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A special class of Finsler metrics

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Abstract. In this paper, we study a special class of Finsler metrics $F = F(x, y)$ in $\mathbb{R}^n$ that satisfy $F(-x, y) = F(x, y)$. We show the induced distance function of F satisfies $d_F(p, q) = d_F(-q, -p)$ for all $p, q \in \mathbb{R}^n$. The geodesics of these metrics have special property and many well-known Finsler metrics belong to this class. We prove that these metrics with constant $\mathbf{S}$-curvature satisfy $\mathbf{S} = 0$.

Keywords: Finsler geodesic, S-curvature, Landsberg curvature.

1. Introduction

The study of distance functions induced by Finsler metrics is one of the important problems in Finsler geometry. The distance functions of some classes of Finsler metrics have special properties. For example the distance function of reversible Finsler manifold (M, F) is symmetric, i.e., $d_F(p, q) = d_F(q, p)$[2]. The Finsler metric is called reversible if F satisfies $F(x, -y) = F(x, y)$. The Riemannian metrics are the most interesting class of reversible Finsler metrics. In this paper, we are going to investigate distance function of Finsler metrics that satisfies

$$F(-x, y) = F(x, y). \tag{1.1}$$

Many interesting Finsler metric belong to this class. We have the following two well-known special cases.

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Example 1.1. ([1]) *Consider Randers metric*

$$F = \frac{\sqrt{(1-|a|^2|x|^4)|y|^2 + (|x|^2 < a, y > -2 < a, x >< x, y >)^2}}{1-|a|^2|x|^4} - \frac{|x|^2 < a, y > -2 < a, x >< x, y >}{1-|a|^2|x|^4},$$

where $a \in \mathbb{R}^n$ is a constant vector, $|.|$ and $<,>$ denote the Euclidean norm and inner product in R^n. The above defined Randers metric F is of isotropic S-curvature and scalar flag curvature, i.e.,

$$\mathbf{S} = (n+1) < a, x > F, \qquad \mathbf{K} = \frac{3 < a, y >}{F} + 3 < a, x >^2 - 2|a|^2|x|^2.$$

This metric satisfies (1.1).

Example 1.2. *Another class of Finsler metric that satisfy (1.1) has the following formula*

$$\alpha_\mu(x,y) := \frac{\sqrt{|y|^2 + \mu\Big(|x|^2|y|^2 - < x, y >^2\Big)}}{1+\mu|x|^2} \qquad y \in R^n$$

where μ is a positive constant. These Riemannian metrics are projectively flat metrics

$$G^i = -\frac{\mu < x, y >}{1+\mu|x|^2} y^i$$

and has constant flag curvature μ. Indeed, $R^i_k = \mu F^2 h^i_k$.

First we prove the following.

Theorem 1.3. *Let (M,F) be a Finsler metric that satisfies (1.1) and d_F is distance function inducted by F. Then d_F satisfies*

$$d_F(p,q) = d_F(-q,-p)$$

for all $p, q \in \mathbb{R}^n$.

A curve $\gamma : [0,1] \to M$ is called a geodesic of (M,F) if it minimizes the Finslerian length for all piecewise C^∞ curves that keep their end points fixed. The Finsler metric F is called with reversible geodesics if and only if for any geodesic $\gamma : [0,1] \to M$ of F, the reverse curve $\bar{\gamma}(t) := \gamma(1-t)$ is also a geodesic of F. All reversible Finsler metrics have this property[2]. Finsler metrics that satisfy $F(x,y) = F(-x,y)$ have similar geodetic properties.

Theorem 1.4. *Let F be a Finsler metrics on $\mathbb{R}^n$ such that $F(-x,y) = F(x,y)$ for any vector $x \in \mathbb{R}^n$. Then $\sigma : [0,1] \to M$ is a geodesic of F if and only if $\bar{\sigma}(t) := -\sigma(1-t)$ is a geodesic of F.*

The **S**-curvature is one of the most important non-Riemannian quantities in RiemannFinsler geometry which was first introduced by Zhongmin Shen when he studied volumn comparison in Finsler geometry [5]. Recent study shows that the **S**-curvature plays a very important role in Finsler geometry [3][4]. The Finsler metrics with constant S-curvature are of some important geometric structures which deserve to be studied deeply. For example, Shen has proved the following rigidity theorem: for a Finsler metric F with constant S-curvature on an n-dimensional closed manifold M, if F has negative flag curvature, then it must be Riemannian [4]. For our main aim, we can prove the following

Theorem 1.5. *Let F be a Finsler metrics on R^n that satisfies (1.1). Suppose that F has constant S-curvature. Then* $\mathbf{S} = 0$.

