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On conformally flat cubic (α, β)-metrics

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Abstract. In this paper, we study the class of conformally flat cubic (α, β)-metrics. We prove that every conformally flat cubic (α, β)-metric with relatively isotropic mean Landsberg curvature must be either Riemannian metrics or locally Minkowski metrics.

Keywords: Cubic metric, (α, β)-metric, Conformally flat metric, relatively isotropic mean Landsberg curvature.

1. Introduction

The Conformal Geometry is the study of the set of angle-preserving transformations on a manifold. The study of Conformal Geometry has an old and beautiful history in Mathematics. Indeed, Conformal Geometry has played an important role in Differential Geometry and Physical Theories. The conformal change of Riemannian metrics and its related subject such as Riemannian curvature and Ricci curvature have been studied by many geometers. There are many important local and global results in Riemannian conformal geometry, which in turn lead to a better understanding on Riemann manifolds.

On the other hand, Finsler geometry is just Riemannian geometry without the quadratic restriction. The well-known Weyl theorem shows that the projective and conformal properties of a Finsler space determine the metric properties uniquely. This means that the conformal properties of a Finsler metric and related subject to it deserve extra attention.

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Let F and $\tilde{F}$ be two arbitrary Finsler metrics on a manifold M Then we say that F is conformal to $\tilde{F}$ if and only if there exists a scalar function $\sigma = \sigma(x)$ such that $F(x, y) = e^{\sigma(x)}\tilde{F}(x, y)$. The scalar function σ is called the conformal factor. A Finsler metric $F = F(x, y)$ on a manifold M is called a conformally flat metric if there exists a locally Minkowski metric $\tilde{F} = \tilde{F}(y)$ such that $F = e^{\kappa(x)}\tilde{F}$, where $\kappa = \kappa(x)$ is a scalar function on M. A new and hot issue is to characterization of conformally flat Finsler metrics. In [2], Asanov constructed a Finslerian metric function on the manifold $N = \mathbb{R}\times M$, where M is a Riemannian manifold endowed with two real functions, and showed that the tangent Minkowski spaces of such a Finsler space are conformally flat. This motivated him to propose a Finslerian extension of the electromagnetic field equations whose solutions are explicit images of the solutions to the ordinary Maxwell equations.

In order to find conformally flat Finsler metrics, we consider the class of m-th root Finsler metrics. Let (M, F) be an n-dimensional Finsler manifold, TM its tangent bundle and (x^i, y^i) the coordinates in a local chart on TM. Let $F : TM \to \mathbb{R}$ be a scalar function defined by

$$F = \sqrt[m]{A},$$

where $A := a_{i_1\dots i_m}(x)y^{i_1}y^{i_2}\dots y^{i_m}$ and $a_{i_1\dots i_m}$ are symmetric in all its indices. Then F is called an m-th root Finsler metric. An m-th root Finsler metric can be regarded as a direct generalization of a Riemannian metric in the sense that the 2-th root metric is a Riemannian metric $F = \sqrt{a_{ij}(x)y^iy^j}$. The fourth root metrics $F = \sqrt[4]{a_{ijkl}(x)y^iy^jy^ky^l}$ are called the quartic metrics. The special quartic metric $F = \sqrt[4]{y^iy^jy^ky^l}$ is called Berwald-Moór metric which plays an important role in theory of space-time structure, gravitation and general relativity. For more progress, see [7], [8], [9] and [11].

In [10], Tayebi-Razgordani proved that every conformally flat weakly Einstein 4-th root (α, β)-metric on a manifold M of dimension $n \geq 3$ is either a Riemannian metric or a locally Minkowski metric. Also, they showed that every conformally flat 4-th root (α, β)-metric of almost vanishing Ξ-curvature on a manifold M of dimension $n \geq 3$ reduces to a Riemannian metric or a locally Minkowski metric.

In this paper, we study conformally flat 3-th root (α, β)-metric with relatively isotropic mean Landsberg curvature. More precisely, we prove the following.

Theorem 1.1. *Let $F = F(x, y)$ be a conformally flat 3-th root (α, β)-metric on a manifold M of dimension $n \geq 3$. Suppose that F has relatively isotropic mean Landsberg curvature*

$$\mathbf{J} + c(x)F\mathbf{I} = 0, \tag{1.1}$$

where $c = c(x)$ is a scalar function on M. Then F reduces to a Riemannian metric or a locally Minkowski metric.

