Journal of Finsler Geometry and its Applications
Vol. 1, No. 2 (2020), pp 73-82
DOI: 10.22098/jfga.2020.1241

On L-Reducible Finsler Manifolds

Sareh Beizavi *

School of Mathematics, Iran University of Science and Technology
Narmak, Tehran, 16846 -13114, Iran

E-mail: h_beizavi@mathdep.iust.ac.ir

Abstract. In this paper, we consider the class of L-reducible Finsler metrics which contains the class of C-reducible metrics and the class of Landsberg metrics. Let (M, F) be a 3-dimensional L-reducible Finsler manifold. Suppose that F has relatively isotropic mean Landsberg curvature. We find a condition on the main scalars of F under which it reduces to a Randers metric or a Landsberg metric..

Keywords: L-reducible Finsler metric, C-reducible metric, Randers metric, Landsberg metric.

1. Introduction

In [10], Takano wrote a paper on the subject of Physics and studied the field equation in Finsler manifolds. He proposed some important geometrical problems in Finsler geometry, namely, he asked to find some interesting special forms of hv-curvature from the standpoint of Physics. Very soon, Matsumoto introduced the notion of L-reducible (P-reducible in the sense of Matsumoto) Finsler metrics as an answer to Takano. This new class of Finsler metrics were a generalization of C-reducible Finsler metrics [6]. Then for a Finsler manifold of dimension $n \geq 3$, he obtained some elegant conditions under which the manifold reduces to a L-reducible manifold. After that, the study of hv-curvature of Finsler connection and its derivatives become urgent necessity for the Finsler Geometry as well as for Theoretical Physics. This cause that Matsumoto-Shimada studied the curvature properties of L-reducible Finsler

*Corresponding Author

AMS 2020 Mathematics Subject Classification: 53B40, 53C30

metrics in [7]. They found an important and almost complex problem which is given as the following open problem:

Is there any non-trivial L-reducible Finsler metric which is not C-reducible?

It is remarkable that, Matsumoto-Hōjō proved that a Finsler metric F is C-reducible if and only if it is a Randers metric or Kropina metric [3]. These metrics defined by $F = \alpha + \beta$ and $F = \alpha^2/\beta$, respectively, where $\alpha = \sqrt{a_{ij}y^iy^j}$ is a Riemann metric and $\beta := b_i(x)y^i$ a 1-form on a manifold M. Let (M, F) be a Finsler manifold. The second derivatives of $\frac{1}{2}F_x^2$ at $y \in T_xM_0$ is an inner product $\mathbf{g}_y$ on T_xM. The third order derivatives of $\frac{1}{2}F_x^2$ at $y \in T_xM_0$ is a symmetric trilinear forms $\mathbf{C}_y$ on T_xM. We call $\mathbf{g}_y$ and $\mathbf{C}_y$ the fundamental form and the Cartan torsion, respectively. The rate of change of $\mathbf{C}_y$ along geodesics is the Landsberg curvature $\mathbf{L}_y$ on T_xM for any $y \in T_xM_0$. F is said to be Landsbergian if $\mathbf{L} = 0$. Taking a trace of $\mathbf{C}$ and $\mathbf{L}$ give us mean Cartan torsion $\mathbf{I}$ and mean Landsberg curvature $\mathbf{J}$, respectively. A Finsler metric F on an n-dimensional manifold M is C-reducible if its Cartan torsion is give by

$$C_{ijk} = \frac{1}{n+1}\Big\{I_ih_{jk} + I_jh_{ik} + I_kh_{ij}\Big\}. \tag{1.1}$$

A Finsler metric F on an n-dimensional manifold M is L-reducible if its Landsberg curvature is give by

$$L_{ijk} = \frac{1}{n+1}\Big\{J_ih_{jk} + J_jh_{ik} + J_kh_{ij}\Big\}. \tag{1.2}$$

By taking a horizontal derivation from (1.1), one can get (1.2). Thus every C-reducible metric is L-reducible. But the converse may not true in general. In [16], Tayebi-Sadegi studied a class of Finsler metrics called generalized P-reducible metrics that contains the class of L-reducible metrics. They proved that every generalized P-reducible (α, β)-metric with vanishing S-curvature reduces to a Berwald metric or C-reducible metric. It results that there is not any concrete L-reducible (α, β)-metric with vanishing S-curvature. In [11], Tayebi-Bahadori-Sadeghi studied the C-reducibility and L-reducibility condition for the class of spherically symmetric Finsler metrics. They proved the following.

