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Douglas (α, β)-metrics on four-dimensional nilpotent Lie groups

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Abstract. In this paper, we give a classification of left-invariant Douglas and Berwald (α, β)-metrics on simply connected four-dimensional nilpotent Lie groups. We show that there are not any bi-invariant Randers metrics on four-dimensional nilpotent Lie groups. Then, we explicitly give the flag curvature formulas and geodesic vectors of these spaces. Finally, we give the formula of S-curvature of left-invariant Randers metrics of Douglas type.

Keywords: nilpotent Lie group, Riemannian metric, (α, β)-metric, flag curvature.

1. Introduction

Nilpotent Lie groups play important roles in mathematical physics, harmonics analysis, and geometric analysis (see [12]). Among nilpotent Lie groups, four-dimensional Lie groups have attracted special attention. For example, left-invariant Riemannian metrics on these spaces are classified and investigated in [15, 20, 22]. Also up to automorphism left-invariant Lorentzian metrics on

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these spaces are classified in [5]. Moreover, the classification of left-invariant metrics of neutral signature is presented in [19].

In this paper we extend the study of the geometry of four-dimensional nilpotent Lie groups to the family of (α,β)-metrics which is a special type of Finsler metrics. These metrics are defined as a combination of Riemannian metrics and one-forms.

An (α,β)-metric on a smooth manifold M is a Finsler metric of the following form:

$$F = \alpha\phi\Big(\frac{\beta}{\alpha}\Big),$$

where $\alpha(x,y)=\sqrt{g(y,y)}$ and $\phi:(-b_0,b_0)\longrightarrow\mathbb{R}^+$ is a C^∞ function such that

$$\phi(s)-s\phi^{'}(s)+(b^2-s^2)\phi^{''}(s)>0, \qquad \forall \qquad |s|\leq b<b_0,$$

and $\|\beta\|_\alpha<b_0$ [6].

If we consider

$$\phi(s)=1+s, \quad \phi(s)=\frac{1}{s}, \quad \text{and} \quad \phi(s)=\frac{1}{1-s},$$

then we have Randers, Kropina and Matsumoto metrics, respectively.

In the definition of (α,β)-metrics, we can replace the one-form β by a vector field X which is defined by the equation $\beta(x,y)=g(X(x),y)$. So for any $x\in M$ and $y\in T_xM$, we can rewrite an (α,β)-metric as follows:

$$F(x,y)=\sqrt{g(y,y)}\phi\left(\frac{g(X(x),y)}{\sqrt{g(y,y)}}\right).$$

Naturally, a Finsler metric F on a Lie group G is named a left-invariant Finsler metric if for any $x\in G$ and $y\in T_xG$ satisfying the condition

$$F(x,y)=F\big(e,dl_{x^{-1}}y\big),$$

where e and l_x are the unit element of G and the left translation, respectively.

Note. In this paper we consider left-invariant (α,β)-metrics which are defined by left-invariant vector fields X and left-invariant Riemannian metrics g.

During the last two decades, the study of invariant Finsler metrics ((α,β)-metrics) on homogeneous spaces and Lie groups has constituted an essential part of the field of Finsler geometry (see [1, 4, 7, 9] and [11]).

Recently, in [9], the classification of left-invariant Douglas (α,β)-metrics on five-dimensional nilpotent Lie groups is given. Here we classify all left-invariant Douglas (α,β)-metrics (where α and β are left-invariant) on four-dimensional nilpotent Lie groups. Also, we study the flag curvature and S-curvature of these spaces.

The structure of the paper is as follows. In Section 2, we first remind the classification of four-dimensional nilpotent Lie groups which are explicitly given in [15] (see also [17]). Then, we give a classification of left-invariant Douglas and Berwald (α,β)-metrics on these spaces. In Section 3, we explicitly give the flag curvature formulas and geodesic vectors of all left-invariant (α,β)-metrics of Douglas type. Finally, we give the formula of S-curvature of left- invariant Randers metrics of Douglas type.

