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On the locally flat Finsler manifolds

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Abstract. It is proved that every locally flat Finsler manifold is a locally flat Riemannian manifold. Some low dimensional locally flat Finsler manifolds are classified. It is also proved that in a categorical sense, there is a correspondence between locally flat Finsler manifolds and locally hessian Riemannian manifolds.

Keywords: Locally flat manifolds, Finsler metric, Bieberbach groups, fourth root metric.

1. Introduction

An n-dimensional Riemannian manifold $(M, \alpha = \sqrt{a_{ij}(x)y^i y^j})$ is said to be *locally flat* (or locally Euclidean) if (M, α) locally isometric with Euclidean space, that is , M admits a covering of coordinates neighborhoods each of which isometric with a Euclidean domain. A Riemannian manifold (M, α) is locally flat if and only if M admits a covering of coordinates neighborhoods on each of, the function $\alpha(x, y)$ is independent of x. A classical result affirms that a Riemannian manifold is locally flat if and only if its Riemann curvature vanishes (equivalently, the sectional curvature $\mathbf{K}_\alpha$ vanishes); This is usually taken as the definition of a locally flat Riemannian manifold in the contexts. The universal Riemannian covering space of a complete locally flat Riemannian manifold is the Euclidean space $\mathbb{E}^n = (\mathbb{R}^n, \alpha_0 = \sqrt{\delta_{ij}y^i y^j})$. Up to local isometry, Bieberbach proved that a compact locally flat Riemannian manifold, is realized as a quotient space $\mathbb{R}^n/\Gamma$, where Γ is a discrete, co-compact and

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torsion free subgroup of the Euclidean group $Isom(\mathbb{E}^n) = \mathbb{E}(n) = O(n) \ltimes \mathbb{R}^n$ group, see [1, 1]; Three dimensional cases was proved by Schoenflies in [7] earlier. 1 dimensional the only complete, locally flat and connected manifolds are $\mathbb{R}$ and $\mathbb{S}^1$. In 2 dimensions, the only complete, locally flat and connected manifolds are cylinder, Möbius strip, Torus and Klein bottle. In 3 dimensions, there are only 10 complete, locally flat and connected manifolds including 6 oriented manifolds and the remainder are non-oriented, cf. [14].

It is remarkable that likewise the Riemannian case, a Finslerian manifold $(M, F = \sqrt{g_{ij}(x,y)y^i y^j})$ is said to be *locally flat* (or locally Minkowskian) if, M admits a covering of coordinates neighborhoods each of which isometric with a single Minkowski normed domain. A Finslerian manifold (M, F) is locally flat if and only if M admits a covering of coordinates neighborhoods on each of, the function $F(x, y)$ is independent of x. The flag curvature $\mathbf{K}$ of any locally flat Finsler manifold vanishes identically; while the converse may not be true generally. It seems that the locally flat Finsler manifolds are multifarious however, we prove the following theorem.

Theorem 1.1. *Every locally flat Finsler manifold is a locally flat Riemannian manifold.*

Therefore, we proved the following result:

Theorem 1.2. *There are only two non-isomorphic Bieberbach groups in 2 dimensions with respect to the fourth root metric.*

We use the natural representation of $\mathbb{E}(n) = Isom(\mathbb{R}^n, d)$ in $Gl(n + 1\mathbb{R})$ given by

$$T(x) = Ax + b \in \mathbb{E}(n) \mapsto \begin{bmatrix} A & b \\ 0 & 1 \end{bmatrix}$$

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2. Riemann-Finsler manifolds

Let M be a n-dimensional smooth connected manifold. The tangent space of M at $x \in M$ is denoted by T_xM and the tangent manifold of M is the disjoint union of tangent spaces $TM := \bigcup_{x\in M} T_xM$. Every element of TM is a pair (x, y) where $x \in M$ and $y \in T_xM$. Denote the slit tangent manifold by $TM_0 = TM \setminus \{0\}$. The natural projection $\pi : TM \to M$ given by $\pi(x, y) := x$ makes TM a vector bundle of rank n over M and TM_0 a fiber bundle over M with fiber type $\mathbb{R}^n \setminus \{0\}$.