Theorem A. Let $F = u\phi(r, s)$ be a spherically symmetric Finsler metric on a domain $\Omega \subseteq \mathbb{R}^n$. Then F is a L-reducible metric if and only if it satisfies the following PDE

$$(\phi - s\phi_s)\, L_1 - 3\phi_{ss}L_2 = 0. \tag{1.3}$$

where

$$\begin{aligned}
P :=& -\frac{1}{\phi}\Big(s\phi+(r^2-s^2)\phi_s\Big)Q+\frac{1}{2r\phi}\Big(s\phi_r+r\phi_s\Big),\\
Q :=& \frac{1}{2r}\frac{-\phi_r+s\phi_{rs}+r\phi_{ss}}{\phi-s\phi_s+(r^2-s^2)\phi_{ss}},\\
L_1 :=& 3\phi_sP_{ss}+\phi P_{sss}+\big(s\phi+(r^2-s^2)\phi_s\big)Q_{sss},\\
L_2 :=& -s\phi P_{ss}+\phi_s(P-sP_s)+\big(s\phi+(r^2-s^2)\phi_s\big)(Q_s-sQ_{ss}).
\end{aligned}$$

It is easy to see that, if $L_1, L_2 \neq 0$ then one can get a L-reducible spherically symmetric Finsler metric which is not C-reducible.

Let (M, F) be a 3-dimensional L-reducible Finsler manifold. Suppose that F has relatively isotropic mean Landsberg curvature. In this paper, we find a condition on the main scalars of F under which it reduces to a Randers metric or a Landsberg metric. More precisely, we prove the following.

Theorem 1.1. *Let (M, F) be a 3-dimensional L-reducible Finsler manifold such that $b_i = b_i(x, y)$ is constant along Finslerian geodesics. Suppose that F has relatively isotropic mean Landsberg curvature*

$$\mathbf{J} = cF\mathbf{I}, \tag{1.4}$$

where $c = c(x)$ is a scalar function on M. Then one of the following holds

(1) *F is a Randers metric;*
(2) *F is a Landsberg metric;*

In this paper, we use the Berwald connection and the h- and v- covariant derivatives of a Finsler tensor field are denoted by " | " and ", " respectively.

2. Preliminaries

Let M be a n-dimensional C^∞ manifold. Denote by T_xM the tangent space at $x \in M$, and by $TM = \cup_{x\in M}T_xM$ the tangent bundle of M. A Finsler metric on M is a function $F : TM \to [0, \infty)$ which has the following properties:
(i) F is C^∞ on $TM_0 := TM \setminus \{0\}$;
(ii) F is positively 1-homogeneous on the fibers of tangent bundle TM;
(iii) for each $y \in T_xM$, the following quadratic form $\mathbf{g}_y$ on T_xM is positive definite,

$$\mathbf{g}_y(u, v) := \frac{1}{2}\frac{\partial^2}{\partial s\partial t}\Big[F^2(y + su + tv)\Big]|_{s,t=0}, \quad u, v \in T_xM.$$

Let $x \in M$ and $F_x := F|_{T_xM}$. To measure the non-Euclidean feature of F_x, define $\mathbf{C}_y : T_xM \times T_xM \times T_xM \to \mathbb{R}$ by

$$\mathbf{C}_y(u, v, w) := \frac{1}{2}\frac{d}{dt}\Big[\mathbf{g}_{y+tw}(u, v)\Big]|_{t=0}, \quad u, v, w \in T_xM.$$

The family $\mathbf{C} := \{\mathbf{C}_y\}_{y\in TM_0}$ is called the Cartan torsion. It is well known that $\mathbf{C=0}$ if and only if F is Riemannian.

