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A new non-Riemannian curvature related to the class of (α, β)-metrics

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Abstract. In this paper, we find a new non-Riemannian quantity for (α, β)-metrics that is closely related to the S-curvature. We call it the $\widetilde{\mathbf{S}}$-curvature. Then we show that an (α, β)-metric is Riemannian if and only if $\widetilde{\mathbf{S}} = 0$. For a Randers metric, we find the relation between $\mathbf{S}$-curvature and $\widetilde{\mathbf{S}}$-curvature.

Keywords: Hopf maximum principle, Elliptic operator, (α, β)-metrics, S-curvature.

1. Introduction

The study of Finsler spaces with (α, β)-metrics is quit old, but it is a very important aspect of Finsler geometry and its applications. An (α, β)-metric is a scalar function on TM defined by $F := \Phi(\frac{\beta}{\alpha})\alpha,\ s = \beta/\alpha$, where $\phi = \phi(s)$ is a C^∞ on $(-b_0, b_0)$ with certain regularity, $\alpha = \sqrt{a_{ij}(x)y^iy^j}$ is a Riemannian metric and $\beta = b_i(x)y^i$ is a 1-form on a manifold M. Then (M, α) is called the associated Riemannian manifold.

Randers metrics are special (α, β)-metrics defined by $\Phi = 1 + s$, i.e, $F = \alpha + \beta$. The most important case of (α, β)-metrics is the Randers metrics which were introduced by Randers in 1941 [8] in the context of general relativity.

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They play a prominent role in Ingarden's study of electron optics [1]. For other properties of Randers metrics see [3] and [4].

In Finsler geometry, there are several important non-Riemannian quantities: the distortion τ, the Cartan torsion $\mathbf{C}$, the Berwald curvature $\mathbf{B}$, the mean Berwald curvature $\mathbf{E}$, the $\mathbf{S}$-curvature and the new non-Riemannian curvature $\mathbf{H}$ in paper [7], etc. They all vanish for Riemannian metrics, hence they are said to be *non-Riemannian.*

In this paper, we first introduce a new non-Riemannian quantity for an (α,β)-metric, by using the geodesic coefficient of α. Indeed, this curvature is obtain for the associated Riemannian manifold (M,α). This new quantity is closely related to the $\mathbf{S}$-curvature. Therefore we call it $\widetilde{\mathbf{S}}$-curvature. Then for a Randers metric $F=\alpha+\beta$, we find the relation between $\mathbf{S}$-curvature and $\widetilde{\mathbf{S}}$-curvature.

For an (α,β)-metric $F=\Phi(\beta/\alpha)\alpha$, we can introduce some non-Riemannian quantity. Let us denote the Levi-Civita connection of α by $\tilde{\nabla}$. We define the function $\widetilde{\mathbf{S}}$ defined over TM_0 as follows:

$$\widetilde{\mathbf{S}}=\tilde{\nabla}_{\hat{\mathbf{v}}}\tau,$$

where $\hat{\mathbf{v}}$ is the Riemannian spray associated to α and the function τ is the so-called distortion.

The curvature $\widetilde{\mathbf{S}}$ is closely related to the S-curvature. $\widetilde{\mathbf{S}}$ is related to (α,β)-metrics, especially to the associated Riemannian manifold (M,α). But we show that $\widetilde{\mathbf{S}}$ is a non-Riemannian quantity and prove the following theorem.

Theorem 1.1. *Let $F=\Phi(\frac{\beta}{\alpha})\alpha$ be an (α,β)-metric and α has positive (negative) sectional curvature. Then $\widetilde{\mathbf{S}}=0$ if and only if F is Riemannian.*

There are many connections in Finsler geometry. One is referred to [5] and [11] for some of these connections. Throughout this paper, we set the Chern connection on Finsler manifolds.