2. Preliminaries

Let M be a n-dimensional C^∞ manifold and $TM = \bigcup_{x\in M} T_xM$ the tangent bundle. Let (M, F) be a Finsler manifold. The following quadratic form $\mathbf{g}_y$ on T_xM is called fundamental tensor

$$\mathbf{g}_y(u, v) = \frac{1}{2}\frac{\partial^2}{\partial s\partial t}\Big[F^2(y + su + tv)\Big]|_{s=t=0}, \quad u, v \in T_xM.$$

Let $x \in M$ and $F_x := F|_{T_xM}$. To measure the non-Euclidean feature of F_x, for a non-zero vector $y \in T_xM_0 := T_xM - \{0\}$, define $\mathbf{C}_y : T_xM \times T_xM \times T_xM \to \mathbb{R}$ by

$$\mathbf{C}_y(u, v, w) := \frac{1}{2}\frac{d}{dt}\Big[\mathbf{g}_{y+tw}(u, v)\Big]_{t=0} = \frac{1}{4}\frac{\partial^3}{\partial r\partial s\partial t}\Big[F^2(y+ru+sv+tw)\Big]_{r=s=t=0},$$

where $u, v, w \in T_xM$. By definition, $\mathbf{C}_y$ is a symmetric trilinear form on T_xM. The family $\mathbf{C} := \{\mathbf{C}_y\}_{y\in TM_0}$ is called the Cartan torsion. Thus $\mathbf{C} = 0$ if and only if F is Riemannian.

For $y \in T_xM_0$, define $\mathbf{I}_y : T_xM \to \mathbb{R}$ by

$$\mathbf{I}_y(u) = \sum_{i=1}^{n} g^{ij}(y)\mathbf{C}_y(u, \partial_i, \partial_j),$$

where $\{\partial_i\}$ is a basis for T_xM at $x \in M$. The family $\mathbf{I} := \{\mathbf{I}_y\}_{y\in TM_0}$ is called the mean Cartan torsion. Thus, $\mathbf{I}_y(u) := I_i(y)u^i$, where $I_i := g^{jk}C_{ijk}$.

On the slit tangent bundle TM_0, the Landsberg curvature $\mathbf{L}_{ijk} := L_{ijk}dx^i \otimes dx^j \otimes dx^k$ is defined by

$$L_{ijk} := C_{ijk;m}y^m,$$

where ";" denotes the horizontal covariant derivative with respect to F.

For an n-dimensional Finsler manifold (M, F), there is a special vector field $\mathbf{G}$ which is induced by F on $TM_0 := TM\backslash\{0\}$. In a standard coordinates (x^i, y^i) for TM_0, it is given by

$$\mathbf{G} = y^i\frac{\partial}{\partial x^i} - 2G^i(x, y)\frac{\partial}{\partial y^i},$$

where

$$G^i := \frac{g^{il}}{4}\Big\{\frac{\partial^2 F^2}{\partial x^k\partial y^l}y^k - \frac{\partial F^2}{\partial x^l}\Big\}$$

The homogeneous scalar functions G^i are called the geodesic coefficients of F. The vector field $\mathbf{G}$ is called the associated spray to (M, F).

The Landsberg curvature can be expressed as following

$$L_{ijk} = -\frac{1}{2}FF_{y^m}[G^m]_{y^iy^jy^k} \tag{2.1}$$

A Finsler metric is called the Landsberg metric if $L_{ijk} = 0$.

The horizontal covariant derivatives of the mean Cartan torsion $\mathbf{I}$ along geodesics give rise to the mean Landsberg curvature $\mathbf{J}_y : T_xM \to \mathbb{R}$ which are defined by $\mathbf{J}_y(u) := J_i(y)u^i$, where

$$J_i := I_{i|s}y^s.$$

Here, "|" denotes the horizontal covariant derivative with respect to the Berwald connection of F. The family $\mathbf{J} := \{\mathbf{J}_y\}_{y\in TM_0}$ is called the mean Landsberg curvature. Also, the mean Landsberg curvature can be expressed as following

$$J_i := g^{jk}L_{ijk} \tag{2.2}$$

A Finsler metric F on a manifold M is called of relatively isotropic mean Landsberg curvature if

$$\mathbf{J} + cF\mathbf{I} = 0,$$

where $c = c(x)$ is a scalar function on M.