A *Finsler metric* on M is a function $F : TM \to [0,\infty)$ satisfying following conditions: (i) F is C^∞ on TM_0, (ii) $F(x,y)$ is positively 1-homogeneous y and (iii) the Hessian matrix of F^2 with entries

$$g_{ij}(x,y) := \frac{1}{2}[F^2(x,y)]_{y^iy^j}$$

is positively defined on TM_0.

Given any Finsler metric F on M, the pair (M,F) is called a *Finsler space*. Traditionally, we denote a Riemannian metric by $\alpha = \sqrt{a_{ij}(x)y^iy^j}$. The Cartan and mean Cartan tensors are defined as follows:

$$\mathbf{C} = C_{ijk}\mathrm{d}x^i \otimes \mathrm{d}x^j \otimes \mathrm{d}x^k, \qquad C_{ijk} = \frac{1}{4}F^2_{y^iy^jy^k}, \qquad \textit{(Cartan tensor)},$$

$$\mathbf{I} = I_k\mathrm{d}x^i, \qquad I_k = g^{ij}C_{ijk} \qquad \textit{(mean Cartan tensor)}.$$

By Deicke's theorem, a Finsler metric $F = F(x,y)$ is Riemannian if and only if $\mathbf{I} = 0$ (see [4]).

The geodesic spray G is naturally induced by F on TM_0 given in any standard coordinate (x^i, y^i) for TM_0 by

$$\mathbf{G} = y^i\frac{\partial}{\partial x^i} - 2G^i(x,y)\frac{\partial}{\partial y^i},$$

where $G^i(x,y)$ are local functions on TM_0 given by

$$G^i := \frac{1}{4}g^{ih}\Big\{y^kF^2_{x^ky^h} - F^2_{x^h}\Big\}.$$

Assume the following conventions:

$$G^i{}_j = \frac{\partial G^i}{\partial y^i}, \quad G^i{}_{jk} = \frac{\partial G^i{}_j}{\partial y^k}, \quad G^i{}_{jkl} = \frac{\partial G^i{}_{jk}}{\partial y^l}.$$

The local functions $G^i{}_j$ are coefficients of a Connection in the pullback bundle $\pi^*TM \longrightarrow M$ which is called the *Berwald connection* denoted by D. By $_{|i}$ we mean the i-th horizontal derivation with respect to D.

The *Busemann-Hausdorff volume form* $dV_F = \sigma_F(x)dx^1 \cdots dx^n$ on any Finsler space (M,F) is defined by

$$\sigma_F(x) := \frac{\mathrm{Vol}(\mathbb{B}^n(1))}{\mathrm{Vol}\Big\{(y^i) \in \mathbb{R}^n \mid F(y^i\frac{\partial}{\partial x^i}|_x) < 1\Big\}}.$$

Assume that

$$\underline{g} = \det(g_{ij}(x,y))$$

and define

$$\tau(x,y) := \ln\frac{\sqrt{\underline{g}}}{\sigma_F(x)}.$$

Then, $\tau = \tau(x, y)$ is a scalar function on TM_0, which is called the *distortion* [15]. For a vector $\mathbf{y} \in T_xM$, let $c(t), -\epsilon < t < \epsilon$, denote the geodesic with $c(0) = x$ and $\dot{c}(0) = \mathbf{y}$. The function

$$\mathbf{S}(\mathbf{y}) := \frac{d}{dt}\Big[\tau(\dot{c}(t))\Big]|_{t=0}$$

is called the **S**-curvature with respect to the Busemann-Hausdorff volume form. A Finsler space is said to be *of isotropic* **S**-*curvature* if there is a function $c = c(x)$ defined on M such that

$$\mathbf{S} = (n+1)c(x)F.$$

It is called a Finsler space *of constant* **S**-*curvature* once c is a constant. Every Berwald space is of vanishing **S**-curvature [15]. Notice that, **S**-curvature are in fact non-Riemannian quantities, namely, they vanish for the Riemannian metrics.

