \mathcal{G}$, explicitly characterizes a diffeomorphism on the Lie symmetry group $\mathcal{G}$. In addition, reckoning on the point that: $\mathcal{Q}_g \circ \mathcal{Q}_{\tilde{g}} = \mathcal{G}_{g\tilde{g}},\ \mathcal{Q}_e = \mathbb{I}_{\mathcal{G}}$, it is identically inferred that $\mathcal{Q}_g$ prescribes a globally defined group action of $\mathcal{G}$ on itself, with each conjugacy map $\mathcal{Q}_g$ defining intrinsically a group homomorphism as follows: $\mathcal{Q}_g\big(h\tilde{h}\big) = \mathcal{Q}_g(h)\mathcal{Q}_g(\tilde{h})$, etc. Accordingly, the corresponding differential $d\mathcal{Q}_g$: $T\mathcal{G}\mid_h \longrightarrow T\mathcal{G}\mid_{\mathcal{Q}_g(h)}$ is ungrudgingly observed to conserve the right-invariance of vector fields, and literally represents a specific linear map on the Lie algebra of $\mathcal{G}$, denoted by the adjoint representation: Ad $g(\mathbf{v}) \equiv d\mathcal{Q}_g(\mathbf{v}),\ \mathbf{v} \in \mathfrak{g}$. The noticeable point is that the adjoint representation definitely expounds a globally defined linear action on $\mathfrak{g}$ expressed by:

$$\mathrm{Ad}\big(g \cdot \tilde{g}\big) = \mathrm{Ad}\ g \circ \mathrm{Ad}\ \tilde{g}, \qquad \mathrm{Ad}\ e = \mathbb{I}.$$

Furthermore, if $\mathbf{v} \in \mathfrak{g}$ creates the associated one-parameter subgroup: $\mathcal{H} = \Big\{\exp\big(\varepsilon\mathbf{v}\big)\ :\ \varepsilon \in \mathbb{R}\Big\}$, then Ad $g(\mathbf{v})$ is straightforwardly observed to provoke the conjugate one-parameter subgroup: $\mathcal{Q}_g(\mathcal{H}) = g\mathcal{H}g^{-1}$. This observation explicitly generalizes to higher dimensional subgroups considering the point that they are totally characterized via their corresponding one-parameter subgroups (refer to [5] for more details).

Subsequently, by virtue of the significant points discussed above, it can be unequivocally deduced that: Let $\mathfrak{S}$ be a group invariant solution which is obtained from a subgroup of the symmetry group, by applying allowed group elements it can be mapped to another solution $\tilde{\mathfrak{S}}$. Two group invariant solutions $\mathfrak{S}_1$ and $\mathfrak{S}_2$ are called *nonsimilar* if they can not be transformed to each other via symmetry transformations. Let $\mathfrak{S}$ be a solution which is invariant under a subgroup $\mathcal{H}$ of the symmetry group $\mathcal{G}$, then for all $h \in \mathcal{H}$, $h\mathfrak{S} = \mathfrak{S}$. In order to determine the subgroup $\mathcal{Q}$ under which the solution $\tilde{\mathfrak{S}} = g\mathfrak{S},\ g \in \mathcal{G}$, is invariant, the fact that the invariance condition under $\mathcal{Q}$ implies that for all $q \in \mathcal{Q}$, $\tilde{\mathfrak{S}} = q\tilde{\mathfrak{S}}$, is considered. Hence,

$$\tilde{\mathfrak{S}} = g\mathfrak{S} = gh\mathfrak{S} = ghg^{-1}g\mathfrak{S} = ghg^{-1}\mathfrak{S},$$

consequently, $q = ghg^{-1}$. So, if $\mathfrak{S}$ is an $\mathcal{H}-$invariant solution, then $g\mathfrak{S}$ would be a $\mathcal{Q}-$invariant solution with $\mathcal{Q} = g\mathcal{H}g^{-1}$. Hence, it is deduced that $\mathcal{Q}$ is the *adjoint or conjugate subgroup* to $\mathcal{H}$ under the symmetry group $\mathcal{G}$. It can be readily observed that the adjoint action implies information about how group invariant solutions transform under the action of the other subgroups of the symmetry group. So, searching for the nonsimilar group invariant solutions is then reduced to the construction of nonsimilar subgroups of the symmetry group under the adjoint action. This procedure is known as the *optimal system* problem which was proposed in 1982 by Ovsiannikov [6] (see also [5]). Since Lie groups can be viewed as a sort of linearization of Lie groups, from a computational standpoint, it is more convenient to work with Lie algebras rather than Lie groups.