2. Finsler metrics on four-dimensional nilpotent Lie groups

Let N be a non-abelian simply connected four-dimensional nilpotent Lie group equipped with a left-invariant Riemannian metric and $\mathcal{N}$ denotes its Lie algebra. In [15] (see also [5]), it is shown that $\mathcal{N}$ is one of the following Lie algebras:

Case 1: $\mathcal{H}_3 \oplus \mathcal{R}$ with corresponding Lie group $H_3 \times \mathbb{R}$, where H_3 is the three-dimensional Heisenberg group, which is a two-step nilpotent Lie group of all 3×3 real matrices of the following form

$$\begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix}.$$

We can see that $\mathcal{H}_3 \oplus \mathcal{R}$ is spanned by the basis $\{X_1,\cdots,X_4\}$ with non-vanishing Lie bracket $[X_1,X_2]=X_3$. It is easy to see that this Lie algebra is a 2-step nilpotent Lie algebra with two-dimensional center

$$Z(\mathcal{H}_3 \oplus \mathcal{R}) = span\{X_4, X_3\}.$$

Moreover, the left-invariant Riemannian metric g with respect to the basis $\{X_1,\cdots,X_4\}$ on $H_3\times\mathbb{R}$ is determined by an inner product $\langle,\rangle$ on $\mathcal{H}_3\oplus\mathcal{R}$ and defined by the following matrix

$$S = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \lambda & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

where $\lambda > 0$. Suppose that,

$$e_1 = X_1, \qquad e_2 = X_2, \qquad e_3 = \frac{X_3}{\sqrt{\lambda}}, \qquad e_4 = X_4.$$

So the set $\{e_1,\cdots,e_4\}$ is an orthonormal basis with respect to the inner product $\langle,\rangle$ and for the non-zero Lie bracket we have

$$[e_1,e_2] = \sqrt{\lambda}e_3.$$

Case 2: $\mathcal{G}_4$ with the corresponding Lie group G_4. This Lie algebra is spanned by the basis $\{X_1, \cdots, X_4\}$ with non-vanishing Lie brackets $[X_1, X_2] = X_3$ and $[X_1, X_3] = X_4$. The Lie algebra $\mathcal{G}_4$ is a 3-step nilpotent Lie algebra, with one-dimensional center $Z(\mathcal{G}_4) = span\{X_4\}$. Moreover, the Riemannian metric g on G_4 with respect to the basis $\{X_1, \cdots, X_4\}$ is determined by an inner product $\langle , \rangle$ on $\mathcal{G}_4$ with the matrix

$$S_A = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & a & b \\ 0 & 0 & b & c \end{pmatrix},$$

where $A = \begin{pmatrix} a & b \\ b & c \end{pmatrix}$, $detA > 0$ and $b \geq 0$, (for more details see [15] and [5]). So if we let

$$\begin{aligned} e_1 &= X_1, \\ e_2 &= X_2, \\ e_3 &= \frac{X_3}{\sqrt{a}}, \\ e_4 &= \frac{\sqrt{a}\left(X_4 - \frac{b}{a}X_3\right)}{\sqrt{ac - b^2}}, \end{aligned}$$

then the set $\{e_1, \cdots, e_4\}$ is an orthonormal basis and for the non-zero Lie brackets we have

$$\begin{aligned} [e_1, e_2] &= \sqrt{a}e_3, \\ [e_1, e_3] &= \frac{\sqrt{ac - b^2}}{a}e_4 + \frac{b}{a}e_3, \\ [e_1, e_4] &= -\frac{b}{a}\left(e_4 + \frac{b}{\sqrt{ac - b^2}}e_3\right), \end{aligned}$$

where $a > 0$. In order to characterize all left-invariant (α, β)-metrics of Berwald type on these spaces we recall that by Theorem 4.1 of [11], a left-invariant (α, β) metric F on a Lie group G, arising from a left-invariant Riemannian metric g and a left-invariant vector field X is of Berwald type if and only if for all $y, z \in \mathcal{G}$, the following two conditions are valid

$$g([y, X], z) + g([z, X], y) = 0, \qquad g([y, z], X) = 0. \tag{2.1}$$

A direct consequence of the above conditions (Theorem 4.1 of [11]) is the next corollary.