2. Preliminaries.

Let M be a n-dimensional C^∞ manifold. T_xM denotes the tangent space of M at x. The tangent bundle of M is the union of tangent spaces $TM := \cup_{x\in M}T_xM$. We will denote the elements of TM by (x,y) where $y\in T_xM$. Let $TM_0=TM\setminus\{0\}$. The natural projection $\pi: TM_0\to M$ is given by $\pi(x,y):=x$.

A *Finsler structure* on M is a function $F:TM\to[0,\infty)$ with the following properties; (i) F is C^∞ on TM_0, (ii) F is positively 1-homogeneous on the fibers of tangent bundle TM, and (iii) the Hessian of F^2 with elements

$$g_{ij}(x,y):=\frac{1}{2}[F^2(x,y)]_{y^iy^j}$$

is positively defined on TM_0. The pair (M,F) is then called a *Finsler manifold.*

Let $x \in M$ and $F_x := F|_{T_xM}$. To measure the non-Euclidean feature of F_x, one can define $\mathbf{C}_y : T_xM \times T_xM \times T_xM \to \mathbb{R}$ by

$$\mathbf{C}_y(u,v,w) := \frac{1}{2}\frac{d}{dt}\Big[\mathbf{g}_{y+tw}(u,v)\Big]_{t=0}, \quad u,v,w \in T_xM.$$

The family $\mathbf{C} := \{\mathbf{C}_y\}_{y\in TM_0}$ is called the Cartan torsion. It is well known that $\mathbf{C} = 0$ if and only if F is Riemannian.

For $y \in T_xM_0$, define $\mathbf{I}_y : T_xM \to \mathbb{R}$ by

$$\mathbf{I}_y(u) := \sum_{i=1}^{n} g^{ij}(y)\mathbf{C}_y(u,\partial_i,\partial_j),$$

where $\{\partial_i\}$ is a basis for T_xM at $x \in M$. The family $\mathbf{I} := \{\mathbf{I}_y\}_{y\in TM_0}$ is called the mean Cartan torsion. By definition, $\mathbf{I}_y(y) = 0$ and $\mathbf{I}_{\lambda y} = \lambda^{-1}\mathbf{I}_y$, $\lambda > 0$. Therefore, $\mathbf{I}_y(u) := I_i(y)u^i$, where $I_i := g^{jk}C_{ijk}$.

F is Riemannian if $g_{ij}(x,y)$ are independent of $y \neq 0$. Then Riemannian metrics are special Finsler metrics. Traditionally, a Riemannian metric is denoted by $a_{ij}(x)dx^i \otimes dx^j$. It is a family of inner products on tangent spaces. Let $\alpha(\mathbf{y}) := \sqrt{g_{ij}(x)y^iy^j}$, $\mathbf{y} = y^i\frac{\partial}{\partial x^i}|_x \in T_xM$. α is a family of Euclidean norms on tangent spaces. Throughout this paper, we also denote a Riemannian metric by $\alpha = \sqrt{a_{ij}(x)y^iy^j}$.

An (α,β)-metric is a scalar function on TM defined by

$$F := \alpha\Phi\Big(\frac{\beta}{\alpha}\Big), \quad s = \beta/\alpha,$$

where $\phi = \phi(s)$ is a C^∞ on $(-b_0, b_0)$ with certain regularity, $\alpha = \sqrt{a_{ij}(x)y^iy^j}$ is a Riemannian metric and $\beta = b_i(x)y^i$ is a 1-form on a manifold M. Randers metrics are special (α,β)-metrics defined by $\Phi = 1+s$, i.e, $F = \alpha+\beta$.