In this paper, we will focus on studying (α, β)-metrics. Let "|" denote the covariant derivative with respect to the Levi-Civita connection of α. Denote

$$r_{ij} := \frac{1}{2}\Big(b_{i|j} + b_{j|i}\Big), \qquad s_{ij} := \frac{1}{2}\Big(b_{i|j} - b_{j|i}\Big)$$
$$s^i{}_j := a^{im}s_{mj}, \qquad r^i{}_j := a^{im}r_{mj}, \quad r_j := b^ir_{ij}, \quad s_j := b^is_{ij},$$

where

$$(a^{ij}) := (a_{ij})^{-1}, \qquad b^j := a^{jk}b_k.$$

We put

$$r_0 := r_iy^i, \quad s_0 := s_iy^i, \quad r_{00} := r_{ij}y^iy^j, \quad s_{i0} := s_{ij}y^j.$$

Let G^i and G^i_α denote the geodesic coefficients of F and α respectively in the same coordinate system. Then we have

$$G^i = G^i_\alpha + \alpha Q s^i{}_0 + \Big\{r_{00} - 2Q\alpha s_0\Big\}\Big\{\Psi b^i + \Theta\alpha^{-1}y^i\Big\}, \tag{2.3}$$

where

$$\begin{aligned} Q &:= \frac{\phi'}{\phi - s\phi'}, \\ \Theta &:= \frac{\phi\phi' - s(\phi\phi'' + \phi'\phi')}{2\phi\Big[\phi - s\phi' + (b^2 - s^2)\phi''\Big]}, \\ \Psi &:= \frac{\phi''}{2\Big[\phi - s\phi' + (b^2 - s^2)\phi''\Big]}. \end{aligned}$$

It is easy to see that if $r_{ij} = s_{ij} = 0$, then

$$G^i = G^i_\alpha.$$

In this case, F reduces to a Berwald metric. For more details, see [4] and [6].

Let

$$\Delta := 1 + sQ + (b^2 - s^2)Q',$$
$$\Phi := -(n\Delta + 1 + sQ)(Q - sQ') - (b^2 - s^2)(1 + sQ)Q'',$$
$$\Psi_1 := \sqrt{b^2 - s^2}\Delta^{\frac{1}{2}} \left[\frac{\sqrt{b^2 - s^2}\Phi}{\Delta^{\frac{3}{2}}}\right]',$$
$$h_j := b_j - \alpha^{-1} s y_j.$$

By (2.1), (2.2), (2.3), the mean Landsberg curvature of the (α, β)-metric $F = \alpha\phi(s)$, $s = \beta/\alpha$, is given by

$$\begin{aligned} J_j = \frac{1}{2\alpha^4\Delta}\Bigg\{ & \frac{2\alpha^3}{b^2 - s^2}\Big[\frac{\Phi}{\Delta} + (n+1)(Q - sQ')\Big](s_0 + r_0)h_j \\ & + \frac{\alpha^2}{b^2 - s^2}\Big[\Psi_1 + s\frac{\Phi}{\Delta}\Big](r_{00} - 2\alpha Q s_0)h_j \\ & + \alpha\Big[-\alpha^2 Q' s_0 h_j + \alpha Q(\alpha^2 s_j - y_j s_0) + \alpha^2\Delta s_{j0} \\ & + \alpha^2(r_{j0} - 2\alpha Q s_j) - (r_{00} - 2\alpha Q s_0)y_j\Big]\frac{\Phi}{\Delta}\Bigg\}. \end{aligned}$$

Here, $y_j = a_{ij}y^i$. See [3] and [5].

In [5], Li-Shen considered weakly Landsberg (α, β)-metric and proved the following.

Theorem 2.1. ([5]) *Let $F = \alpha\phi(\beta/\alpha)$ be an almost regular non-Riemannian (α, β)-metric on a manifold M of dimension $n \geq 3$. Then F is a weakly Landsberg metric if and only if β satisfies*

$$r_{ij} = k\Big\{b^2 a_{ij} - b_i b_j\Big\}, \qquad s_{ij} = 0, \tag{2.4}$$

where $k = k(x)$ is a scalar function, and $\phi = \phi(s)$ satisfies

$$\Phi = \frac{\lambda}{\sqrt{b^2 - s^2}}\Delta^{\frac{3}{2}}, \tag{2.5}$$

where λ is a constant.

3. Proof of Theorem 1.1

in this section, we are going to prove Theorem 1.1. To prove it, we need the following.