Let (M, α) be a Riemannian space and $\beta = b_i(x)y^i$ be a 1-form defined on M such that $\|\beta\|_x := \sup_{y \in T_xM} \beta(y)/\alpha(y) < 1$. The Finsler metric $F = \alpha + \beta$ is called a Randers metric on a manifold M. It is know that a Randers metric $F = \alpha + \beta$ is a solution of the following Zermelo navigation problem:

$$h\Big(x, \frac{y}{F} - V_x\Big) = 1,$$

where $h = \sqrt{h_{ij}(x)y^iy^j}$ is a Riemannian metric and $V = V^i(x)\frac{\partial}{\partial x^i}$ is a vector field with

$$h(x, -V_x) = \sqrt{h_{ij}(x)V^i(x)V^j(x)} < 1.$$

In fact, α and β are given by

$$\alpha = \frac{\sqrt{\lambda h^2 + V_0}}{\lambda}, \quad \text{and} \quad \beta = -\frac{V_0}{\lambda},$$

respectively where

$$\lambda = 1 - h(x, -V_x), \quad \text{and} \quad V_0 = V^iy^jh_{ij}.$$

It is proved that for every Randers metric $F = \alpha+\beta$ on n-dimensional manifold M expressed in terms of a Riemannian metric h and a vector field V, F has isotropic S-curvature $\mathbf{S} = (n + 1)c(x)F(x, y)$ if and only if V is a conformal vector field,

$$\mathcal{L}_V h = -4c(x)h.$$

A locally flat manifold is an n-dimensional manifold M admitting a locally flat liner connection. A Riemannian manifold (M, α) is said to be a Riemannian locally flat manifold if the sectional curvature of α vanishes identically. We may observe immediately that the following conditions are equivalent:

(1) (M, α) is a locally flat Riemannian manifold,
(2) The Cristoffel symbols of α are locally vanishing,

(3) The Riemann curvature tensor vanishes,
(4) M can be covered by an atlas whose coordinate neighborhoods are isometric with an open set of $\mathbb{R}^n$ equipped withe Euclidean metric,
(5) M can be covered by an atlas so that $\alpha(x, y)$ does note depend to x on any coordinates neighbor hood.

Notice that, the conditions (2)-(5) above are not equivalent for Finsler manifolds. However, we prefer to define the locally flat Finsler manifolds in the simplest way as follows: *A Finsler manifold (M, F) is said to be locally flat if M can be covered by an atlas so that $F(x, y)$ does not depend to x on any coordinates neighborhood.*

2.1. **Proof of Theorem 1.1.** Let us suppose that (M, F) be a Finsler space. Given any point $x \in M$, the tangent space T_xM can be equipped with the Riemannian metric $g_x = g_{ij}(x, v)\mathrm{d}v^i \otimes \mathrm{d}v^j$ and the canonical volume form

$$\Omega_x = \sqrt{\det(g_{ij}(x, v))}\mathrm{d}v^1 \wedge \mathrm{d}v^2 \wedge ... \wedge \mathrm{d}v^n.$$

The volume form Ω_x can be pulled back by its the natural embedding $S_{x_0} \hookrightarrow T_xM$ to a volume form ω_{x_0} on the indicatrix S_x. Without loss of generality, we may further assume the

$$\mathrm{Vol}(S_x) = \int_{S_x} \omega_x = 1.$$

One may consider several geometric objects on the manifold M using integration on each indicatrix S_x. The *averaged Riemannian metric of F* is denoted by $\alpha = \sqrt{a_{ij}(x)v^j v^j}$ and is defined by

$$\alpha^2(x, v) = a_{ij}(x)v^i v^j := \int_{S_x} v^i v^j g_{ij}(x, v)\omega_x. \tag{2.1}$$

One may also refer [17] to be informed about other potential averaged structures on Finsler spaces. Now, suppose that the Finsler space (M, F) is locally flat. It follows that M can be covered by an atlas so that $F(x, y)$ does not depend to x on any coordinates neighborhood. It results immediately that the average Riemannian metric $\alpha(x, y)$ given by (2.1) does not depend to x either. Therefore, α is a locally flat Riemannian metric on M and M is a locally flat Riemannian space. □

3. Bieberbach Theorems

Three Bieberbach theorems suggest a good recognition of algebraic structure of crystallographic groups.