For $y \in T_xM_0$, define mean Cartan torsion $\mathbf{I}_y$ by $\mathbf{I}_y(u) := I_i(y)u^i$, where

$$I_i := g^{jk}C_{ijk}$$

and $u = u^i\partial/\partial x^i|_x$. By Diecke Theorem, F is Riemannian if and only if $\mathbf{I}_y = 0$.

For $y \in T_xM_0$, define the Matsumoto torsion $\mathbf{M}_y : T_xM \times T_xM \times T_xM \to \mathbb{R}$ by $\mathbf{M}_y(u, v, w) := M_{ijk}(y)u^iv^jw^k$ where

$$M_{ijk} := C_{ijk} - \frac{1}{n+1}\Big\{I_ih_{jk} + I_jh_{ik} + I_kh_{ij}\Big\},$$

and

$$h_{ij} := FF_{y^iy^j} = g_{ij} - \frac{1}{F^2}g_{ip}y^pg_{jq}y^q$$

is the angular metric. A Finsler metric F is said to be C-reducible if $\mathbf{M} = 0$. This quantity is introduced by Matsumoto [6]. Matsumoto proves that every Randers metric satisfies that $\mathbf{M} = 0$. It is remarkable that, a Randers metric $F = \alpha + \beta$ on a manifold M is just a Riemannian metric $\alpha = \sqrt{a_{ij}y^iy^j}$ perturbated by a one form $\beta = b_i(x)y^i$ on M such that $\|\beta\|_\alpha < 1$. Later on, Matsumoto-Hōjō proves that the converse is true too.

Lemma 2.1. ([3]) A Finsler metric F on a manifold of dimension $n \geq 3$ is a Randers metric if and only if $\mathbf{M}_y = 0$, $\forall y \in TM_0$.

A Finsler metric F o an n-dimensional manifold M is called semi-C-reducible if its Cartan tensor is given by

$$C_{ijk} = \frac{p}{1+n}\Big\{h_{ij}I_k + h_{jk}I_i + h_{ki}I_j\Big\} + \frac{q}{||\mathbf{I}||^2}I_iI_jI_k,$$

where $p = p(x, y)$ and $q = q(x, y)$ are scalar function on TM and $||\mathbf{I}||^2 = I^iI_i$. Multiplying the definition of semi-C-reducibility with g^{jk} shows that p and q must satisfy $p + q = 1$. If $p = 0$, then F is called $C2$-like metric. In [4], Matsumoto and Shibata proved that every (α, β)-metric is semi-C-reducible. Let us remark that an (α, β)-metric is a special Finsler metric on M defined by $F := \alpha\phi(s)$, $s = \beta/\alpha$, where $\phi = \phi(s)$ is a C^∞ function on the $(-b_0, b_0)$ with certain regularity, $\alpha = \sqrt{a_{ij}y^iy^j}$ is a Riemannian metric and $\beta = b_i(x)y^i$ is a 1-form on M on such that $\|\beta\|_\alpha < 1$ (see [1], [5], [9], [12], [13], [14], [15], [16] and [17]).

Theorem 2.2. ([4][5]) Let $F = \phi(s)\alpha$, $s = \beta/\alpha$, be a non-Riemannian (α, β)-metric on a manifold M of dimension $n \geq 3$. Then F is semi-C-reducible.