3.2. **One-dimensional Optimal System of the Symmetry Algebra of Killing Vector Fields.** Taking into account [5], we consider $\mathcal{H}$ and $\tilde{\mathcal{H}}$ as connected s-dimensional Lie subgroups of the Lie group $\mathcal{G}$. Besides, suppose $\mathfrak{h}$ and $\tilde{\mathfrak{h}}$ denote the analogous Lie subalgebras of the Lie algebra $\mathfrak{g}$ of $\mathcal{G}$. Consequently, $\tilde{\mathcal{H}} = g\mathcal{H}g^{-1}$ are conjugate subgroups if and only if $\tilde{\mathfrak{h}} = \mathrm{Ad}\ g(\mathfrak{h})$ are conjugate subalgebras. It is worth mentioning that the subsequent outstanding achievement of the above discussion can be unambiguously expressed as follows: For one-dimensional subalgebras indeed this classification is characteristically analogous to the classifying the orbits of the adjoint representation. Subsequently, whenever only one representative is explicitly designated from each family of equivalent subalgebras, an optimal set of subalgebras is unreservedly constituted. Ultimately, a minimal list of invariant solutions from which all the other invariant solutions of one-dimensional subalgebras can be achieved, is generated in a straightforward manner via transformations.
On the other hand, the adjoint representation of a Lie group on its corresponding Lie algebra is instantly formulated through its related infinitesimal generators. Therefore, if $\mathbf{v}$ prompts canonically the one-parameter subgroup $\Big\{\exp\big(\varepsilon\mathbf{v}\big)\Big\}$, then we suppose ad $\mathbf{v}$ as the vector field on $\mathfrak{g}$ totally determining the associated one-parameter group of adjoint transformations:

$$\mathrm{ad}\ \mathbf{v}\ |_{\mathbf{w}} \equiv \frac{d}{d\varepsilon}\ |_{\varepsilon=0}\ \ \mathrm{Ad}\big(\exp(\varepsilon\mathbf{v})\big)\mathbf{w}, \quad \mathbf{w} \in \mathfrak{g}.$$

Conversely, if the infinitesimal adjoint action ad $\mathfrak{g}$ of a Lie algebra $\mathfrak{g}$ on itself is recognized, then the adjoint representation Ad $\mathcal{G}$ of the underlying Lie group, can be exhaustively reestablished either by integrating the following system of linear ordinary differential equations expressed by:

$$\frac{d\mathbf{w}}{d\varepsilon} = \mathrm{ad}\ \mathbf{v}\ |_{\mathbf{w}}, \quad \mathbf{w}(0) = \mathbf{w}_0,$$

with solution $\mathbf{w}(\varepsilon) = \mathrm{Ad}\big(\exp(\varepsilon\mathbf{v})\big)\mathbf{w}_0$, or unambiguously by virtue of summing the Lie series [5]:

$$\begin{aligned}\mathrm{Ad}\big(\exp(\varepsilon\mathbf{v})\big)\mathbf{w}_0 &= \textstyle\sum_{n=0}^{\infty} \frac{\varepsilon^n}{n!}\big(\mathrm{ad}\ \mathbf{v}\big)^n(\mathbf{w}_0) \\ &= \mathbf{w}_0 - \varepsilon[\mathbf{v},\mathbf{w}_0] + \frac{\varepsilon^2}{2}\big[\mathbf{v},[\mathbf{v},\mathbf{w}_0]\big] - \cdots\end{aligned} \tag{3.1}$$