1. First theorem asserts that Γ subgroup of all translations in the Euclidean

space $\mathbb{E}^n$ is a normal subgroup of Isom($\mathbb{E}^n$) which is of finite index. This subgroup is inclusive n translation which are linear independent, with their help we have a short exact sequence

$$1 \to \mathbb{Z}^n \to \Gamma \to G \to 1.$$

Above $\mathbb{Z}^n$ is identical with subgroup of all translation in Γ and G corresponds with $\Gamma/\mathbb{Z}^n$ is finite. By this short exact sequence we can make a homomorphism

$$h_\Gamma : G \longrightarrow GL\,(n, \mathbb{Z})\,.$$

h_Γ was named Holonomy description. If a crystallographic group is torsion free, then it calls Bieberbach group.

2. These groups have important properties, because their quotient spaces, which represented $M^n = \mathbb{R}^n/\Gamma$ are manifolds with fundamental groups Γ. This sentence means as Γ acts on $\mathbb{R}^n$ by isometries contemporary M^n manifold enjoys Riemannian structure by Euclidean space. this means M^n is a locally flat manifold. On the other words two crystallographic groups are equivalent if and only if these are isomorphic in algebra.

3. Since in third Bieberbach Theorem said that, in any finite dimension there are only finite number of crystallographic group (with isomorphism), can classification these groups.

4. Bieberbach Groups and Fourth Root Metric

First, we remark that every discrete and compact subgroup of $\mathbb{E}(n) = O(n) \ltimes \mathbb{R}^n = Isom(\mathbb{R}^n, d_0)$ is called a *crystallographic group*, where d_0 denotes the Euclidean metric, cf. [14]. Three Bieberbach theorems suggest a good recognition of the algebraic structure of the crystallographic groups. First theorem asserts that given any crystallographic group Γ, the set of translations $\Gamma \cap (\{I_n\} \times \mathbb{R}^n)$ is a torsion free and finitely generated Abelian group of rank n and finite index in $\mathbb{E}(n)$. This subgroup contains n translation(s) which are linearly independent and Γ satisfies the following short exact sequence (cf. [8]):

$$1 \to \mathbb{Z}^n \to \Gamma \to G \to 1,$$

where, $G = \Gamma/\mathbb{Z}^n$ is a finite group. This short exact sequence yields a homomorphism $h_\Gamma : G \longrightarrow GL\,(n, \mathbb{Z})$. h_Γ which is called the *Holonomy description*. Every torsion free crystallographic group is called a *Bieberbach group*. These groups are interesting since given any Bieberbach group $\Gamma \subseteq \mathbb{R}^n$, the quotient space, $\mathbb{R}^n/\Gamma$ is an n-diemnsional manifold with fundamental groups Γ and are locally isometric with $(\mathbb{R}^n, d_0)$, in particular, $\mathbb{R}^n/\Gamma$ is a locally flat manifold. Indeed, the second Bieberbach's Theorem asserts that, two crystallographic groups of dimension n are isomorphic if and only if they are conjugate in the affine group $\mathbb{A}(n)$; Moreover, the third Bieberbach's Theorem asserts that, in any finite dimensions there are only finitely many crystallographic groups (up to isomorphism).

The above procedure can be reconstructed starting any other Minkowski norm-derived metrics on $\mathbb{R}^n$. An isometry $(\mathbb{R}^n, d)$ is a surjective mapping $f : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ such that we have $d(x, y) = d(f(x), f(y)),\ x, y \in \mathbb{R}^n$. The set of isometries of $\mathbb{R}^n, d)$ forms the isometry group $Isom(\mathbb{R}^n, d)$. We would like to classify the discrete and cocompact subgroups of $\mathbb{R}^n, d)$ with respect to other important metrics; These subgroups are in fact counterparts for the crystallographic groups. We may go further and distinguish such groups possessing torsion freeness; These groups are also alternatives for the Bieberbach group. Any discrete and cocompact subgroup of $Isom(\mathbb{R}^n, d)$ is called a d-crystallographic group. The following classical result ensures that any d-crystallographic group contains only affine transformations in $\mathbb{A}(n)$.