Given a Finsler manifold (M,F), then a global vector field G is induced by F on TM_0, which in a standard coordinate (x^i,y^i) for TM_0 is given by

$$\mathbf{G} = y^i\frac{\partial}{\partial x^i} - 2G^i(x,y)\frac{\partial}{\partial y^i},$$

where $G^i(x,y)$ are local functions on TM_0 satisfying $G^i(x,\lambda y) = \lambda^2 G^i(x,y)$ $\lambda > 0$. $\mathbf{G}$ is called the associated *spray* to (M,F). The projection of an integral curve of G is called a *geodesic* in M. In local coordinates, a curve $c(t)$ is a geodesic if and only if its coordinates $(c^i(t))$ satisfy

$$\ddot{c}^i + 2G^i(\dot{c}) = 0.$$

If F is Riemannian, then $G^i(x,y) = \frac{1}{2}\Gamma^i_{jk}(x)y^jy^k$ are quadratic in (y^i) at every point $x \in M$. A Finsler metric is called a *Berwald metric* if the geodesic coefficients have this property.

For a Finsler metric F on an n-dimensional manifold M, the Busemann-Hausdorff volume form $dV_F = \sigma_F(x)dx^1 \cdots dx^n$ is defined by

$$\sigma_F(x) := \frac{\mathrm{Vol}(\mathbb{B}^n(1))}{\mathrm{Vol}\Big\{(y^i) \in \mathbb{R}^n \;\Big|\; F\Big(y^i \frac{\partial}{\partial x^i}|_x\Big) < 1\Big\}}.$$

In general, the local scalar function $\sigma_F(x)$ can not be expressed in terms of elementary functions, even F is locally expressed by elementary functions [9].

Let

$$\tau(x,y) := \ln\Big[\frac{\sqrt{\det\Big(g_{ij}(x,y)\Big)}}{\mathrm{Vol}(\mathrm{B}^n(1))} \cdot \mathrm{Vol}\Big\{(y^i) \in \mathbb{R}^n \Big|\; F\Big(y^i \frac{\partial}{\partial x^i}|_x\Big) < 1\Big\}\Big].$$

$\tau = \tau(x,y)$ is a scalar function on TM_0, which is called the *distortion* [9]. For a vector $\mathbf{y} \in T_xM$, let $c(t), -\epsilon < t < \epsilon$, denote the geodesic with $c(0) = x$ and $\dot{c}(0) = \mathbf{y}$. Define

$$\mathbf{S}(\mathbf{y}) := \frac{d}{dt}\Big[\tau\Big(\dot{c}(t)\Big)\Big]|_{t=0}.$$

We call $\mathbf{S}$ the S-curvature. This quantity was first introduced in [10] for a volume comparison theorem.

Let $G^i(x,y)$ denote the geodesic coefficients of F in the same local coordinate system. The S-curvature can be express by

$$\mathbf{S}(\mathbf{y}) = \frac{\partial G^i}{\partial y^i}(x,y) - y^i \frac{\partial}{\partial x^i}\Big[\ln \sigma_F(x)\Big],$$

where $\mathbf{y} = y^i \frac{\partial}{\partial x^i}|_x \in T_xM$. It is proved that $\mathbf{S} = 0$ if F is a Berwald metric [10]. There are many non-Berwald metrics satisfying $\mathbf{S} = 0$.

Now, we recall the definition of Riemann curvature. Let F be a Finsler metric on an n-manifold and G^i denote the geodesic coefficients of F. For a vector $\mathbf{y} = y^i \frac{\partial}{\partial x^i}|_x \in T_xM$, define $\mathbf{R_y} = R^i{}_k(x,y)dx^k \otimes \frac{\partial}{\partial x^i}|_x : T_xM \to T_xM$ by

$$R^i{}_k := 2\frac{\partial G^i}{\partial x^k} - y^j \frac{\partial^2 G^i}{\partial x^j \partial y^k} + 2G^j \frac{\partial^2 G^i}{\partial y^j \partial y^k} - \frac{\partial G^i}{\partial y^j}\frac{\partial G^j}{\partial y^k}.$$

Let us put

$$R^i{}_{kl} := \frac{1}{3}\Big\{\frac{\partial R^i_k}{\partial y^l} - \frac{\partial R^i_l}{\partial y^k}\Big\}, \quad R^i_{j\ kl} := \frac{1}{3}\Big\{\frac{\partial^2 R^i_k}{\partial y^j \partial y^l} - \frac{\partial^2 R^i_l}{\partial y^j \partial y^k}\Big\}.$$