Corollary 2.1. *The only four-dimensional nilpotent Lie group N which admits a left-invariant (α, β)-metric of Berwald type is the Lie group $H_3 \times \mathbb{R}$. This* *metric is defined by the inner product* $\langle,\rangle$ *on* $\mathcal{N}$ *and the left-invariant vector field* $X = x_4e_4$*, with* $|x_4| < b_0$*.*

For example, the previous corollary shows that the only Berwaldian left-invariant Randers, Matsumoto and Kropina metrics on four-dimensional Lie groups are of the following forms respectively:

$$F(y) = \sqrt{y_1^2+y_2^2+y_3^2+y_4^2} + y_4x_4, \qquad \text{with } |x_4| < 1. \tag{2.2}$$

$$F(y) = \frac{y_1^2+y_2^2+y_3^2+y_4^2}{\sqrt{y_1^2+y_2^2+y_3^2+y_4^2} - y_4x_4}, \quad \text{with } |x_4| < \frac{1}{2} \tag{2.3}$$

and

$$F(y) = \frac{\sqrt{y_1^2+y_2^2+y_3^2+y_4^2}}{y_4x_4}, \tag{2.4}$$

where

$$y := y_1e_1 + \cdots + y_4e_4$$

is an arbitrary left-invariant vector field on $\mathcal{H}_3 \oplus \mathcal{R}$ and $X = x_4e_4$.

Remark 2.2. We mention that if the Ricci curvature $Ric(x,y)$ of a Finsler manifold (M,F) is quadratic in y then the Finsler metric F is called Ricci-quadratic. By Theorem 7.9 of [7] a homogeneous Randers space is Ricci quadratic if and only if it is of Berwald type. Thus, by Corollary 2.1, a left-invariant Randers metric on a four-dimensional Lie group N, which is defined by a left-invariant Riemannian metric g and a left-invariant vector field X, is Ricci-quadratic if and only if $X = x_4e_4$.

Remark 2.3. Recall that a Finsler space (M,F) is named a generalized Berwald space if there exists a covariant derivative ∇ on M such that its parallel translations preserve F. In [2], it is shown that any left-invariant Finsler metric on a Lie group is a generalized Berwald space. So the Corollary 2.1 shows that any left-invariant (α,β)-metric on a four-dimensional Lie group N which is defined by a left-invariant Riemannian metric g and a left-invariant vector field $X = \sum_{i=1}^4 x_ie_i$, with the condition that $x_i \neq 0$ for some $i = 1,2,3$, is a non-Berwardian generalized Berwald space.

By attention to the Theorem 1.1 of [16], a left-invariant Douglas (α,β)-metric must be a Berwaldian Finsler metric or a Douglas Randers metric. In the Corollary 2.1, we classify all left-invariant Berwaldian (α,β)-metrics on four-dimensional Lie groups. Now, using Theorem 1.1 of [16], to give a complete description of left-invariant Douglas (α,β)-metrics on four-dimensional Lie groups, it is sufficient to classify all non-Berwaldian left-invariant Randers metrics of Douglas type on these spaces.

Remark 2.4. *Suppose that G is an arbitrary Lie group. We recall that a left-invariant Randers metric F on G, which is defined by a left-invariant vector field X and a left-invariant Riemannian metric g ($\langle , \rangle$), is a Douglas metric if and only if*

$$\langle [y, z], X\rangle = 0, \text{ for all } y, z \in \mathfrak{g}, \tag{2.5}$$

where $\mathfrak{g}$ denotes the Lie algebra of G (for more details see Proposition 7.4 of [7]*).*

Using the previous remark we have:

Corollary 2.5. *A four-dimensional nilpotent Lie group N admits a left-invariant non-Berwaldian Randers metric of Douglas type $F(x, y) = \sqrt{g(y, y)}+g(X(x), y)$, which is defined by a left-invariant Riemannian metric g and a left-invariant vector field X, if and only if*

(1) $N = H_3 \times \mathbb{R}$, $X = u_1e_1 + u_2e_2 + u_4e_4$ *and*

$$0 < \sqrt{u_1^2 + u_2^2 + u_4^2} < 1,$$

where u_1 and u_2 are not at the same time zero;

(2) $N = G_4$ *and* $X = u_1e_1 + u_2e_2$, *where* $0 < \sqrt{u_1^2 + u_2^2} < 1$.