The horizontal covariant derivatives of $\mathbf{C}$ along geodesics give rise to the Landsberg curvature $\mathbf{L}_y : T_xM \otimes T_xM \otimes T_xM \to \mathbb{R}$ defined by

$$\mathbf{L}_y(u,v,w) := L_{ijk}(y)u^i v^j w^k,$$

where

$$L_{ijk} := C_{ijk|s}y^s,$$

$u = u^i\partial/\partial x^i|_x$, $v = v^i\partial/\partial x^i|_x$ and $w = w^i\partial/\partial x^i|_x$. The family $\mathbf{L} := \{\mathbf{L}_y\}_{y\in TM_0}$ is called the Landsberg curvature. A Finsler metric is called a Landsberg metric if $\mathbf{L} = 0$.

Given a Finsler manifold (M, F), then a global vector field $\mathbf{G}$ is induced by F on TM_0, and in a standard coordinate (x^i, y^i) for TM_0 is given by

$$\mathbf{G} = y^i\frac{\partial}{\partial x^i} - 2G^i(x,y)\frac{\partial}{\partial y^i},$$

where $G^i = G^i(x, y)$ are scalar functions on TM_0 given by

$$G^i := \frac{1}{4}g^{ij}\left\{\frac{\partial^2[F^2]}{\partial x^k\partial y^j}y^k - \frac{\partial[F^2]}{\partial x^j}\right\}, \quad y \in T_xM.$$

The vector field $\mathbf{G}$ is called the spray associated with (M, F).

For $y \in T_xM$, define $\mathbf{J}_y : T_xM \to \mathbb{R}$ by $\mathbf{J}_y(u) := J_i(y)u^i$, where

$$J_i := g^{jk}L_{ijk}.$$

By definition, $\mathbf{J}_y(y) = 0$. $\mathbf{J}$ is called the mean Landsberg curvature or J-curvature. A Finsler metric F is called a weakly Landsberg metric if $\mathbf{J}_y = 0$. By definition, every Landsberg metric is a weakly Landsberg metric. Mean Landsberg curvature can be defined as following

$$J_i := y^m\frac{\partial I_i}{\partial x^m} - I_m\frac{\partial G^m}{\partial y^i} - 2G^m\frac{\partial I_i}{\partial y^m}.$$

By definition, we get

$$\mathbf{J}_y(u) := \frac{d}{dt}\Big[\mathbf{I}_{\dot\sigma(t)}\big(U(t)\big)\Big]_{t=0},$$

where $y \in T_xM$, $\sigma = \sigma(t)$ is the geodesic with $\sigma(0) = x$, $\dot\sigma(0) = y$ and $U(t)$ is a linearly parallel vector field along σ with $U(0) = u$. In local coordinate, it defines as follows

$$J_i = I_{i|m}y^m.$$

Then the mean Landsberg curvature $\mathbf{J}_y$ is the rate of change of $\mathbf{I}_y$ along geodesics for any $y \in T_xM_0$. It has been shown that on a weakly Landsberg manifold, the volume function $Vol(x)$ is a constant.

A Finsler metric F on a manifold M is called of relatively isotropic mean Landsberg curvature if

$$\mathbf{J} + cF\mathbf{I} = 0,$$

where $c = c(x)$ is a scalar function on M.

For $y \in T_xM_0$, define $\mathbf{M}'_y : T_xM \times T_xM \times T_xM \to \mathbb{R}$ by $\mathbf{M}'_y(u, v, w) := M'_{ijk}(y)u^iv^jw^k$ where

$$M'_{ijk} := L_{ijk} - \frac{1}{n+1}\Big\{J_ih_{jk} + J_jh_{ik} + J_kh_{ij}\Big\}.$$

A Finsler metric F is said to be L-reducible if $\mathbf{M}' = 0$. It is easy to see that every C-reducible metric is a L-reducible metric.