Let us put

$$F(x,y) := \frac{\sqrt{|y|^2 - (|x|^2|y|^2 - <x,y>^2)} + <x,y>}{1-|x|^2}, \quad y \in T_x\mathrm{B}^{\mathrm{n}} = \mathrm{R}^{\mathrm{n}}, \quad (1.2)$$

F is called the *Funk metric*. The Funk metric has interesting curvature properties. It satisfies

$$L_{ijk} = cFC_{ijk} \quad (1.3)$$

where c is a constant. In this paper, we prove every Finsler metric that satisfies (1.1) and (1.3) is a Landsberg metric.

Theorem 1.6. *Let F be a Finsler metric that satisfies (1.1). Suppose F is also general relative isotropic Landsberg metric,*

$$L_{ijk} = cFC_{ijk}, \quad (1.4)$$

where c is a constant. Then F is a Landsberg metric.

2. Preliminary

Let M be an n-dimensional C^∞ manifold. Denote by T_xM the tangent space at $x \in M$, and by $TM = \cup_{x\in M} T_xM$ the tangent bundle of M. A Finsler metric on M is a function $F : TM \to [0,\infty)$ which has the following properties:
(i) F is C^∞ on $TM_0 := TM \setminus \{0\}$;
(ii) F is positively 1-homogeneous on the fibers of tangent bundle TM; (iii) for each $y \in T_xM$, the following quadratic form $\mathbf{g}_y$ on T_xM is positive definite,

$$\mathbf{g}_y(u,v) := \frac{1}{2}\frac{\partial^2}{\partial s \partial t}\left[F^2(y+su+tv)\right]|_{s,t=0}, \quad u,v \in T_xM.$$

Let (M,F) be a Finsler manifold. The third-order derivatives of $\frac{1}{2}F_x^2$ at $y \in T_xM_0$ is a symmetric trilinear form $\mathbf{C}_y$ on T_xM which is called Cartan torsion. The rate of change of Cartan torsion $\mathbf{C}$ along geodesics is called the Landsberg curvature $\mathbf{L}$. A Finsler metric satisfies $\mathbf{L} = 0$ is called a Landsberg metric. A Finsler metric F on a manifold M is said to be general relative isotropic Landsberg metric , if

$$L_{ijk} = \lambda C_{ijk} \tag{2.1}$$

where λ is a positively 1-homogeneous scalar function.

If $\gamma : [0,1] \to M$ is a piecewise C^∞ curve on M, then its Finslerian length is defined as

$$\mathcal{L}_F(\gamma) := \int_0^1 F(\gamma(t), \dot{\gamma}(t))\, dt.$$

and the Finslerian distance function $d_F : M \times M \to [0, \infty)$ is defined by

$$d_F(p,q) = \inf_\gamma \mathcal{L}_F(\gamma),$$

where the infimum is taken over all piecewise C^∞ curves on M joining the points $p, q \in M$. Given a Finsler manifold (M, F), then a global vector field $\mathbf{G}$ is induced by F on TM_0, which in a standard coordinate (x^i, y^i) for TM_0 is given by $\mathbf{G} = y^i \frac{\partial}{\partial x^i} - 2G^i(x,y)\frac{\partial}{\partial y^i}$, where

$$G^i := \frac{1}{4} g^{il}\Big[\frac{\partial^2 F^2}{\partial x^k \partial y^l} y^k - \frac{\partial F^2}{\partial x^l}\Big], \qquad y \in T_xM, \tag{2.2}$$

where $g^{ij} := (g_{ij})^{-1}$. $\mathbf{G}$ is called the spray associated to (M, F). A C^∞ map $\sigma := \sigma(t)$ in (M, F) is called a geodesic of F if it satisfies

$$\ddot{\sigma}^i(t) + 2G^i\Big(\sigma(t), \dot{\sigma}(t)\Big) = 0$$

For a Finsler metric F on an n-dimensional manifold M, the Busemann-Hausdorff volume form $dV_F = \sigma_F(x) dx^1 \cdots dx^n$ is defined by

$$\sigma_F(x) := \frac{\mathrm{Vol}(\mathbb{B}^n(1))}{\mathrm{Vol}\big[(y^i) \in R^n \,\big|\, F\big(y^i|_x\big) < 1\big]}.$$

Let G^i denote the geodesic coefficients of F in the same local coordinate system. Then for $\mathbf{y} = y^i|_x \in T_xM$, the S-curvature is defined by

$$\mathbf{S}(\mathbf{y}) := \frac{G^i}{y^i}(x,y) - y^i \frac{}{x^i}\big[\ln \sigma_F(x)\big].$$

This quantity was first introduced by Shen for a volume comparison theorem [5]. A Finsler metric F on an n-dimensional manifold M is said to have isotropic S-curvature if there is a scalar function $c = c(x)$ on M such that

$$\mathbf{S} = (n+1)cF$$

F is said to have constant $\mathbf{S}$-curvature if $c = constant$.