Therefore the Corollary 2.1 together with the Corollary 2.5, classify all left-invariant Douglas (α, β)-metrics on four-dimensional Lie groups.

In the case (1) of the previous corollary if

$$u_1 = u_2 = 0$$

then the Randers metric is of Berwald type. On the other hand, we know that every bi-invariant Finsler metric is Berwaldian (see [14]). In the following theorem we show that the Randers metrics given in Corollary 2.5 are not bi-invariant.

Theorem 2.6. *There does not exist any bi-invariant Randers metric F on a four-dimensional nilpotent Lie group N.*

Proof. Suppose that F is a bi-invariant Randers metric on a four-dimensional nilpotent Lie group N. By Theorem 3.1 of [13], F is Berwaldian. So F is a Randers metric of the form Corollary 2.1. Then, by Theorem 3.2 of [13] for any $x, y, z \in \mathcal{N}$, we must have

$$\Big\langle [x, y], z\Big\rangle + \Big\langle [x, z], y\Big\rangle = 0.$$

But if we replace x, y, z by e_1, e_2, e_3, then we have $\lambda = 0$, which is a contradiction. □

3. Flag curvature and geodesic vectors

Flag curvature is a quantity in Finsler geometry which takes the place of the quantity of sectional curvature in Riemannian geometry. Let

$$\mathbf{g}_y(u,v) = \frac{1}{2}\frac{\partial^2}{\partial s \partial t}\Big[F^2(y+su+tv)\Big]\,|_{s=t=0}$$

and ∇^{Ch} denote the fundamental tensor and the Chern connection of a Finsler manifold (M, F), respectively. Then the flag curvature is defined by

$$\mathbf{K}(y,P) = \frac{\mathbf{g}_y\left(R_y\left(u\right),u\right)}{\mathbf{g}_y(y,y)\mathbf{g}_y(u,u) - \mathbf{g}_y^2(y,u)}, \tag{3.1}$$

where $P = span\{u, y\}$ and

$$R_y(u) = R(u,y)y = \nabla_u^{Ch}\nabla_y^{Ch}y - \nabla_y^{Ch}\nabla_u^{Ch}y - \nabla_{[u,y]}^{Ch}y.$$

For more details, see [7].

Recently, the formula of the flag curvature of Berwaldian (α, β)-metrics, in general case, are given in [11]. So the flag curvature formulas of the metrics given in Corollary 2.1 already exist. Therefore, here we compute the flag curvature of Douglas Randers metrics which are given in the Corollary 2.5. In order to compute the flag curvature formulas, we need to compute the formulas of sectional curvature of the base Riemannian metrics.

Lemma 3.1. *Let N be a four-dimensional nilpotent Lie group, which is equipped with a left-invariant Riemannian metric g. Suppose that $P = span\{y, t\}$ is a two-dimensional subspace of the tangent space at the unit element of N, where $\{y = \sum_{i=1}^4 y_i e_i, t = \sum_{i=1}^4 t_i e_i\}$ is an orthonormal set with respect to $\mathbf{g}$. Then the sectional curvature $\mathbf{K}^{\mathbf{g}}(P)$ for the Lie groups $H_3 \times \mathbb{R}$ and G_4 are respectively given by*

$$\mathbf{K}^{\mathbf{g}}(P) = \frac{\lambda}{4}\Big(-3(y_1t_2 - y_2t_1)^2 + (y_1t_3 - y_3t_1)^2 + (y_2t_3 - y_3t_2)^2\Big), \tag{3.2}$$