Theorem 4.1. ([6]) (Mazur-Ulam 1932) Every bijective isometry between real normed spaces is affine.

For example, we may endow $\mathbb{R}^n$ with the norm-derived metric: the 4th root metric d_4. This metric is given by:

$$\begin{cases} \|\cdot\|_4 : \mathbb{R}^n \longrightarrow \mathbb{R}, \\ \|x\|_4 = \|(x_1, x_2, \ldots, x_n)\| = \sqrt[4]{x_1^4 + x_2^4 + \ldots + x_n^4}. \end{cases}$$

$$\begin{cases} d : \mathbb{R}^n \times \mathbb{R}^n \longrightarrow \mathbb{R}, \\ d(x, y) = \|x - y\|_4 = \sqrt[4]{\sum_{i=1}^n (x_i - y_i)^4}. \end{cases}$$

5. Proof of Theorem 1.2

Consider the subgroup of $GL(n, \mathbb{R})$ containing matrices that their entries belong to $\{-1, 0, 1\}$ There are 81 different such matrices where $n = 2$, including the zero matrix, 8 matrices having 3 zero entries, 16 matrices have a zero column or a zero row, so their determinant are equal to zero and these matrices are not in $Gl(2, \mathbb{R})$. 4 matrices have 4 different entries, but two row or two column are the same, so their determinant are equal to zero and these matrices are not in $Gl(2, \mathbb{R})$. 52 remaining matrices are nonsingular and the following remaining 8 matrices are $\|.\|_4$ preserving, i.e $\|Ax\|_4 = \|X\|_4, (x \in \mathbb{R}^n)$, A is a square matrix that its order is 2:

$$A_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, A_2 = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}, A_3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, A_4 = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix},$$
$$A_5 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, A_6 = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}, A_7 = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, A_8 = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}.$$

Every isometry is presented by a pair (A, b) of a matrix $A \in GL(2, \mathbb{R})$ and a vector $b \in \mathbb{R}^2$ 4.1. It is well-known that, as a subgroup of $Aff(\mathbb{R}^2)$, the isometry group has a $Gl(3,\mathbb{R}^3)$-representation, [8]. The possible Bieberbach groups are listed by $G_1, ..., G_7$ noticing their generators as follows:

$$G_1 = \Big\{I, \Gamma_n | n \in \mathbb{Z}\Big\},$$

where

$$\Gamma_n := \begin{bmatrix} 1 & 0 & na \\ 0 & 1 & nb \\ 0 & 0 & 1 \end{bmatrix},$$

where $a, b \in \mathbb{R}$. Thus Γ_1 is torsion free group.

$$G_2 = \Big\{I, J, \Gamma_2, I_3\Big\}, \quad (a, b \in \mathbb{R}),$$

where

$$J := \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}, \quad \Gamma_2 := \begin{bmatrix} -1 & 0 & a \\ 0 & -1 & b \\ 0 & 0 & 1 \end{bmatrix}$$

So Γ_2, is a torsion group. It is meaning that Γ_2 can't be Bieberbach group.

$$G_3 = \left\{I, \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\right\},$$

$$\Gamma_3 = \left\{ \begin{bmatrix} 1 & 0 & na \\ 0 & -1 & b \\ 0 & 0 & 1 \end{bmatrix} n \textit{ is odd}, \begin{bmatrix} 1 & 0 & na \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} n \textit{ is even} | n \in \mathbb{Z} \right\}, \qquad (a, b \in \mathbb{R}).$$

So Γ_3, is a torsion free group.

$$G_4 = \left\{I, \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}\right\}$$