Then

$$R^i_k = R^i_{j\ kl}y^j y^l, \qquad R^i{}_{kl} = R^i_{j\ kl}y^j, \qquad R^i_{j\ kl} + R^i_{j\ lk} = 0,$$

$$R^h{}_{ijk} + R^h{}_{jki} + R^h{}_{kij} = 0.$$

3. **Proof of Theorem 1.1.**

Let (M,F) be an n-dimensional Finsler space. For every $x\in M$, let

$$S_xM=\Big\{y\in T_xM|F(x,y)=1\Big\}.$$

S_xM is called the indicatrix of F at $x\in M$ and it is a compact hyper surface of T_xM, for every $x\in M$. Let

$$v:S_xM\hookrightarrow T_xM$$

be its canonical embedding, where $\|v\|=1$. Let (t,U) be a coordinate system on S_xM. Then, S_xM is represented locally by $v^i=v^i(t^\alpha)$, $\alpha=1,2,...,(n-1)$. One can show that:

$$\frac{\partial}{\partial v^i}=F\frac{\partial}{\partial y^i}$$

The $(n-1)$ vectors $\{(v^i{}_\alpha)\}$ form a basis for the tangent space of S_xM in each point, where

$$v^i{}_\alpha=\frac{\partial v^i}{\partial t^\alpha},\quad \alpha=1,2,...,(n-1).$$

For the sake of simplicity, put

$$\partial_\alpha=\frac{\partial}{\partial t^\alpha}.$$

One can easily show that

$$\partial_\alpha=Fv^i{}_\alpha\frac{\partial}{\partial y^i}$$

$g=g_{ij}(x,y)dy^idy^j$ is a Riemannian metric on T_xM. Inducing g on S_xM, one gets the Riemannian metric $\bar g=\bar g_{\alpha\beta}dt^\alpha dt^\beta$, where

$$\bar g_{\alpha\beta}=v^i{}_\alpha v^i{}_\beta g_{ij}.$$

The canonical unit vertical vector field $V(x,y)=y^i\frac{\partial}{\partial y^i}$ together the $(n-1)$ vectors ∂_α, form the local basis for T_xM, $\mathcal{B}=\{u^1,u^2,...,u^n\}$, where, $u^\alpha=(v^i{}_\alpha)$ and $u^n=V$. We conclude that

$$g(V,\partial_\alpha)=0,$$

that is

$$y_iv^i{}_\alpha=0.$$

For an (α,β)-metric $F=\Phi(\beta/\alpha)\alpha$, we can introduce some non-Riemannian quantity. Let us denote the Levi-Civita connection and the Rieman curvature of α by $\tilde\nabla$ and $\tilde R^i{}_{jkl}$, respectively. Put

$$\hat{\underline{u}}=\underline{u}^i\frac{\hat\delta}{\hat\delta x^i},\quad \hat u=u^i\frac{\delta}{\delta x^i},\quad \underline{u}=\frac{v}{\alpha},\quad u=\frac{v}{F}$$

where $\{\frac{\delta}{\delta x^i}\}$ and $\{\frac{\hat\delta}{\hat\delta x^i}\}$ are the natural locally horizontal basis of TTM_0 with respect to F and α, respectively.

We define the function $\widetilde{\mathbf{S}}$ defined over TM_0 as follows:

$$\widetilde{\mathbf{S}} := \tilde{\nabla}_{\hat{\mathbf{v}}}\tau,$$

where $\hat{\mathbf{v}}$ is the Riemannian spray associated to α and the function τ is the so-called distortion. Define:

$$\tau_i = \frac{\partial \tau}{\partial y^i}, \qquad \tau_{ij} = \frac{1}{2}\frac{\partial^2 \tau}{\partial y^j \partial y^i}.$$

The $\widetilde{S}$-curvature can be express by

$$\widetilde{\mathbf{S}}(\mathbf{y}) = \frac{\partial \widetilde{G}^i}{\partial y^i}(x,y) - y^i \frac{\partial}{\partial x^i}\Big[\ln \sigma_F(x)\Big],$$

where $\widetilde{G}^i(x)$ denote the geodesic coefficients of α in the same local coordinate system and $\sigma_F(x)$ is the volume form of the Finslerian manifold (M, F).