and

$$\begin{aligned}\mathbf{K}^{\mathbf{g}}(P) = \frac{1}{4}\Bigg\{ & \frac{1}{ac-b^2}(a^2c(y_3t_1 - y_1t_3)^2 + c^2((y_1t_4 - y_4t_1)^2 + (y_3t_4 - y_4t_3)^2 \\ & - 3(y_1t_3 - y_3t_1)^2) - \frac{2\sqrt{ac}}{\sqrt{ac-b^2}}(y_2y_4t_1^2 - y_1y_4t_1t_2 + y_3y_4t_2t_3 \\ & - y_2y_4t_3^2 - y_1y_2t_1t_4 + y_1^2t_2t_4 - y_3^2t_2t_4 + y_2y_3t_3t_4) + a((y_2t_3 - y_3t_2)^2\end{aligned}$$

$$
\begin{aligned}
&-3(y_1t_2-y_2t_1)^2)+\frac{ab^2}{b^2-ac}(y_3t_1-y_1t_3)^2+\frac{4bc}{a(ac-b^2)}\\
&(b((y_1t_3-y_3t_1)^2-(y_1t_4-y_4t_1)^2)+2\sqrt{ac-b^2}(y_3y_4t_1^2-y_1y_4t_1t_3\\
&-y_1y_3t_1t_4+y_1^2t_3t_4))+\frac{4b}{\sqrt{a}\sqrt{ac-b^2}}((bt_4(2y_1^2-y_3^2)+by_4(-2y_1t_1\\
&+y_3t_3)+\sqrt{ac-b^2}(2y_1y_3t_1-2y_1^2t_3+y_4^2t_3-y_3y_4t_4))t_2\\
&+y_2(bt_4(-2y_1t_1+y_3t_3)+by_4(2t_1^2-t_3^2)+\sqrt{ac-b^2}(-2y_3t_1^2\\
&+2y_1t_1t_3-y_4t_3t_4+y_3t_4^2))\Bigg\}. \qquad (3.3)
\end{aligned}
$$

Proof. In the case of $N=H_3\times\mathbb{R}$, the Levi-Civita connection is given by

$$
\begin{aligned}
\nabla_{e_1}^{e_2}&=-\nabla_{e_2}^{e_1}=\frac{\sqrt{\lambda}}{2}e_3,\\
\nabla_{e_1}^{e_3}&=\nabla_{e_3}^{e_1}=\frac{-\sqrt{\lambda}}{2}e_2,\\
\nabla_{e_2}^{e_3}&=\nabla_{e_3}^{e_2}=\frac{\sqrt{\lambda}}{2}e_1.
\end{aligned}
$$

So for the non-zero Riemannian curvature tensor components $R_{ijk}:=R(e_i,e_j)e_k$, we have

$$
\begin{aligned}
e_1&=-\frac{4}{3\lambda}R_{122}=\frac{4}{3\lambda}R_{212}=\frac{4}{\lambda}R_{133}=-\frac{4}{\lambda}R_{313},\\
e_2&=-\frac{4}{3\lambda}R_{211}=\frac{4}{3\lambda}R_{121}=\frac{4}{\lambda}R_{233}=-\frac{4}{\lambda}R_{323},\\
e_3&=-\frac{4}{\lambda}R_{131}=\frac{4}{\lambda}R_{311}=\frac{4}{\lambda}R_{322}=-\frac{4}{\lambda}R_{232}.
\end{aligned}
$$

Now, it is sufficient to use the sectional curvature formula

$$
\mathbf{K}(y,t)=\frac{\langle R(y,t)t,y\rangle}{\langle y,y\rangle\langle t,t\rangle-\langle y,t\rangle^2},
$$

and the fact that the set $\{y,t\}$ is an orthonormal set with respect to $\mathbf{g}$. Also, in the case of $N=G_4$ we have a similar proof. □

The above result enables us to explicitly obtain the flag curvature formulas for all left-invariant Douglas Randers metrics (non-Berwaldian Douglas (α,β)-metrics) on four-dimensional nilpotent Lie groups as follows.