Γ_4 is isomorphic with Γ_3, so Γ_4 is a torsion free group.

$$G_5 = \left\{I, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\right\}$$

$$\Gamma_5 = \left\{ A_n \textit{ if } n \textit{ is odd}, \quad \textit{and} \quad B_n \textit{ if } n \textit{ is even} \Big| \; n \in \mathbb{Z}, \quad (a, b \in \mathbb{R}) \right\}$$

where

$$A_n := \begin{bmatrix} 0 & 1 & \frac{n+1}{2}a + \frac{n-1}{2}b \\ 1 & 0 & \frac{n+1}{2}b + \frac{n-1}{2}a \\ 0 & 0 & 1 \end{bmatrix}$$

and

$$B_n := \begin{bmatrix} 1 & 0 & \frac{n}{2}a + \frac{n}{2}b \\ 0 & 1 & \frac{n}{2}a + \frac{n}{2}b \\ 0 & 0 & 1 \end{bmatrix}.$$

Γ_5 is a torsion free group and Γ_5 is isomorphic with Γ_3.

$$G_6 = \left\{ I, \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix} \right\}$$

$$\Gamma_6 = \Big\{ C_n \quad n \text{ is odd}, \quad \text{and} \quad D_n \quad n \text{ is even} | \quad n \in \mathbb{Z} \Big\}, \qquad (a, b \in \mathbb{R}),$$

where

$$C_n := \begin{bmatrix} 0 & -1 & \frac{n+1}{2}a - \frac{n-1}{2}b \\ -1 & 0 & \frac{n+1}{2}b - \frac{n-1}{2}a \\ 0 & 0 & 1 \end{bmatrix}$$

and

$$D_n := \begin{bmatrix} 1 & 0 & \frac{n}{2}a - \frac{n}{2}b \\ 0 & 1 & \frac{n}{2}b - \frac{n}{2}a \\ 0 & 0 & 1 \end{bmatrix}$$

Γ_6, is a torsion free group too, and Γ_6 is isomorphic with Γ_3.

$$G_7 = \left\{ I, \quad \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \right\}$$

$$\Gamma_7 = \left\{ \begin{bmatrix} 0 & 1 & a \\ -1 & 0 & b \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} -1 & 0 & a+b \\ 0 & -1 & b-a \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & -1 & b \\ 1 & 0 & -a \\ 0 & 0 & 1 \end{bmatrix}, \ (a, b \in \mathbb{R}) \right\}$$

Γ_7, is a torsion group, so Γ_7 isn't a Bieberbach group.

A subgroup of $\mathbb{E}(n)$ is cocompact if the quotient space $\mathbb{E}(n)/\Gamma$ is compact. Now, we may study the structure of quotient spaces $\mathbb{E}(2)/\Gamma_i$, where, $i = 1, \cdots, 7$. we notice that some of arising quotient spaces are isomorphic.

Corollary 5.1. *Every discrete and cocompact subgroup of* $Isom(\mathbb{R}^n, d_4)$ *is a crystallographic group.*

In first position:

$$\frac{\mathbb{E}(2)}{\Gamma_1} = \frac{\mathbb{O}(2) \ltimes \mathbb{R}^2}{\Gamma_2} \cong \frac{\mathbb{GP}(2) \ltimes \mathbb{R}^2}{\mathbb{R}^2} \cong \mathbb{GP}(2)$$

1- Subgroup Γ_1 is *cocompact*, because $\mathbb{GP}(2)$(the group of generalized permutation matrices) is compact (see[14] and [11]). Also $\mathbb{GP}(2)$ (is discrete, with induced topology that is consequence of homeomorphisem between $\mathbb{M}(2, \mathbb{R})$ and $\mathbb{R}^4$. Consequently Γ_1 is crystallographic group. Hence Γ_1 is Bieberbach group, because Γ_1 is torsion free too. Torus is result of quotient space of Γ_1.

2- Subgroups Γ_2 and Γ_7 aren't Bieberbach group, since these are torsion group.