Elliptic differential operator: In an n-dimensional coordinate neighborhood U, we consider a linear partial differential equation of second order called Elliptic type,

$$L(\varphi) = g^{ik}\frac{\partial^2 \varphi}{\partial x^i \partial x^k} + h^i \frac{\partial \varphi}{\partial x^i},$$

where $g^{jk}(x)$ and $h^i(x)$ are continuous function of point $p(x)$ in U, and quadratic form $g^{jk}Z_jZ_k$ is supposed to be positive definite every where in U. Then we call L the elliptic differential operator.

Principle maximum of Hopf Theorem. In coordinate neighborhood U, if a function $\varphi(p)$ of class C^2 satisfies

$$L(\varphi) \geq 0$$

where $\varphi : M \to R^n$, and if there exist a fixed point p_0 in U such that $\varphi(p) \leq \varphi(p_0)$, $\forall p \in U$, then we have $\varphi(p) = \varphi(p_0)$, $\forall p \in U$. If φ have absolute maximum in U, then φ is constant on U.

Proof of Theorem 1.1: Let the $\widetilde{\mathbf{S}} = 0$ then, it results that the tensor τ_{ij} be $\tilde{\nabla}$-parallel. Writing the Ricci identity of tensor τ_{ij}

$$0 = \tilde{\nabla}_k \tilde{\nabla}_l \tau_{jm} - \tilde{\nabla}_l \tilde{\nabla}_k \tau_{jm} = -\tau_{rm}\tilde{R}^r{}_{jkl} - \tau_{jr}\tilde{R}^r{}_{mkl} - \frac{\partial \tau_{jm}}{\partial y^r}\tilde{R}^r{}_{0kl}. \qquad (3.1)$$

A simple use of Bianchi identity for $\tilde{\nabla}$, results that

$$\nabla_i \tau_{jk} = 0.$$

Multiplying the above relation in v^j, v^l and $\tilde{a}^{km}$, it results:

$$\begin{aligned}\mathbf{D}(\tau) &= \tilde{R}^{r\ \ m}_{\ 0\ \ 0}\frac{1}{2}\frac{\partial^2\tau}{\partial y^r\partial y^m}\\ &= \tau_{rm}\tilde{R}^{r\ \ m}_{\ 0\ \ 0}=0.\end{aligned} \tag{3.2}$$

Let $x\in M$ and denote by $\bar{\tau}$ the restriction of ρ on the indicatrix S_xM of F, we have

$$\partial_\alpha\tau = F\ v^i_\alpha\frac{\partial\tau}{\partial y^i}. \tag{3.3}$$

and then

$$\partial_\beta\partial_\alpha\tau = F\ \partial_\beta v^i_\alpha\ \frac{\partial\tau}{\partial y^i}+F^2\ v^i_\alpha v^j_\beta\ \frac{\partial^2\tau}{\partial y^i\partial y^j}+Lv^j_\beta\ \ \frac{\partial F}{\partial y^j}\ v^i_\alpha\ \frac{\partial\tau}{\partial y^i}, \tag{3.4}$$

But, we have

$$v^j_\beta\ \frac{\partial F}{\partial y^j}=0.$$

Thus,

$$\partial_\beta\partial_\alpha\tau = F\ \partial_\beta v^i_\alpha\ \frac{\partial\tau}{\partial y^i}+F^2\ v^i_\alpha v^j_\beta\ \frac{\partial^2\tau}{\partial y^i\partial y^j} \tag{3.5}$$