At this stage, according to the comprehensive discussion declared above, by constructing the adjoint representation group, which clearly insinuates a conjugate relation in the set of all one-dimensional symmetry subalgebras, an optimal system of Killing subalgebras of $\tilde{\mathcal{K}}$ (the Lie algebra characterized by the direct sum of indecomposable Lie subalgebras) is thoroughly inaugurated. Accordingly, taking into account (3.1) each $\mathbf{F_i}$, $i = 1, \cdots, 7$, of the basis symmetries generates an adjoint representation (or interior automorphism) $\mathrm{Ad}\big(\exp(\varepsilon \mathbf{F_i})\big)$ determined by the Lie series:

$$\mathrm{Ad}\big(\exp(\varepsilon \cdot \mathbf{F_i})\big) \cdot \mathbf{F_j} = \mathbf{F_j} - \varepsilon \cdot [\mathbf{F_i}, \mathbf{F_j}] + \frac{\varepsilon^2}{2} \cdot \big[\mathbf{F_i}, [\mathbf{F_i}, \mathbf{F_j}]\big] - \cdots \tag{3.2}$$

where $[\mathbf{F_i}, \mathbf{F_j}]$ is the commutator for the Lie algebra, ε is a parameter, and $i, j = 1, \cdots, 7$. In table (2) all the adjoint representations of the Lie algebra of Killing vector fields $\tilde{\mathcal{K}}$ is presented, with the (i, j) entry indicating $\mathrm{Ad}\big(\exp \cdot (\varepsilon \mathbf{F_i})\big) \cdot \mathbf{F_j}$.

TABLE 2. Adjoint representation generated by the basis symmetries of the Lie algebra of Killing vector fields $\tilde{\mathcal{K}}$

Ad	$\mathbf{F_1}$	$\mathbf{F_2}$	$\mathbf{F_3}$
$\mathbf{F_1}$	$\mathbf{F_1}$	$\cosh(\varepsilon)\mathbf{F_2} + \sinh(\varepsilon)\mathbf{F_4}$	$\cosh(\varepsilon)\mathbf{F_3} - \sinh(\varepsilon)\mathbf{F_6}$
$\mathbf{F_2}$	$\cosh(\varepsilon)\mathbf{F_1} - \sinh(\varepsilon)\mathbf{F_4}$	$\mathbf{F_2}$	$\cosh(\varepsilon)\mathbf{F_3} - \sinh(\varepsilon)\mathbf{F_5}$
$\mathbf{F_3}$	$\mathbf{F_1} + \varepsilon\mathbf{F_6}$	$\mathbf{F_2} + \varepsilon\mathbf{F_5}$	$\mathbf{F_3}$
$\mathbf{F_4}$	$\cos(\varepsilon)\mathbf{F_1} - \sin(\varepsilon)\mathbf{F_2}$	$\sin(\varepsilon)\mathbf{F_1} + \cos(\varepsilon)\mathbf{F_2}$	$\mathbf{F_3}$
$\mathbf{F_5}$	$\mathbf{F_1}$	$\mathbf{F_2} + \varepsilon\mathbf{F_3}$	$\mathbf{F_3}$
$\mathbf{F_6}$	$\mathbf{F_1} + \varepsilon\mathbf{F_3}$	$\mathbf{F_2}$	$\mathbf{F_3}$
$\mathbf{F_7}$	$\mathbf{F_1}$	$\mathbf{F_2}$	$\mathbf{F_3}$

Ad	$\mathbf{F_4}$	$\mathbf{F_5}$
$\mathbf{F_1}$	$\sinh(\varepsilon)\mathbf{F_2} + \cosh(\varepsilon)\mathbf{F_4}$	$\mathbf{F_5}$
$\mathbf{F_2}$	$-\sinh(\varepsilon)\mathbf{F_1} + \cosh(\varepsilon)\mathbf{F_4}$	$-\sinh(\varepsilon)\mathbf{F_3} + \cosh(\varepsilon)\mathbf{F_5}$
$\mathbf{F_3}$	$\mathbf{F_4}$	$\mathbf{F_5}$
$\mathbf{F_4}$	$\mathbf{F_4}$	$\cos(\varepsilon)\mathbf{F_5} + \sin(\varepsilon)\mathbf{F_6}$
$\mathbf{F_5}$	$\mathbf{F_4} - \varepsilon\mathbf{F_6}$	$\mathbf{F_5}$
$\mathbf{F_6}$	$\mathbf{F_4} + \varepsilon\mathbf{F_5}$	$\mathbf{F_5}$
$\mathbf{F_7}$	$\mathbf{F_4}$	$\mathbf{F_5}$