3. Proof of Theorems

Proof of Theorem 1.3: By definition we have

$$d_F(p,q) = \inf_{\gamma} \int_0^1 F\Big(\gamma(t), \dot{\gamma}(t)\Big)\, dt, \tag{3.1}$$

for any curve $\gamma : [0,1] \to M$ that $\gamma(0) = p$ and $\gamma(1) = q$. It is easy to see that

$$d_F(p,q) = \inf_{\gamma} \int_0^1 F\Big(\gamma(t), \dot{\gamma}(t)\Big)\, dt = \inf_{\gamma} \int_0^1 F\Big(\gamma(1-t), \dot{\gamma}(1-t)\Big)\, dt. \tag{3.2}$$

Since F satisfy (1.1), from (3.2) one can see that

$$d_F(p,q) = \inf_{\gamma} \int_0^1 F\Big(-\gamma(1-t), \dot{\gamma}(1-t)\Big)\, dt = d_F(-q,-p).$$

Thus we get the proof. $\square$

In order to prove Theorem 1.4, we first need to prove the following

Lemma 3.1. *Let F be a Finsler metrics on $\mathbb{R}^n$ such that $F(-x,y) = F(x,y)$ for any vector $x \in \mathbb{R}^n$. The spray coefficients of F satisfy:*

$$G^i(-x,y) = -G^i(x,y) \tag{3.3}$$

Proof. It is easy to see that

$$g_{ij}(-x,y) = g_{ij}(x,y), \tag{3.4}$$

$$g^{ij}(-x,y) = g^{ij}(x,y), \tag{3.5}$$

$$[F^2]_{x^m}(-x,y) = -[F^2]_{x^m}(x,y). \tag{3.6}$$

By (2.2), (3.5) and (3.6), we get (3.3). $\square$

Proof of Theorem 1.4: A direct computation yields

$$\ddot{\bar{\sigma}}^i(t) + 2G^i\Big(\bar{\sigma}(t), \dot{\bar{\sigma}}(t)\Big) = -\ddot{\sigma}^i(1-t) + 2G^i\Big(-\sigma(1-t), \dot{\sigma}(1-t)\Big). \tag{3.7}$$

By the Lemma 3.1, we have

$$\ddot{\bar{\sigma}}^i(t) + 2G^i\Big(\bar{\sigma}(t), \dot{\bar{\sigma}}(t)\Big) = -\Big[\ddot{\sigma}^i(1-t) + 2G^i\Big(\sigma(1-t), \dot{\sigma}(1-t)\Big)\Big]. \tag{3.8}$$

Since $\sigma = \sigma(t)$ is a geodesic of F, then

$$\ddot{\sigma}^i(1-t) + 2G^i\Big(\sigma(1-t), \dot{\sigma}(1-t)\Big) = 0. \tag{3.9}$$

By (3.8) and (3.9) one can see that $\bar{\sigma}$ is a geodesic of F. $\square$

Proof of Theorem 1.5: Let G^i denote the spray coefficients of F. It is easy to see that

$$\sigma_F(x) = \sigma_F(-x)$$

Therefore by Lemma 3.1 we obtain

$$\mathbf{S}(-x,y) = -\mathbf{S}(x,y) \tag{3.10}$$

Since F has constant $S-$ curvature, we have

$$\mathbf{S}(x,y) = (n+1)cF(x,y) \tag{3.11}$$

where c is a constant. Thus

$$\mathbf{S}(-x,y) = (n+1)cF(-x,y) \tag{3.12}$$

From (3.10), (3.11) and (3.12) one can see that $c = 0$. This completes the proof of Theorem 1.5. □

Proof of Theorem 1.6: It is easy to see that

$$C_{ijk}(-x,y) = C_{ijk}(x,y). \tag{3.13}$$

Since $L_{ijk} = C_{ijk|l}y^l$, so we have

$$L_{ijk}(-x,y) = -L_{ijk}(x,y). \tag{3.14}$$

By (1.4), we have

$$L_{ijk}(-x,y) = cF(-x,y)C_{ijk}(-x,y).$$

From (3.13) and (3.14) one can see that $c = 0$. This completes the proof of Theorem 1.6.

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