Lemma 3.1. ([3]) For an (α, β)-metric $F = \alpha\phi(s)$, $s = \beta/\alpha$, the mean Cartan torsion is given by

$$I_i = -\frac{1}{2F}\frac{\Phi}{\Delta}(\phi - s\phi')h_i. \tag{3.1}$$

In [3], the following was proved.

Lemma 3.2. ([3]) An (α,β)-metric F is a Riemannian metric if and only if $\Phi = 0$.

In [3], the following formula obtained

$$\begin{aligned} J_j + c(x)FI_j = -\frac{1}{2\alpha^4\Delta}\Bigg\{ & \frac{2\alpha^3}{b^2-s^2}\left[\frac{\Phi}{\Delta} + (n+1)(Q-sQ')\right](s_0+r_0)h_j \\ & + \frac{\alpha^2}{b^2-s^2}\left[\Psi_1 + s\frac{\Phi}{\Delta}\right](r_{00}-2\alpha Q s_0)h_j \\ + \alpha\Big[& -\alpha^2 Q' s_0 h_j + \alpha Q(\alpha^2 s_j - y_j s_0) + \alpha^2\Delta s_{j0} + \alpha^2(r_{j0} - 2\alpha Q s_j) \\ & -(r_{00} - 2\alpha Q s_0)y_j\Big]\frac{\Phi}{\Delta} + c(x)\alpha^4\Phi(\phi - s\phi_{\prime})h_j\Bigg\}. \quad (3.2) \end{aligned}$$

Also, we remark the following key lemma.

Lemma 3.3. ([1]) Let $F = \alpha\phi(s)$, $s = \beta/\alpha$, be an (α,β)-metric. Then F is locally Minkowskian if and only if α is a flat Riemannian metric and β is parallel with respect to α.

Also, the following holds.

Lemma 3.4. ([3]) If $\phi = \phi(s)$ satisfies $\Psi_1 = 0$, then F is Riemannian.

Now, assume that $F = \alpha\phi(s)$, $s = \beta/\alpha$, is a conformally flat Finsler metric, that is, F is conformally related to a Minkowski metric $\tilde{F}$. Then there exists a scalar function $\sigma = \sigma(x)$ on the manifold, so that $\tilde{F} = e^{\sigma(x)}F$. It is easy to see that $\tilde{F} = \tilde{\alpha}\phi(\tilde{s})$, $\tilde{s} = \tilde{\beta}/\tilde{\alpha}$. We have

$$\tilde{\alpha} = e^{\sigma(x)}\alpha, \qquad \tilde{\beta} = e^{\sigma(x)}\beta$$

which are equivalent to

$$\tilde{a_{ij}} = e^{2\sigma(x)}a_{ij}, \qquad \tilde{b_i} = e^{\sigma(x)}b_i.$$

Let "$\|$" denote the covariant derivative with respect to the Levi-Civita connection of $\tilde{\alpha}$. Put

$$\sigma_i := \frac{\partial\sigma}{\partial x^i}, \qquad \sigma^i := a^{ij}\sigma_j.$$

The Christoffel symbols Γ^i_{jk} of α and the Christoffel symbols $\tilde{\Gamma}^i_{jk}$ of $\tilde{\alpha}$ are related by

$$\tilde{\Gamma}^i_{jk} = \Gamma^i_{jk} + \delta^i_j\sigma_k + \delta^i_k\sigma_j - \sigma^i a_{jk}.$$

Hence, one can obtain

$$\tilde{b}_{i\|j} = \frac{\partial \tilde{b}_i}{\partial x^j} - \tilde{b}_s \tilde{\Gamma}^i_{jk} = e^{\sigma}(b_{i|j} - b_j\sigma_i + b_r\sigma^r a_{ij}). \tag{3.3}$$