Ad	$\mathbf{F_6}$	$\mathbf{F_7}$
$\mathbf{F_1}$	$-\sinh(\varepsilon)\mathbf{F_3} + \cosh(\varepsilon)\mathbf{F_6}$	$\mathbf{F_7}$
$\mathbf{F_2}$	$\mathbf{F_6}$	$\mathbf{F_7}$
$\mathbf{F_3}$	$\mathbf{F_6}$	$\mathbf{F_7}$
$\mathbf{F_4}$	$-\sin(\varepsilon)\mathbf{F_5} + \cos(\varepsilon)\mathbf{F_6}$	$\mathbf{F_7}$
$\mathbf{F_5}$	$\mathbf{F_6}$	$\mathbf{F_7}$
$\mathbf{F_6}$	$\mathbf{F_6}$	$\mathbf{F_7}$
$\mathbf{F_7}$	$\mathbf{F_6}$	$\mathbf{F_7}$

We can expect to simplify a given arbitrary element,

$$\mathbf{F} = a_1\mathbf{F_1} + a_2\mathbf{F_2} + a_3\mathbf{F_3} + a_4\mathbf{F_4} + a_5\mathbf{F_5} + a_6\mathbf{F_6} + a_7\mathbf{F_7}. \tag{3.3}$$

of the Lie algebra of Killing vector fields $\tilde{\mathcal{K}}$. Note that the elements of $\tilde{\mathcal{K}}$ can be represented by vectors $a = (a_1, \cdots, a_7) \in \mathbb{R}^7$ since each of them can be written in the form (3.3) for some constants $a_1, ..., a_7$. Hence, the adjoint action can be regarded as (in fact is) a group of linear transformations of the vectors $(a_1, ..., a_7)$.

Consequently, we can express the following theorem:

Theorem 3.1. *An optimal system of one-dimensional Lie subalgebras of $\tilde{\mathcal{K}}$ i.e. the Lie algebra of Killing vector fields created by the sum of indecomposable Lie subalgebras is provided by those generated by:*

$$\begin{aligned}
&(\mathbf{1}) : \mathbf{F_6} + \alpha\mathbf{F_2} + \beta\mathbf{F_3} + \gamma\mathbf{F_5} + \zeta\mathbf{F_7}\\
&= -\Big(k^2r^2+1\Big)^{1/4}\ \mathbf{E_2} + \Bigg(\alpha kyr\Big(k^2r^2+1\Big)^{1/4}\\
&+\big(\beta k^2+\zeta\big)\ \frac{r\Big(k^2r^2+1\Big)^{1/4}}{k}\Bigg)\ \mathbf{E_3} + \Bigg(\alpha t\Big(k^2r^2+1\Big)^{1/4} - \gamma\Big(k^2r^2+1\Big)^{1/4}\Bigg)\ \mathbf{E_4}\\
&+\Bigg(\alpha y\Big(k^2r^2+1\Big)^{3/4} + \frac{\beta}{\Big(k^2r^2+1\Big)^{1/4}} + \big(\beta k^2+\zeta\big)\ \frac{r^2}{\Big(k^2r^2+1\Big)^{1/4}}\Bigg)\ \mathbf{E_5}.
\end{aligned}$$