3. Proof of Theorem 1.1

A Finsler metric F is called $C3$-like if its Cartan tensor is given by

$$C_{ijk} = \Big\{A_ih_{jk} + A_jh_{ki} + A_kh_{ij}\Big\} + \Big\{B_iI_jI_k + I_iB_jI_k + I_iI_jB_k\Big\}, \qquad (3.1)$$

where $A_i = A_i(x, y)$ and $B_i = B_i(x, y)$ are y-homogeneous scalar functions on TM of degree -1 and 1, respectively. By definition, we have some special cases as follows:

(i) If $A_i = 0$, then we have

$$C_{ijk} = \Big\{B_iI_jI_k + I_iB_jI_k + I_iI_jB_k\Big\}.$$

Contracting it with g^{ij} implies that

$$B_i = \frac{1}{3||\mathbf{I}||^2}I_i.$$

Then F is a $C2$-like metric;

(ii) If $B_i = 0$, then we have

$$C_{ijk} = \Big\{A_ih_{jk} + A_jh_{ki} + A_kh_{ij}\Big\}.$$

Contracting it with g^{ij} implies that

$$A_i = \frac{1}{n+1}I_i.$$

Then F is a C-reducible metric;

(iii) Let us put

$$A_i = \frac{p}{n+1}I_i, \qquad B_i = \frac{q}{3||\mathbf{I}||^2}I_i,$$

where $p = p(x, y)$ and $q = q(x, y)$ are scalar functions on TM, then F is a semi-C-reducible metric.

It is remarkable that, in [4] Matsumoto-Shibata introduced the notion of semi-C-reducibility and proved that every non-Riemannian (α,β)-metric on a manifold M of dimension $n \geq 3$ is semi-C-reducible. Therefore the study of the class of $C3$-like Finsler spaces will enhance our understanding of the geometric meaning of (α,β)-metrics.

Proof of Theorem 1.1: In [8], Moór introduced a special orthonormal frame field (ℓ^i, m^i, n^i) in the three dimensional Finsler space. The first vector of the frame is the normalized supporting element, the second is the normalized mean Cartan torsion vector, and third is the unit vector orthogonal to them. Let (M,F) be a 3-dimensional Finsler manifold. Suppose that $\ell_i := F_{y^i}$ is the unit vector along the element of support, m_i is the unit vector along mean Cartan torsion I_i, i.e.,

$$m_i := \frac{1}{||\mathbf{I}||} I_i,$$

where $||\mathbf{I}|| := \sqrt{I_i I^i}$ and n_i is a unit vector orthogonal to the vectors ℓ_i and m_i. Then the triple (ℓ_i, m_i, n_i) is called the Moór frame. It is proved that the Cartan torsion of 3-dimensional Finsler metric F is given by following

$$C_{ijk} = \Big\{A_i h_{jk} + A_j h_{ki} + A_k h_{ij}\Big\} + \Big\{B_i I_j I_k + B_j I_i I_k + B_k I_i I_j\Big\}, \tag{3.2}$$

where $A_i = A_i(x,y)$ and $B_i = B_i(x,y)$ are scalar functions on TM and given by where

$$A_i := \frac{1}{3F}\Big[3\mathcal{I} m_i + \mathcal{J} n_i\Big], \qquad B_i := \frac{F}{3(\mathcal{H}+\mathcal{I})^2}\Big[(\mathcal{H}-3\mathcal{I})m_i - 4\mathcal{J} n_i\Big]. \tag{3.3}$$

Here, $\mathcal{H} = \mathcal{H}(x,y)$, $\mathcal{I} = \mathcal{I}(x,y)$ and $\mathcal{J} = \mathcal{J}(x,y)$ are the main scalars of F. It is easy to see that $A_i y^i = 0$ and $B_i y^i = 0$. Multiplying (3.2) with g^{ij} implies that

$$4A_i = (1 - 2B_m I^m) I_i - ||\mathbf{I}||^2 B_i, \tag{3.4}$$

where $||\mathbf{I}||^2 = I^m I_m$. Putting (3.4) in (3.2) yields

$$\begin{aligned} C_{ijk} = & \ \frac{1}{4}\Big\{I_i h_{jk} + I_j h_{ki} + I_k h_{ij}\Big\} - \frac{1}{2}\Big\{I_i h_{jk} + I_j h_{ki} + I_k h_{ij}\Big\} I^m B_m \\ & - \frac{||\mathbf{I}||^2}{4}\Big\{B_i h_{jk} + B_j h_{ki} + B_k h_{ij}\Big\} + \Big\{B_i I_j I_k + B_j I_i I_k + B_k I_i I_j\Big\}, \end{aligned} \tag{3.5}$$