Multiplying the above relation in $\tilde{R}^{\alpha\beta}=\tilde{R}^{\alpha\ \beta}_{\ n\ n}$ we have

$$\tilde{R}^{\alpha\beta}\partial_\beta\partial_\alpha\tau = F^2\tilde{R}^{i\ \ j}_{\ n\ n}\ \frac{\partial^2\tau}{\partial y^i\partial y^j}+F\tilde{R}^{\alpha\beta}\ \partial_{_\beta}v^i_\alpha\ \ \frac{\partial\tau}{\partial y^i}. \tag{3.6}$$

Put

$$B^\alpha = v^\alpha_i H^i_{\beta n\gamma}\tilde{a}^{\beta\gamma}.$$

Therefore, rewrite (3.2) on S_xM

$$\tilde{\mathbf{D}}(\tau) := \tilde{R}^{\alpha\beta}\partial_\beta\partial_\alpha\tau - B^\alpha\ \partial_{_\alpha}\tau = 0\ ,\quad (\alpha,\ \beta=1,...,n-1) \tag{3.7}$$

S_xM is compact and from the hypothesis of the theorem, we know that the quantity $H^{\alpha\beta}X_\alpha X_\beta$ is positive (or negative) for any vector X tangent to S_xM. In this case, the partial differential operator $\tilde{\mathbf{D}}$ is an elliptic operator. Therefore, from the last equation and the maximum principle of Hopf it results that ρ is constant on S_xM and therefore,

$$\tau(x,y)=f(x).$$

It means that F is a Riemannian metric. In this case τ is a constant. The converse of the theorem is trivial. $\square$

4. $\widetilde{\mathbf{S}}$-curvature of Randers Metrics

Randers metrics are among the simplest non-Riemannian Finsler metrics, so that many well-known geometric quantities are computable. In this section, we compute the non-Riemannian quantity $\widetilde{\mathbf{S}}$ for a Randers metric. Let $F = \Phi(\beta/\alpha)\alpha$ be an (α, β)-metric and $\tilde{\nabla}$ and ∇ denote the Levi-Civita and Chern connections associated to α and F, respectively. Put

$$\widetilde{\mathbf{S}} = \tilde{\nabla}_{\hat{\mathbf{v}}}\tau,$$

where $\hat{\mathbf{v}}$ denotes the Riemannian spray associated of α. Suppose that we denote the geodesic spray coefficients of α and F by the notions $\widetilde{G}^i$ and G^i, respectively. Let $F = \alpha + \beta$ be a Randers metric on a manifold M, where

$$\alpha(y) = \sqrt{a_{ij}(x)y^i y^j}, \qquad \beta(y) = b_i(x)y^i$$

with $\|\beta\|_x := \sup_{y\in T_xM} \beta(y)/\alpha(y) < 1$. Define $b_{i|j}$ by

$$b_{i|j}\theta^j := db_i - b_j\theta_i^{\ j},$$

where $\theta^i := dx^i$ and $\theta_i^{\ j} := \tilde{\Gamma}^j_{ik}dx^k$ denote the Levi-Civita connection forms of α. Let

$$r_{ij} := \frac{1}{2}\Big(b_{i|j} + b_{j|i}\Big), \qquad s_{ij} := \frac{1}{2}\Big(b_{i|j} - b_{j|i}\Big),$$

$$s^i_{\ j} := a^{ih}s_{hj}, \qquad s_j := b_i s^i_{\ j}, \qquad e_{ij} := r_{ij} + b_i s_j + b_j s_i.$$

Then G^i are given by

$$G^i = \widetilde{G}^i + \frac{e_{00}}{2F}y^i - s_0 y^i + \alpha s^i_{\ 0}, \tag{4.1}$$

where

$$e_{00} := e_{ij}y^i y^j, \quad s_0 := s_i y^i, \quad s^i_{\ 0} := s^i_{\ j}y^j$$

and $\bar{G}^i$ denote the geodesic coefficients of α. See [1].