$$\begin{aligned}
&(\mathbf{2}) : \mathbf{F_3} + \alpha\mathbf{F_4} + \beta\mathbf{F_5} + \gamma\mathbf{F_7}\\
&= \alpha y\Big(k^2r^2+1\Big)^{1/4}\ \mathbf{E_2} + \big(k^2+\gamma\big)\ \frac{r\Big(k^2r^2+1\Big)^{1/4}}{k}\ \mathbf{E_3} - \Bigg(\alpha z\Big(k^2r^2+1\Big)^{1/4}\\
&+\beta\Big(k^2r^2+1\Big)^{1/4}\Bigg)\ \mathbf{E_4} + \Bigg(\frac{1}{\Big(k^2r^2+1\Big)^{1/4}} + \big(k^2+\gamma\big)\ \frac{r^2}{\Big(k^2r^2+1\Big)^{1/4}}\Bigg)\ \mathbf{E_5}.
\end{aligned}$$

$$\begin{aligned}
&(\mathbf{3}) : \mathbf{F_5} + \alpha\mathbf{F_1} + \beta\mathbf{F_4} + \gamma\mathbf{F_7}\\
&= \Bigg(\alpha t\Big(k^2r^2+1\Big)^{1/4} + \beta y\Big(k^2r^2+1\Big)^{1/4}\Bigg)\ \mathbf{E_2} + \Bigg(\alpha kzr\Big(k^2r^2+1\Big)^{1/4}\\
&+\gamma\ \frac{r\Big(k^2r^2+1\Big)^{1/4}}{k}\Bigg)\ \mathbf{E_3} - \Bigg(\Big(k^2r^2+1\Big)^{1/4} + \beta z\Big(k^2r^2+1\Big)^{1/4}\Bigg)\ \mathbf{E_4}\\
&+\Bigg(\alpha z\Big(k^2r^2+1\Big)^{3/4} + \frac{\gamma r^2}{\Big(k^2r^2+1\Big)^{1/4}}\Bigg)\ \mathbf{E_5}.
\end{aligned}$$

$$
\begin{aligned}
&(\mathbf{4}) : \mathbf{F_1} + \alpha\mathbf{F_4} + \beta\mathbf{F_7} \\
&= \left(t\Big(k^2r^2+1\Big)^{1/4} + \alpha y\Big(k^2r^2+1\Big)^{1/4} \right) \mathbf{E_2} + \left(kzr\Big(k^2r^2+1\Big)^{1/4} \right. \\
&\left. +\beta \ \frac{r\Big(k^2r^2+1\Big)^{1/4}}{k} \right) \mathbf{E_3} - \alpha z\Big(k^2r^2+1\Big)^{1/4} \ \mathbf{E_4} \\
&+\left(z\Big(k^2r^2+1\Big)^{3/4} + \frac{\beta r^2}{\Big(k^2r^2+1\Big)^{1/4}} \right) \mathbf{E_5}.
\end{aligned}
$$

$$
\begin{aligned}
&(\mathbf{5}) : \alpha\mathbf{F_2} + \beta\mathbf{F_4} + \gamma\mathbf{F_7} \\
&= \beta y\Big(k^2r^2+1\Big)^{1/4} \ \mathbf{E_2} + \left(\alpha kyr\Big(k^2r^2+1\Big)^{1/4} + \gamma \ \frac{r\Big(k^2r^2+1\Big)^{1/4}}{k} \right) \mathbf{E_3} \\
&+\left(\alpha t\Big(k^2r^2+1\Big)^{1/4} - \beta z\Big(k^2r^2+1\Big)^{1/4} \right) \mathbf{E_4} \\
&+\left(\alpha y\Big(k^2r^2+1\Big)^{3/4} + \frac{\gamma r^2}{\Big(k^2r^2+1\Big)^{1/4}} \right) \mathbf{E_5}.
\end{aligned}
$$

where $\alpha, \beta, \gamma, \zeta$ are arbitrary real constants.