3- Subgroups Γ_3, Γ_4, Γ_5 and Γ_6 are isomorphic, so their Bieberbach groups are equal. This means their quotient spaces are homeomorphic to the Klein bottle. □

Now, we discuss the case in 3 dimensions. Consider the subgroup of $Gl(3, \mathbb{R})$ containing the matrices whose entries belong to $\{-1, 0, 1\}$. So if $n = 3$, for isometry condition, we have for A matrix [14]:

1) $G_1 = \{I\}$ and its Isometry group is

$$\Gamma_1 = \Big\{E_n | \ n \in \mathbb{Z}\Big\},$$

where

$$E_n := \begin{bmatrix} 1 & 0 & 0 & na \\ 0 & 1 & 0 & nb \\ 0 & 0 & 1 & nc \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

that is torsion free and Bieberbach group of isometries. So quotient space is a torus.

2) Let

$$G_2 = \Big\{I_{(3)}, F_n\Big\},$$

where

$$F_n := \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

and its Isometry group is

$$\Gamma_2 = \Big\{a_n, b_n | \ n \in \mathbb{Z}\Big\},$$

where

$$a_n := \begin{bmatrix} 1 & 0 & 0 & \dfrac{2n-1}{2} \\ 0 & 0 & 1 & n \\ 0 & 1 & 0 & n-1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

and

$$b_n := \begin{bmatrix} 1 & 0 & 0 & n \\ 0 & 1 & 0 & n \\ 0 & 0 & 1 & n \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

One can see that Γ_2 is torsion free.

3) Let

$$G_3 = \Big\{I_{(3)}, d_n\Big\},$$

where

$$d_n := \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}.$$

Then for Isometry group, we have

$$\Gamma_3 = \Big\{e_n, f_n | n \in \mathbb{Z}\Big\},$$

where

$$e_n := \begin{bmatrix} 1 & 0 & 0 & \frac{2n-1}{2} \\ 0 & 1 & 0 & 2n-1 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad f_n := \begin{bmatrix} 1 & 0 & 0 & n \\ 0 & 1 & 0 & 2n \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

4) Let

$$G_4 = \Big\{I_{(3)}, w\Big\},$$

where

$$w := \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}.$$

Then, the isometry group is

$$\Gamma_4 = \Big\{h_n, k_n | \ n \in \mathbb{Z}\Big\},$$

where

$$h_n := \begin{bmatrix} 1 & 0 & 0 & \frac{2n-1}{2} \\ 0 & -1 & 0 & 1 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad \text{and} \quad k_n := \begin{bmatrix} 1 & 0 & 0 & n \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

This group is torsion free.

5) Now, let

$$G_5 = \Big\{I_{(3)}, b_1, A_1 = b_2, A_2 = b_3\Big\},$$

where

$$b_1 := \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}, \quad b_2 := \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}, \quad b_3 := \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Then Γ_5 is a generated group, i.e.,

$$\Gamma_5 = \langle w_1, w_2 \rangle,$$

where

$$w_1 := \begin{bmatrix} 1 & 0 & 0 & \frac{1}{2} \\ 0 & 1 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \qquad w_2 := \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & 1 & \frac{1}{2} \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

and $a \in \mathbb{R}$. Γ_5 is torsion free.

6) Let

$$G_6 = \Big\{ I_{(3)}, A_1 = s_1, A_2 = s_2 \Big\},$$

where

$$s_1 := \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & -1 \end{bmatrix}, \qquad s_2 := \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 1 \\ 0 & -1 & 0 \end{bmatrix}$$

and

$$\Gamma_6 = \Big\{ t_{1n}, t_{2n}, t_{3n} \Big|\ n \in \mathbb{Z} \Big\},$$

where

$$t_{1n} := \begin{bmatrix} 1 & 0 & 0 & \dfrac{3n-1}{3} \\ 0 & 0 & -1 & 1 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad t_{2n} := \begin{bmatrix} 1 & 0 & 0 & \dfrac{3n-2}{3} \\ 0 & -1 & 1 & 1 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$t_{3n} := \begin{bmatrix} 1 & 0 & 0 & n \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

One can see that Γ_6 is torsion free.