Now, we calculate $\widetilde{S}$ for a Randers metric:

$$\begin{aligned}\widetilde{\mathbf{S}} &= \tilde{\nabla}_{\hat{\mathbf{v}}}\tau = \tilde{\nabla}_{\hat{\mathbf{v}}}\ln\sqrt{det(g_{ij})} - \tilde{\nabla}_{\hat{\mathbf{v}}}\ln\sigma_F \\ &= \frac{1}{2}g^{ij}\frac{\partial g_{ij}}{\partial x^k}y^k - 2g^{ij}C_{ijk}\widetilde{G}^k - \frac{y^m}{\sigma_F}\frac{\partial \sigma_F}{\partial x^m},\end{aligned} \tag{4.2}$$

where $C_{ijk} = \frac{1}{2}[F^2]_{y^iy^jy^k}$. By the relation (4.1) and (4.2), we get

$$\widetilde{\mathbf{S}} = \frac{1}{2}g^{ij}\frac{\partial g_{ij}}{\partial x^k}y^k - 2g^{ij}C_{ijk}G^k + 2g^{ij}C_{ijk}\alpha s^k_{\ 0} - \frac{y^m}{\sigma_F}\frac{\partial \sigma_F}{\partial x^m} \tag{4.3}$$

Since

$$\mathbf{S} = \frac{1}{2}g^{ij}\frac{\partial g_{ij}}{\partial x^k}y^k - 2g^{ij}C_{ijk}G^k - \frac{y^m}{\sigma_F}\frac{\partial \sigma_F}{\partial x^m},$$

then we have

$$\widetilde{\mathbf{S}} = \mathbf{S} + 2I_k\alpha s^k_{\ 0}. \tag{4.4}$$

Corollary 4.1. *Let $F = \alpha + \beta$ be a Randers metric on an n-manifold M, where $\alpha = \sqrt{a_{ij}(x)y^i y^j}$ and $\beta = b_i(x)y^i$. Then $\widetilde{\mathbf{S}} = 0$ if and only if*

$$\mathbf{S} = -2I_k \alpha s^k{}_0.$$

Moreover, if β is a close 1-form then $\widetilde{\mathbf{S}} = \mathbf{S}$.

Example 4.2. ([9]) The Funk metric on a strongly convex domain $\Omega \subset \mathbb{R}^n$ is a nonnegative function on $T\Omega = \Omega \times \mathbb{R}^n$, which in the special case $\Omega = \mathbb{B}^n$ (the unit ball in the Euclidean space $\mathbb{R}^n$) is defined by the following explicit formula:

$$F(y) := \frac{\sqrt{|y|^2 - (|x|^2|y|^2 - <x, y>^2)} + <x, y>}{1 - |x|^2}, \quad y \in T_x\mathbb{B}^n = \mathbb{R}^n$$

where $|.|$ and $<,>$ denote the Euclidean norm and inner product in $\mathbb{R}^n$, respectively. The Funk metric on $\mathbb{B}^n$ is a Randers metric. For Funk metric we have:

$$G^i(y) = \frac{1}{2}F(y)y^i.$$

Then for every Funk metric we have $\mathbf{S} = \frac{n+1}{2}F$. Thus

$$\widetilde{\mathbf{S}} = \frac{n+1}{2}F + 2I_k \alpha s^k{}_0. \tag{4.5}$$

Regarding the Berwald curvature of Funk metric, Cheng-Shen introduced the notion of isotropic Berwald metrics [6]. A Finsler metric F is said to be isotropic Berwald metric if its Berwald curvature is in the following form

$$B^i{}_{jkl} = \sigma\Big\{F_{y^j y^k}\delta^i{}_l + F_{y^k y^l}\delta^i{}_j + F_{y^l y^j}\delta^i{}_k + F_{y^j y^k y^l}y^i\Big\}, \tag{4.6}$$

for some scalar function $\sigma = \sigma(x)$ on M. Berwald metrics are trivially isotropic Berwald metrics. Funk metrics are also non-trivial isotropic Berwald metrics $\sigma = \frac{1}{2}$.