Proof. First of all, we precisely concentrate on the linear maps $\mathcal{A}_i^s : \ \tilde{\mathcal{K}} \to \tilde{\mathcal{K}}$ defined by $\mathbf{F} \mapsto \mathrm{Ad}\big(\exp(s_i\mathbf{F_i})\big) \cdot \mathbf{F}$, for $i = 1, \cdots, 7$. The matrix M_i^s of $\mathcal{A}_i^s$, $i = 1, \cdots, 7$, with respect to the basis $\Big\{\mathbf{F_1}, \cdots, \mathbf{F_7}\Big\}$ is expressed as follows:

$$
M_1^{s_1} = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \cosh(s_1) & 0 & \sinh(s_1) & 0 & 0 & 0 \\
0 & 0 & \cosh(s_1) & 0 & 0 & -sinh(s_1) & 0 \\
0 & \sinh(s_1) & 0 & \cosh(s_1) & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & -\sinh(s_1) & 0 & 0 & \cosh(s_1) & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix}
$$

$$
M_2^{s_2} = \begin{pmatrix}
\cosh(s_2) & 0 & 0 & -sinh(s_2) & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \cosh(s_2) & 0 & -\sinh(s_2) & 0 & 0 \\
-\sinh(s_2) & 0 & 0 & \cosh(s_2) & 0 & 0 & 0 \\
0 & 0 & -\sinh(s_2) & 0 & \cosh(s_2) & 0 & 0 \\
0 & 0 & s & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix}
$$

$$M_3^{s_3} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & s_3 & 0 \\ 0 & 1 & 0 & 0 & s_3 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

$$M_4^{s_4} = \begin{pmatrix} \cos(s_4) & -\sin(s_4) & 0 & 0 & 0 & 0 & 0 \\ \sin(s_4) & \cos(s_4) & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \cos(s_4) & \sin(s_4) & 0 \\ 0 & 0 & 0 & 0 & -\sin(s_4) & \cos(s_4) & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

$$M_5^{s_5} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & s_5 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & -s_5 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

$$M_6^{s_6} = \begin{pmatrix} 1 & 0 & s_6 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & s_6 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

$$M_7^{s_7} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

respectively. Let $\mathbf{F} = \sum_{i=1}^{7} a_i \mathbf{F_i}$, then it is explicitly observed that:

$$
\begin{aligned}
&\mathcal{A}_7^{s_7} \circ \mathcal{A}_6^{s_6} \circ \cdots \circ \mathcal{A}_1^{s_1} \ : \ \mathbf{F} \ \mapsto \\
&\quad \Bigg(\cosh(s_2)\cos(s_4)a_1 - \cosh(s_2)\sin(s_4)a_2 + \Big(\cosh(s_2)\cos(s_4)s_6 \\
&- \cosh(s_2)\sin(s_4)s_5 \Big) a_3 - \sinh(s_2)a_4 + \Big(- \sinh(s_2)s_6 - \cosh(s_2)s_3\sin(s_4) \Big) a_5 \\
&+ \Big(\sinh(s_2)s_5 + \cosh(s_2)s_3\cos(s_4) \Big) a_6 \Bigg) \mathbf{F_1} + \Bigg(\Big(- \sinh(s_1)\sinh(s_2)\cos(s_4) \\
&+ \cosh(s_1)\sin(s_4) \Big) a_1 + \Big(\sinh(s_1)\sinh(s_2)\sin(s_4) + \cosh(s_1)\cos(s_4) \Big) a_2 \\
&+ \Big(\big(- \sinh(s_1)\sinh(s_2)\cos(s_4) + \cosh(s_1)\sin(s_4) \big) s_6 \\
&+ \big(\sinh(s_1)\sinh(s_2)\sin(s_4) + \cosh(s_1)\cos(s_4) \big) s_5 \Big) a_3 \\
&+ \sinh(s_1)\cosh(s_2)a_4 + \Big(\sinh(s_1)\cosh(s_2)s_6 + \cosh(s_1)s_3\cos(s_4) \\
&+ \sinh(s_1)\sinh(s_2)s_3\sin(s_4) \Big) a_5 + \Big(- \sinh(s_1)\cosh(s_2)s_5 \\
&+ \cosh(s_1)s_3\sin(s_4) - \sinh(s_1)\sinh(s_2)s_3\cos(s_4) \Big) a_6 \Bigg) \mathbf{F_2} \\
&+ \Bigg(\cosh(s_1)\cosh(s_2)a_3 + \Big(- \cosh(s_1)\sinh(s_2)\cos(s_4) + \sinh(s_1)\sin(s_4) \Big) a_5 \\
&+ \Big(- \cosh(s_1)\sinh(s_2)\sin(s_4) - \sinh(s_1)\cos(s_4) \Big) a_6 \Bigg) \mathbf{F_3} \\
&+ \Bigg(\Big(- \cosh(s_1)\sinh(s_2)\cos(s_4) + \sinh(s_1)\sin(s_4) \Big) a_1 \\
&+ \Big(\cosh(s_1)\sinh(s_2)\sin(s_4) + \sinh(s_1)\cos(s_4) \Big) a_2 \\
&+ \Big(\big(- \cosh(s_1)\sinh(s_2)\cos(s_4) + \sinh(s_1)\sin(s_4) \big) s_6 \\
&+ \big(\cosh(s_1)\sinh(s_2)\sin(s_4) + \sinh(s_1)\cos(s_4) \big) s_5 \Big) a_3 + \cosh(s_1)\cosh(s_2)a_4 \\
&+ \Big(\cosh(s_1)\cosh(s_2)s_6 + \sinh(s_1)s_3\cos(s_4)\cosh(s_1)\sinh(s_2)s_3\sin(s_4) \Big) a_5 \\
&+ \Big(- \cosh(s_1)\cosh(s_2)s_5 + \sinh(s_1)s_3\sin(s_4) - \cosh(s_1)\sinh(s_2) \\
&\times s_3\cos(s_4) \Big) a_6 \Bigg) \mathbf{F_4} + \Bigg(- \sinh(s_2)a_3 + \cosh(s_2)\cos(s_4)a_5 \\
&+ \cosh(s_2)\sin(s_4)a_6 \Bigg) \mathbf{F_5} + \Bigg(- \sinh(s_1)\cosh(s_2)a_3 \\
&+ \Big(\sinh(s_1)\sinh(s_2)\cos(s_4) - \cosh(s_1)\sin(s_4) \Big) a_5 \\
&+ \Big(\sinh(s_1)\sinh(s_2)\sin(s_4) + \cosh(s_1)\cos(s_4) \Big) a_6 \Bigg) \mathbf{F_6} + a_7\mathbf{F_7}.
\end{aligned}
$$

In order to classify the one-dimensional Lie subalgebras of $\tilde{\mathcal{K}}$, the following cases are planned such that in each case, by acting a finite number of the adjoint representations $M_i^{s_i}$ $(i = 1, \cdots, 7)$ on $\mathbf{F}$, by proper selection of parameters s_i in each stage, it is gradually tried to make the coefficients of $\mathbf{F}$ vanish and to acquire the most simple form of $\mathbf{F}$.

♣ At first, we suppose that $a_6 \neq 0$. This assumption proposes *Case (1)* as follows:

Case (1): Since $a_6 \neq 0$, then we act on $\mathbf{F}$ by $\mathrm{Ad}(\exp(-\frac{a_1}{a_6})\mathbf{F_3})$ and hence we can make the coefficient of $\mathbf{F_1}$ vanish. Then we tend to the new form

$$\mathbf{F}' = a_2'\mathbf{F_2} + a_3\mathbf{F_3} + a_4\mathbf{F_4} + a_5\mathbf{F_5} + a_6\mathbf{F_6} + a_7\mathbf{F_7}$$

For certain scalars a_2' depending on a_1, a_2, a_5 and a_6. By acting $\mathrm{Ad}(\exp(\frac{a_4}{a_6})\mathbf{F_5})$ on $\mathbf{F}'$ we tend to

$$\mathbf{F}'' = a_2''\mathbf{F_2} + a_3\mathbf{F_3} + a_5\mathbf{F_5} + a_6\mathbf{F_6} + a_7\mathbf{F_7}$$