or equivalently

$$\begin{aligned} M_{ijk} = -\frac{1}{2}\Big\{I_i h_{jk} + I_j h_{ki} + I_k h_{ij}\Big\} B_m I^m - \frac{1}{4}\Big\{B_i h_{jk} + B_j h_{ki} + B_k h_{ij}\Big\} ||\mathbf{I}||^2 \\ + \Big\{B_i I_j I_k + I_i B_j I_k + I_i I_j B_k\Big\}. \end{aligned} \tag{3.6}$$

By taking a horizontal derivation of (3.6), we have

$$\begin{aligned}M'_{ijk} = &-\frac{1}{2}\Big(J^m B_m + I^m B'_m\Big)\Big\{I_i h_{jk} + I_j h_{ki} + I_k h_{ij}\Big\}\\ &-\frac{1}{2}\Big\{J_i h_{jk} + J_j h_{ki} + J_k h_{ij}\Big\}B_m I^m\\ &-\frac{1}{4}\Big\{B'_i h_{jk} + B'_j h_{ki} + B'_k h_{ij}\Big\}||\mathbf{I}||^2\\ &-\frac{1}{4}\Big(J^m I_m + I^m J_m\Big)\Big\{B_i h_{jk} + B_j h_{ki} + B_k h_{ij}\Big\}\\ &+B_i\Big\{J_j I_k + I_j J_k\Big\} + B_j\Big\{J_i I_k + I_i J_k\Big\}\\ &+B_k\Big\{J_i I_j + I_i J_j\Big\} + \Big\{B'_i I_j I_k + B'_j I_i I_k + B'_k I_i I_j\Big\}, \qquad (3.7)\end{aligned}$$

where

$$B'_i = B_{i|s} y^s$$

and

$$M'_{ijk} = L_{ijk} - \frac{1}{4}\Big\{J_i h_{jk} + J_j h_{ki} + J_k h_{ij}\Big\}.$$

Since B_i is constant along geodesics, i.e., $B'_i = 0$, then (3.7) reduces to following

$$\begin{aligned}M'_{ijk} = \ &-\frac{1}{2}B_m\Big[\Big\{I_i h_{jk} + I_j h_{ki} + I_k h_{ij}\Big\}J^m + \Big\{J_i h_{jk} + J_j h_{ki} + J_k h_{ij}\Big\}I^m\Big]\\ &-\frac{1}{4}\Big(J^m I_m + I^m J_m\Big)\Big\{B_i h_{jk} + B_j h_{ki} + B_k h_{ij}\Big\}\\ &+B_i\Big\{J_j I_k + I_j J_k\Big\} + B_j\Big\{J_i I_k + I_i J_k\Big\}\\ &+B_k\Big\{J_i I_j + I_i J_j\Big\}. \qquad (3.8)\end{aligned}$$

Let F is of relatively isotropic mean Landsberg curvature

$$\mathbf{J} = cF\mathbf{I}. \qquad (3.9)$$

By putting (3.9) in (3.8), we get

$$\begin{aligned}M'_{ijk} = &\ 2cF\Big\{B_i I_j I_k + B_j I_i I_k + B_k I_i I_j\Big\}\\ &-cF\Big\{I_i h_{jk} + I_j h_{ki} + I_k h_{ij}\Big\}B_m I^m\\ &-\frac{1}{2}cF\Big\{B_i h_{jk} + B_j h_{ki} + B_k h_{ij}\Big\}||\mathbf{I}||^2. \qquad (3.10)\end{aligned}$$