By Lemma 3.38, for Minkowski metric $\tilde{F}$, we have

$$\tilde{b}_{i\|j} = 0.$$

Thus

$$b_{i|j} = b_j\sigma_i - b_r\sigma^r a_{ij}, \tag{3.4}$$

$$r_{ij} = \frac{1}{2}(\sigma_i b_j + \sigma_j b_i) - b_r\sigma^r a_{ij}, \tag{3.5}$$

$$s_{ij} = \frac{1}{2}(\sigma_i b_j + \sigma_j b_i), \tag{3.6}$$

$$r_j = -\frac{1}{2}(b_r\sigma^r)b_j + \frac{1}{2}\sigma_j b^2, \tag{3.7}$$

$$s_j = \frac{1}{2}(b_r\sigma^r)b_j - \sigma_j b^2, \tag{3.8}$$

$$r_{i0} = \frac{1}{2}[\sigma_i\beta + (\sigma_r y^r)b_i] - \sigma_r b^r y_i, \tag{3.9}$$

$$s_{i0} = \frac{1}{2}[\sigma_i\beta + (\sigma_r y^r)b_i]. \tag{3.10}$$

Further, we have

$$r_{00} = (\sigma_r y^r)\beta - (\sigma_r y^r)\alpha^2, \tag{3.11}$$

$$r_0 = \frac{1}{2}(\sigma_r y^r)b^2 - \frac{1}{2}(\sigma_r b^r)\beta, \tag{3.12}$$

$$s_0 = \frac{1}{2}(\sigma_r y^r)\beta - \frac{1}{2}(\sigma_r y^r)b^2. \tag{3.13}$$

By (3.12) and (3.13), the conformally flat (α, β)-metrics satisfying

$$r_0 + s_0 = 0$$

which is equivalent to the length of β with respect to α being a constant.

We take an orthonormal basis at any point x with respect to α such that

$$\alpha = \sqrt{\sum_{i=1}^{n}(y^i)^2} \quad \text{and} \quad \beta = by^1,$$

where $b := \|\beta_x\|_\alpha$. By using the same coordinate transformation

$$\psi : (s, u^A) \longrightarrow (y^i)$$

in T_xM, we get

$$y_1 = \frac{s}{\sqrt{b^2 - s^2}}\bar{\alpha}, \quad y^A = u^A, \quad 2 \le A \le n, \tag{3.14}$$

where

$$\bar{\alpha} = \sqrt{\sum_{i=2}^{n}(u^A)^2}.$$

We have

$$\alpha = \frac{b}{\sqrt{b^2 - s^2}}\bar{\alpha}, \quad \beta = \frac{bs}{\sqrt{b^2 - s^2}}\bar{\alpha}. \tag{3.15}$$

Put

$$\bar{\sigma_0} := \sigma_A u^A.$$

Then, by (3.4)-(3.8), (3.14) and (3.15) we have

$$r_{00} = -b\sigma_1\bar{\alpha}^2 + \frac{bs\bar{\sigma_0}\bar{\alpha}}{\sqrt{b^2 - s^2}}, \tag{3.16}$$

$$r_0 = \frac{1}{2}b^2\bar{\sigma_0} = -s_0, \tag{3.17}$$

$$r_{10} = \frac{1}{2}b\bar{\sigma_0}, \tag{3.18}$$

$$r_{A0} = \frac{1}{2}\frac{\sigma_A bs\bar{\alpha}}{\sqrt{b^2 - s^2}} - (b\sigma_1)u_A, \tag{3.19}$$

$$s_1 = 0, \qquad s_A = -\frac{1}{2}\sigma_A b^2, \tag{3.20}$$

$$s_{10} = \frac{1}{2}b\bar{\sigma_0}, \qquad s_{A0} = \frac{1}{2}\frac{\sigma_A bs\bar{\alpha}}{\sqrt{b^2 - s^2}}, \tag{3.21}$$

$$h_1 = b - \frac{s^2}{b}, \qquad h_A = -\frac{\sqrt{b^2 - s^2}su_A}{b\bar{\alpha}}. \tag{3.22}$$

Proof of Theorem 1.1: We remark that $\tilde{b} = constant$. If $\tilde{b} = 0$, then $F = e^{k(x)}\tilde{\alpha}$ is a Riemannian metric. Now, let F is not Riemannian metric. Assume that F is a conformally flat (α, β)-metric with relatively isotropic mean Landsberg curvature. By (3.2) and $r_0 + s_0 = 0$, we get

$$\begin{aligned}
&\frac{\alpha^2}{b^2 - s^2}\Big[\Psi_1 + s\frac{\Phi}{\Delta}\Big](r_{00} - 2\alpha Q s_0)h_j + \alpha\Big[-\alpha^2 Q' s_0 h_j + \alpha Q(\alpha^2 s_j - y_j s_0)\\
&+ \alpha^2\Delta s_{j0} + \alpha^2(r_{j0} - 2\alpha Q s_j) - (r_{00} - 2\alpha Q s_0)y_j\Big]\frac{\Phi}{\Delta}\\
&+ c(x)\alpha^4\Phi(\phi - s\phi')h_j = 0.
\end{aligned} \tag{3.23}$$