Theorem 3.2. *Let N be a four-dimensional nilpotent Lie group with a left-invariant non-Berwaldian Douglas Randers metric which is given in Corollary 2.5. Then, in each case, for the flag curvature $\mathbf{K}^F(P,y)$ we have*

(1) *If* $N = H_3\times\mathbb{R}$ *then,*

$$\begin{aligned}\mathbf{K}^F(P,y) &= \frac{(\sum_{i=1}^4 y_i^2)\mathbf{K}^{\mathbf{g}}(P)}{2(\sqrt{\sum_{i=1}^4 y_i^2}+\sum_{i=1}^3 y_iu_i)^2}\\ &\quad+\frac{1}{4(\sqrt{\sum_{i=1}^4 y_i^2}+\sum_{i=1}^3 y_iu_i)^4}\\ &\quad\times\Bigg(2y_3^2(\sqrt{\sum_{i=1}^4 y_i^2}+\sum_{i=1}^3 y_iu_i)(y_1u_1+y_2u_2)\\ &\qquad+3(u_1y_1y_3-y_1y_3u_2)^2\Bigg),\end{aligned}$$

(2) *If* $N = G_4$ *then,*

$$\begin{aligned}\mathbf{K}^F(P,y) &= \frac{1}{4(\sqrt{\sum_{i=1}^4 y_i^2}+\sum_{i=1}^2 y_iu_i)^4}\\ &\quad\times\Bigg(4(\sum_{i=1}^4 y_i^2)\mathbf{K}^{\mathbf{g}}(P)(\sqrt{\sum_{i=1}^4 y_i^2}+\sum_{i=1}^2 y_iu_i)^2\\ &\quad+3[(y_3y_4+y_2y_3)u_1-y_1y_3u_2]^2\\ &\quad-2(\sqrt{\sum_{i=1}^4 y_i^2}+\sum_{i=1}^2 y_iu_i)\big(-y_1(y_2y_4+y_3^2+y_4^2)u_1\\ &\quad+(y_1^2y_4-y_3^2y_4-y_2y_3^2)u_2\big)\Bigg),\end{aligned}$$

where $\{y = y_1e_1+\cdots+y_4e_4, t = t_1e_1+\cdots+t_4e_4\}$ *is an orthonormal basis of* P *with respect to* g *and* $K^g(P)$ *denotes the sectional curvature given in Lemma 3.2.*

Proof. Suppose that F is the Randers metric which is given in the case (i) of Corollary 2.5. Define U by the following equation,

$$2\langle U(x,y),z\rangle = \langle [z,x],y\rangle + \langle [z,y],x\rangle,$$

where $x,y,z\in\mathcal{H}_3\times\mathcal{R}$.

So we have

$$U(y,t) = \frac{1}{2}\Big\{\lambda(y_2t_3+t_2y_3)e_1-\lambda(y_1t_3+t_1y_3)e_2\Big\},$$

which implies that

$$\langle U(y,y),X\rangle = u_1y_2y_3-y_1y_3u_2, \tag{3.4}$$

where X is considered as Corollary 2.5. Also we see that

$$\Big\langle U\big(y, U(y, y)\big), X\Big\rangle = \frac{-y_3^2}{2}\Big(y_1 u_1 + y_2 u_2\Big). \tag{3.5}$$

Now, it is sufficient to use the following formula which is given in Theorem 2.1 of [8]

$$\mathbf{K}(P, y) = \frac{\langle y, y\rangle^2}{F^2(y)}\mathbf{K}^{\mathbf{g}} + \frac{1}{4F(y)^4}\Bigg(3\langle U(y, y), X\rangle^2 - 4F(y)\langle U(y, U(y, y)), X\rangle\Bigg), \tag{3.6}$$

where $\mathbf{K}^{\mathbf{g}}$ is the sectional curvature of the left-invariant Riemannian metric g. In the case (ii), we have a similar proof. □

Geodesics of left-invariant Riemannian metrics on Lie groups, which have important applications in mechanics, were studied by Arnold (see [3]). In [21], the geodesic vectors of invariant (α, β)-metrics on homogeneous spaces are investigated. Here we study the geodesic vectors of four-dimensional nilpotent Lie groups which are equipped with a left-invariant Douglas (α, β)-metric F.