7) Let

$$G_7 = \Big\{ I_{(3)}, A_1, A_2, A_3 \Big\},$$

where

$$A_1 := \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix}, \quad A_2 := \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}, \quad A_3 := \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

and

$$\Gamma_7 = \Big\{ r_{1n}, r_{2n}, r_{3n}, r_{4n} \Big|\ n \in \mathbb{Z} \Big\},$$

where

$$r_{1n} := \begin{bmatrix} 1 & 0 & 0 & \frac{4n-3}{4} \\ 0 & 0 & 1 & 1 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad r_{2n} := \begin{bmatrix} 1 & 0 & 0 & \frac{4n-1}{4} \\ 0 & 0 & -1 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

$$r_{3n} := \begin{bmatrix} 1 & 0 & 0 & \frac{4n-2}{4} \\ 0 & -1 & 0 & 2 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad r_{4n} := \begin{bmatrix} 1 & 0 & 0 & n \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

It is easy to see that Γ_7 is torsion free.

8) Let

$$G_8 = \Big\{I_{(3)}, m_1, m_2, m_3, m_4, m_5\Big\},$$

where

$$m_1 := \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 1 & 0 \end{bmatrix}, \quad m_2 := \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & -1 \end{bmatrix}, \quad m_3 := \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

$$m_4 := \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 1 \\ 0 & -1 & 0 \end{bmatrix}, \quad m_5 := \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 1 \end{bmatrix}.$$

One can see that

$$\Gamma_8 = \Big\{k_1, k_2, k_3, k_4, k_5, k_6 \,|\; n \in \mathbb{Z}\Big\},$$

where

$$k_1 := \begin{bmatrix} 1 & 0 & 0 & n \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad k_2 := \begin{bmatrix} 1 & 0 & 0 & \frac{6n-5}{6} \\ 0 & 1 & -1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

$$k_3 := \begin{bmatrix} 1 & 0 & 0 & \frac{6n-4}{6} \\ 0 & 0 & -1 & 1 \\ 0 & 1 & -1 & 2 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad k_4 := \begin{bmatrix} 1 & 0 & 0 & \frac{6n-3}{6} \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 2 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

$$k_5 := \begin{bmatrix} 1 & 0 & 0 & \frac{6n-2}{6} \\ 0 & -1 & 1 & -1 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad k_6 := \begin{bmatrix} 1 & 0 & 0 & \frac{6n-1}{6} \\ 0 & 0 & 1 & -1 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

This group is torsion free.

9) Let

$$G_9 = \Big\{ I_{(3)}, A_1, A_2, A_3 \Big\},$$

where

$$A_1 := \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad A_2 := \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}, \quad A_3 := \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}.$$

Thus Γ_9 is a generated group:

$$\Gamma_9 = \langle L_1, L_2 \rangle.$$

where

$$L_1 := \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & \frac{1}{2} \\ 0 & 0 & -1 & \frac{1}{2} \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad L_2 := \begin{bmatrix} 1 & 0 & 0 & \frac{1}{2} \\ 0 & -1 & 0 & \frac{1}{2} \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

However Γ_9 is torsion free.

10) Finally, let

$$G_{10} = \Big\{ I_{(3)}, A_1, A_2, A_3 \Big\},$$

where

$$A_1 := \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad A_2 := \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}, A_3 := \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

It is easy to see that Γ_{10} is a generated group too:

$$\Gamma_{10} = \langle X_1, X_2 \rangle,$$

where

$$X_1 := \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & 1 & \frac{1}{2} \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad X_2 := \begin{bmatrix} 1 & 0 & 0 & \frac{1}{2} \\ 0 & 1 & 0 & \frac{1}{2} \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

This group is torsion free. so all of these group are torsion free. If $\mathbb{E}(3)/\Gamma_i$ that i=1,2, ...,10, become compact for that whose Γ_i become cocompact. In addition to, they should be discrete. These properties are needed to their become Bieberbach groups (see [16]).

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