For certain scalars a_2'' depending on a_2, a_3, a_4 and a_6. At this stage, by acting the adjoint representations $M_i^{s_i}$, $i = 1 \cdots 7$ on $\mathbf{F}''$, we find that no more simplification of $\mathbf{F}''$ is possible. Thus each of a_2'', a_3, a_5, a_6 and a_7 are arbitrary. By scaling if necessary, we can assume that $a_6 = 1$. Therefore, any one-dimensional Lie subalgebra generated by $\mathbf{F}$ (by knowing that $a_6 \neq 0$) is equivalent to the Lie subalgebra spanned by

$$\mathbf{F_6} + a_2''\mathbf{F_2} + a_3\mathbf{F_3} + a_5\mathbf{F_5} + a_7\mathbf{F_7}$$

which is equal to section (1) of the theorem.

Each of the following cases is prepared by a similar method to *Case(1)* and by eliminating unnecessary details, we would just present the conditions.

♣ At the second, we suppose that $a_6 = 0$ and consider *Case(2),...,Case(5)* prepared by this assumption.

Case(2): If $a_3 \neq 0$, then we can make the coefficients of $\mathbf{F_1}$ and $\mathbf{F_2}$ vanish by $\mathcal{A}_6^{s_6}$ and $\mathcal{A}_5^{s_5}$. By setting $s_6 = -\dfrac{a_1}{a_3}$ and $s_5 = -\dfrac{a_2}{a_3}$, respectively. Scaling $\mathbf{F}$ if necessary we can assume that $a_3 = 1$ and then $\mathbf{F}$ is reduced to (2).

Case(3): If $a_3 = 0$ and $a_5 \neq 0$, then we can make the coefficients of $\mathbf{F_2}$ vanish by $\mathcal{A}_3^{s_3}$. By setting $s_3 = -\dfrac{a_2}{a_5}$. Scaling $\mathbf{F}$ if necessary we can assume that $a_5 = 1$ and then $\mathbf{F}$ is reduced to (3).

Case(4): If $a_3 = a_5 = 0$ and $a_1 \neq 0$, then we can make the coefficients of $\mathbf{F_2}$ vanish by $\mathcal{A}_4^{s_4}$, by setting $s_4 = -\arctan\left(\dfrac{a_2}{a_1}\right)$. Scaling $\mathbf{F}$ if necessary we can assume that $a_1 = 1$ and then $\mathbf{F}$ is reduced to (4).

Case(5): Finally, if in the last case $a_6 = a_3 = a_5 = a_1 = 0$, no further simplification is possible and then $\mathbf{F}$ is reduced to (5).

Consequently, there is not any more possible case for studying and the proof is complete. □

Conclusion

The general theory of relativity which can be regarded as the field theory of gravitation is fundamentally governed by the Einstein field equations (EFE). These equations are extremely nonlinear and are demonstrated in terms of the Lorentzian metric g_{ab}. Taking into account this nonlinearity, obtaining their exact solutions is totally difficult. Hence, it has been one of the basic problems in general relativity to analyze the solutions of the Einstein field equations by means of the symmetries they possess. In the current research, we have exhaustively analyzed the structure of the Lie algebra of Killing vector fields for a specific cosmological solution in standard Einstein theory. This physically remarkable five dimensional spacetime, describes the total behaviour of rotating fluids. Particularly, by reexpressing the analyzed metric in the orthogonal coframe, the corresponding Killing vector fields are thoroughly calculated. Significantly, for the resulted Lie algebra of Killing vector fields, the associated basis for the original Lie algebra is determined in which the Lie algebras will be appropriately decomposed into an internal direct sum of subalgebras, where each summand is indecomposable. Principally, a thorough classification of the symmetry subalgebras of Killing vector fields for our analyzed privileged cosmological solution is formulated via the adjoint representation group. This preliminary group classification generically insinuates a conjugate relation in the set of all one-dimensional subalgebras. Accordingly, the corresponding set of invariant solutions can be reckoned canonically as the mimimal list from which all the other invariant solutions of one-dimensional subalgebras are comprehensively designated unambiguously by virtue of transformations.

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Received: 17.02.2022
Accepted: 06.05.2022