Theorem 3.3. *Suppose that N is a four-dimensional nilpotent Lie group equipped with a left-invariant Douglas Randers metric F which is given in Corollary 2.5. Then,*

1. *if $N = H_3 \times \mathbb{R}$, then Y is a geodesic vector of (N, F) if and only if $Y \in span\{E_1, E_2, E_4\}$ or $Y \in span\{E_3, E_4\}$,*
2. *if $N = G_4$, then Y is a geodesic vector of (N, F) if and only if $Y \in span\{E_1, E_2\}$ or $Y \in span\{E_3, E_4\}$.*

Proof. Let F be the Randers metric defined in case (i) of Corollary 2.5. We see that F satisfies the conditions of Theorem 2.3 of [21]. So Y is a geodesic vector of the Randers metric F if and only if Y is a geodesic vector of the Riemannian metric g. We notice that, by geodesic lemma of [10], Y is a geodesic vector of (N, g) if and only if $\langle [y, z], y\rangle = 0$ for any $z \in \mathcal{N}$. Hence $Y = \sum_{i=1}^{4} y_i e_i$ is a geodesic vector of (N, F) if and only if we have

$$\Big\langle [\sum_{i=1}^{4} y_i e_i, e_i], \sum_{i=1}^{4} y_i e_i\Big\rangle = 0$$

which implies that

$$\sqrt{\lambda} y_2 y_3 = 0 \ \text{ and } \ \sqrt{\lambda} y_1 y_3 = 0.$$

Thus we consider two cases $y_3 = 0$ and $y_3 \neq 0$. If $y_3 = 0$, then we have $Y \in span\{E_1, E_2, E_4\}$ and if $y_3 \neq 0$, then we have $Y \in span\{E_3, E_4\}$. For the case (ii) of Corollary 2.5 ($N = G_4$) we have a similar proof. □

4. S-curvature of left-invariant Randers metrics

In [18], Shen introduced the concept of S-curvature which is an important non-Riemannian quantity in Finsler geometry. It is a scalar function on TM, which describes the rate of changes of the distortion along geodesics. In the case of left-invariant Randers metrics on Lie groups there is an explicit formula for S-curvature (see [7]).
Assume that G is a Lie group equipped with a left-invariant Randers metric F which is defined by an inner product $\langle , \rangle$ on the Lie algebra $\mathcal{G}$ of G and a left-invariant vector field X. Then the S-curvature is given by

$$\mathbf{S}(e, y) = \frac{n+1}{2}\left\{\frac{\langle [X, y], \langle y, X\rangle X - y\rangle}{F(y)} - \langle [X, y], X\rangle\right\}. \tag{4.1}$$

Now using the above formula we have the following theorem.

Theorem 4.1. *Let N be a four-dimensional nilpotent Lie group with a left-invariant Randers metric of Douglas type F which is defined by an inner product $\langle , \rangle$ on $\mathcal{N}$ and the left-invariant vector field X which is given in Corollary 2.5.*

i: *If $N = H_3 \times \mathbb{R}$, the S-curvature formula is given by,*

$$\mathbf{S}(e, y) = \frac{5\sqrt{\lambda} y_3(u_2 y_1 - u_1 y_2)}{2(\sqrt{y_1^2 + y_2^2 + y_3^2 + y_4^2} + u_1 y_1 + u_2 y_2 + u_4 y_4)}.$$

ii: *If $N = G_4$, then the S-curvature formula is determined by*

$$\mathbf{S}(e, y) = -\frac{5}{2aV}\left\{\sqrt{a^3}(u_1 y_2 y_3 - u_2 y_1 y_3) + u_1\left(\frac{-b^2 y_3 y_4}{\sqrt{ac - b^2}} + y_3 y_4\sqrt{ac - b^2} + b(y_3^2 - y_4^2)\right)\right\},$$

where

$$V := (\sqrt{y_1^2 + y_2^2 + y_3^2 + y_4^2} + u_1 y_1 + u_2 y_2)$$

and $y = y_1 e_1 + \cdots + y_4 e_4$ is a vector in the Lie algebra $\mathcal{N}$ of N.

Proof. Notice that X is given in Corollary 2.5. Thus using the formula (4.1) and some calculations completes the proof. □

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