Letting $j = 1$ in (3.23), we have

$$\begin{aligned}
&\frac{\alpha^2}{b^2 - s_2}\Big[\Psi_1 + s\frac{\Phi}{\Delta}\Big](r_{00} - 2\alpha Q s_0)h_1 + \alpha\Bigg[-\alpha^2 Q' s_0 h_1 + \alpha Q(\alpha^2 s_1 - y_1 s_0)\\
&+ \alpha^2\Delta s_{10} + \alpha^2(r_{10} - 2\alpha Q s_1) - (r_{00} - 2\alpha Q s_0)y_1\Bigg]\frac{\Phi}{\Delta}\\
&+ c(x)\alpha^4\Phi(\phi - s\phi')h_1 = 0.
\end{aligned} \tag{3.24}$$

Putting (3.15)-(3.22) into (3.24) and multiplying the result with $2\Delta(b^2 - s^2)^{5/2}$ implies that

$$\begin{aligned}
&2b^2(b^2-s^2)^{3/2}\Delta(b\Phi\phi c - b\Phi s\phi' c - \Psi_1\sigma_1)\bar{\alpha}^4 \\
&\quad + b^2(b^2-s^2)\bar{\sigma}_0(b^4\Phi Q' - b^2\Phi\Delta - b^2\Phi Q' s^2) \\
&\quad + 2b^2\Psi_1\Delta Q + b^2\Phi + b^2\Phi Q s + 2\Psi_1\Delta s)\bar{\alpha}^3 = 0. \qquad (3.25)
\end{aligned}$$

From (3.25), we get

$$\Delta\Big[b\Phi\phi c - b\Phi s\phi' c - \Psi_1\sigma_1\Big] = 0, \qquad (3.26)$$

$$\begin{aligned}
&\bar{\sigma}_0(b^4\Phi Q' - b^2\Phi\Delta - b^2\Phi Q' s^2) + 2b^2\Psi_1\Delta Q \\
&\qquad + b^2\Phi + b^2\Phi Q s + 2\Psi_1\Delta s = 0. \qquad (3.27)
\end{aligned}$$

Note that $\Delta = Q'(b^2 - s^2) + sQ + 1$. Simplifying (3.27) yields

$$(b^2\Psi_1\Delta Q + \Psi_1\Delta s)\bar{\sigma_0} = 0,$$

that is

$$\Psi_1\Delta(b^2 Q + s)\bar{\sigma_0} = 0. \qquad (3.28)$$

Letting $j = A$ in (3.23), we have

$$\begin{aligned}
&\frac{\alpha^2}{b^2-s^2}\Big[\Psi_1 + s\frac{\Phi}{\Delta}\Big](r_{00} - 2\alpha Q s_0)h_A + \alpha\Big[-\alpha^2 Q' s_0 h_A + \alpha Q(\alpha^2 s_A - y_A s_0) \\
&+ \alpha^2\Delta s_{A0} + \alpha^2(r_{A0} - 2\alpha Q s_A) - (r_{00} - 2\alpha Q s_0)y_A\Big]\frac{\Phi}{\Delta} \\
&+ c(x)\alpha^4\Phi(\phi - s\phi')h_A = 0. \qquad (3.29)
\end{aligned}$$

Putting (3.15)-(3.22) into (3.29) and by using the similar method used in the case of $j = 1$, we get

$$\begin{aligned}
&-(s\Delta + s + b^2 Q)b^2\Phi\sigma_A\bar{\alpha}^2 + \Big[(s\Delta + s + b^2Q)b^2\Phi \\
&\qquad + 2s(b^2Q + s)\Psi_1\Delta\Big]\bar{\sigma}_0 u_A = 0, \qquad (3.30)
\end{aligned}$$

and

$$s(b^2 - s^2)\Big[b(\phi - s\phi')\Phi c - \Psi_1\sigma_1\Big]\Delta u_A = 0. \qquad (3.31)$$

It is easy to see that (3.31) is equivalent to (3.26). Further, multiplying (3.30) with u^A implies that

$$s(b^2Q + s)\Psi_1\Delta\bar{\sigma}_0\bar{\alpha}^2 = 0. \qquad (3.32)$$

It is easy to see that (3.32) is equivalent to (3.28). In summary, conformally flat (α, β)-metrics with relatively isotropic mean Landsberg curvature satisfy (3.26) and (3.28). According to (3.28), we have some cases as follows:

Case (i): If $b^2Q + s = 0$, then we have

$$\phi = \kappa\sqrt{b^2 - s^2},$$

which is a contradiction with the assumption of cubic metric. Then we have $b^2Q + s \neq 0$.