In [12], it is proved that every Finsler metric of isotropic Berwald curvature (4.6) has isotropic S-curvature. Then we conclude the following.

Corollary 4.3. *Let $F = \alpha + \beta$ be a Randers metric on an n-manifold M, where $\alpha = \sqrt{a_{ij}(x)y^i y^j}$ and $\beta = b_i(x)y^i$. Suppose that F has isotropic Berwald curvature* (4.6)*. Then*

$$\widetilde{\mathbf{S}} = (n+1)cF + 2I_k \alpha s^k{}_0.$$

A Finsler metric on an open subset in $\mathbb{R}^n$ is said to be projectively flat if all geodesics of F are straight in the domain. A Finsler metric on a manifold M is said to be locally projectively flat if at any point, there is a local coordinate system (x^i) in which F is projectively flat. Let F be a smooth and strongly convex Finsler metric on a convex domain $\mathcal{U} \subset R^n$. Then F is projectively flat if only if there exists scalar homogeneous function $P : T\mathcal{U} \to \mathbb{R}$ such that the its spray coefficients satisfy

$$G^i(x,y) = P(x,y)y^i.$$

In this case, $P = P(x,y)$ is called the projective factor.

Now, let $F = \alpha + \beta$ be a locally projectively flat Randers metric on an n-manifold M. Therefore by proposition 4.3.5, page 51 of Chern-Shen, α is locally projectively flat and then

$$s^k_0 = 0.$$

In this case, we get $\widetilde{\mathbf{S}} = \mathbf{S}$.

The Douglas metrics are extension of Berwald metrics, which introduced by Douglas as a projective invariant in Finsler geometry. A Finsler metric is called a Douglas metric if

$$G^i = \frac{1}{2}\Gamma^i_{jk}(x)y^j y^k + P(x,y)y^i,$$

where $\Gamma^i_{jk} = \Gamma^i_{jk}(x)$ is a scalar function on M and $P = P(x,y)$ is a homogeneous function of degree one with respect to y on TM_0. Equivalently, a Finsler metric is a Douglas metric if and only if $G^i y^j - G^j y^i$ are homogeneous polynomials in (y^i) of degree three. If $P = 0$, then F reduces to a Berwald metric. If $\Gamma = 0$, then F is a projectively flat Finsler metric.

For non-zero vector $y \in T_xM_0$, define $\mathbf{D}_y : T_xM \otimes T_xM \otimes T_xM \to T_xM$ by $\mathbf{D}_y(u,v,w) := D^i{}_{jkl}(y)u^i v^j w^k \frac{\partial}{\partial x^i}|_x$, where

$$D^i{}_{jkl} := \frac{\partial^3}{\partial y^j \partial y^k \partial y^l}\left[G^i - \frac{2}{n+1}\frac{\partial G^m}{\partial y^m}y^i\right].$$

$\mathbf{D}$ is called the Douglas curvature. F is called a Douglas metric if $\mathbf{D} = \mathbf{0}$ [2]. By definition, it follows that the Douglas tensor $\mathbf{D}_y$ is symmetric trilinear form and has the following properties

$$\mathbf{D}_y(y,u,v) = 0, \quad trace(\mathbf{D}_y) = 0.$$

We have the following.

Corollary 4.4. *Let $F = \alpha + \beta$ be a Douglas-Randers metric on an n-manifold M, where $\alpha = \sqrt{a_{ij}(x)y^i y^j}$ and $\beta = b_i(x)y^i$. Then $\widetilde{\mathbf{S}} = \mathbf{S}$.*

Proof. In [2], it is proved that a Randers metric $F = \alpha + \beta$ is a Douglas metric if and only if β is a closed one-form. Then by (4.4), we get the proof. □

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