By considering (3.5) and (3.10), we have

$$M'_{ijk} = 2cFC_{ijk} - \frac{cF}{2}\Big\{I_i h_{jk} + I_j h_{ki} + I_k h_{ij}\Big\}$$

or equivalently

$$\mathbf{M}' = 2cF\mathbf{M}. \qquad (3.11)$$

Since F is L-reducible then by (3.11), we get

$$cF\mathbf{M} = 0.$$

We have two main cases as follows:

If $c = 0$, then we get

$$\mathbf{J} = 0.$$

By assumption, F is L-reducible

$$L_{ijk} = \frac{1}{n+1}\Big\{J_i h_{jk} + J_j h_{ik} + J_k h_{ij}\Big\}.$$

Then it reduces to a Landsberg metric.

If $c \neq 0$, then we get $\mathbf{M} = 0$. Therefore by Matsumoto-Hōjō's Lemma, F is a Randers metric. This completes the proof. □

References

1. M. Atashafrouz , *Characterization of 3-dimensional left-invariant locally projectively flat Randers metrics*, Journal of Finsler Geometry and its Applications, **1**(2020), 96-102.
2. M. Z. Laki, *On generalized symmetric Finsler spaces with some special (α, β)-metrics*, Journal of Finsler Geometry and its Applications, **1**(2020), 45-53.
3. M. Matsumoto and S. Hōjō, *A conclusive theorem for C-reducible Finsler spaces*, Tensor. N. S. **32**(1978), 225-230.
4. M. Matsumoto and C. Shibata, *On semi-C-reducibility, T-tensor and S4-1ikeness of Finsler spaces*, J. Math. Kyoto Univ. **19**(1979), 301-314.
5. M. Matsumoto, *Theory of Finsler spaces with (α, β)-metric*, Rep. Math. Phys. **31**(1992), 43-84.
6. M. Matsumoto, *On Finsler spaces with Randers metric and special forms of important tensors*, J. Math. Kyoto Univ. **14**(1974), 477-498.
7. M. Matsumoto and H. Shimada, *On Finsler spaces with the curvature tensors P_{hijk} and S_{hijk} satisfying special conditions*, Rep. Math. Phys. **12**(1977), 77-87.
8. A. Moór, *Über die Torsion-Und Krummungs invarianten der drei reidimensionalen Finslerchen Räume*, Math. Nach, **16**(1957), 85-99.
9. T. Rajabi, *On the norm of Cartan torsion of two classes of (α, β)-metrics*, Journal of Finsler Geometry and its Applications, **1**(2020), 66-72.
10. Y. Takano, *On the theory of fields in Finsler spaces*, Intern. Symp. Relativity and Unified Field Theory, Calcutta, 1975.
11. A. Tayebi, M. Bahadori and H. Sadeghi, *On spherically symmetric Finsler metrics with some non-Riemannian curvature properties*, J. Geom. Phys. **163** (2021), 104125.
12. A. Tayebi and M. Barzegari, *Generalized Berwald spaces with (α, β)-metrics*, Indagationes Mathematicae, **27**(2016), 670-683.
13. A. Tayebi and B. Najafi, *On homogeneous Landsberg surfaces*, J. Geom. Phys. **168**(2021), 104314.
14. A. Tayebi and B. Najafi, *Classification of 3-dimensional Landsbergian (α, β)-mertrics*, Publ. Math. Debrecen, **96**(2020), 45-62.
15. A. Tayebi and M. Razgordani, *On conformally flat fourth root (α, β)-metrics*, Differ. Geom. Appl. **62**(2019), 253-266.

16. A. Tayebi and H. Sadeghi, *Generalized P-reducible (α,β)-metrics with vanishing S-curvature*, Ann. Polon. Math. **114**(1) (2015), 67-79.
17. A. Tayebi and H. Sadeghi, *On generalized Douglas-Weyl (α,β)-metrics*, Acta. Math. Sinica. English. Series. **31**(2015), 1611-1620.

Received: 17.07.2020
Accepted: 19.12.2020