Case (ii): If $\Psi_1 = 0$, then by Lemma 3.4, F is Riemannian.

Case (iii): If $\Psi_1 \neq 0$, then $\sigma_A = 0$. In the following, we prove that if $\Psi_1 \neq 0$, then by (3.26) one can get $\sigma_1 = 0$.

Now, assume that

$$\phi = \sqrt[3]{a_1 s + a_2 s^3} \tag{3.33}$$

here a_1, a_2 are numbers independent of s and $a_i \neq 0, i = 1, 2$. Simplifying (3.26) and multiplying it by Δ^2, we get

$$\Big\{[-s\Phi + (b^2 - s^2)\Phi']\Delta - \frac{3}{2}(b^2 - s^2)\Phi\Delta'\Big\}\sigma_1 - b\Delta^2\Phi(\phi - s\phi')c = 0. \tag{3.34}$$

Putting (3.33) into (3.34) and multiplying it by

$$\vartheta := \frac{1}{27a_1^2 s^4 (a_2 s^2 + a_1)^3}, \tag{3.35}$$

we can express the result as a polynomial of s

$$E_{15}s^{15} + E_{14}s^{14} + ...E_1 s + E_0 = 0 \tag{3.36}$$

where $E_i (0 \leq i \leq 15)$, are polynomials of $a_1,\ a_2,\ b,\ c$, and σ_1. Equation (3.36) is equivalent to the following two equations

$$E_{15}s^{15} + E_{13}s^{13} + ... + E_3 s^3 + E_1 s = 0, \tag{3.37}$$

$$E_{14}s^{14} + E_{12}s^{12} + ... + \Pi_2 s^2 + E_0 = 0, \tag{3.38}$$

where

$$E_0 = 6a_1^3 b^6 (35a_1^3 + 6a_1^2 + 40)\sigma_1.$$

(3.38) implies that $E_0 = 0$, because $b \neq 0$ and $a_1 \neq 0$, then $E_0 = 0$ implies that $\sigma_1 = 0$. Together with $A = 0$, it follows that σ is a constant, which means that F is a locally Minkowski metric. This completes the proof. $\square$

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References

1. P.L. Antonelli, R.S. Ingarden and M. Matsumoto, *The Theory of Sprays and Finsler Spaces with Applications in Physics and Biology*, Kluwer Academic Publishers, Dordrecht, 1993.
2. G.S. Asanov, *Finslerian metric functions over product $R \times M$ and their potential applications*, Rep. Math. Phys. **41**(1998), 117-132.
3. X. Cheng, H. Wang and M. Wang, *(α, β)-metrics with relatively isotropic mean Landsberg curvature*, Publ. Math. Debrecen, **72**(2008), 475-485.
4. S. S. Chern and Z. Shen, *Riemann-Finsler Geometry*, World Scientiflc, Singapore, 2005.
5. B. Li and Z. Shen, *On a class of weak Landsberg metrics*, Science in China Series A, **50**(2007), 75-85.
6. T. Tabatabaeifar, *On generalized 4-th root Finsler metrics*, Journal of Finsler Geometry and its Applications, **1**(1) (2020), 54-59.
7. A. Tayebi, *On generalized 4-th root metrics of isotropic scalar curvature*, Mathematica Slovaca, **68**(2018), 907-928.
8. A. Tayebi and B. Najafi, *On m-th root Finsler metrics*, J. Geom. Phys. **61**(2011), 1479-1484.
9. A. Tayebi and B. Najafi, *On m-th root metrics with special curvature properties*, C. R. Acad. Sci. Paris, Ser. I. **349**(2011), 691-693.
10. A. Tayebi and M. Razgordani, *On conformally flat fourth root (α, β)-metrics*, Differ. Geom. Appl. **62**(2019), 253-266.
11. A. Tayebi and M. Shahbazi Nia, *A new class of projectively flat Finsler metrics with constant flag curvature* $\mathbf{K} = 1$, Differ. Geom. Appl. **41**(2015), 123-133.

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