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EINSTEIN METRICS ON MIXED 3-STRUCTURE STATISTICAL MANIFOLDS

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ABSTRACT

This paper presents a comprehensive study of curvature properties on statistical manifolds endowed with mixed 3-Sasakian and mixed 3-cosymplectic structures. We derive necessary and sufficient conditions for mixed 3-Sasakian statistical manifolds to be Einstein, establishing fundamental identities for the Ricci tensor and scalar curvature. Furthermore, we introduce and analyze mixed 3-cosymplectic statistical manifolds, proving that they are necessarily Ricci-flat. The interplay between statistical duality and these geometric structures yields rich curvature constraints that are thoroughly analyzed. We provide explicit examples of both mixed 3-Sasakian Einstein and mixed 3-cosymplectic Ricci-flat statistical manifolds, discussing their physical implications in the context of general relativity and field theories. The results generalize previous work on Sasakian and cosymplectic geometry to the statistical setting, opening new paths for research in information geometry and theoretical physics.

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1. Introduction

Einstein manifolds play a fundamental role in mathematical physics and specially in general relativity, where they represent solutions to the vacuum Einstein field equations:

$$\mathbf{Ric} - \frac{1}{2} S\mathbf{g} + \Lambda \mathbf{g} = 0.$$

In the presence of a cosmological constant, we have $\mathbf{Ric} = \lambda \mathbf{g}$ with λ constant. The study of Einstein metrics on manifolds with additional structures has been particularly fruitful in physics. Sasakian-Einstein manifolds, for instance, have applications in string theory and AdS/CFT correspondence. The search for Einstein metrics on manifolds with additional geometric structures has been a driving force in modern differential geometry, with seminal contributions from Berger, Besse, and many others [3, 4, 11, 14].

The theory of statistical manifolds, since its formalization by Lauritzen [10], has emerged as a profound bridge connecting information geometry with differential geometry. A statistical manifold is a Riemannian or semi-Riemannian manifold equipped with a pair of torsion-free affine connections dual with respect to the metric. This elegant framework naturally generalizes the geometric properties of statistical models, where the metric corresponds to the Fisher information metric and the connections to the α-connections introduced by Amari [1]. The dualistic structure has found applications in various fields including statistical inference, machine learning, and recently, in theoretical physics.

Parallel to this development, the study of almost contact metric structures and their higher-dimensional analogues has deep roots in differential geometry [12]. Sasakian geometry, named after Shigeo Sasaki [13], represents an odd-dimensional counterpart to Kähler geometry. The generalization to 3-Sasakian manifolds, initiated by Kuo [9] and further developed by Boyer, Galicki, and others [4], provides a rich framework with remarkable curvature properties. Mixed 3-Sasakian structures, incorporating both paracontact and contact elements, offer an even broader landscape for geometric investigation.

The intersection of these domains—statistical manifolds, contact geometry, and Einstein conditions—remains largely unexplored. Previous work has examined Sasakian statistical manifolds and their submanifold theory [6], while 3-Sasakian statistical structures were introduced in [2, 8]. However, the Einstein properties of mixed 3-Sasakian statistical manifolds and the Ricci-flat properties of mixed 3-cosymplectic statistical manifolds have not been systematically investigated until now.

This paper aims to fill this gap by providing a comprehensive analysis of curvature properties on both mixed 3-Sasakian and mixed 3-cosymplectic statistical manifolds. These results extend the work of Caldarella and Pastore [5] to the statistical setting, while also contributing to the growing interaction between differential geometry and information theory. Our main contributions include:

(1) Derivation of necessary and sufficient conditions for mixed 3-Sasakian statistical manifolds to be Einstein
(2) Introduction of mixed 3-cosymplectic statistical manifolds and proof that they are necessarily Ricci-flat
(3) Establishment of fundamental identities relating the statistical structure to curvature properties
(4) Construction of explicit examples illustrating both Einstein and Ricci-flat cases

(5) Analysis of physical implications in the context of general relativity and field theories

The paper is organized as follows: Section 2 provides necessary preliminaries on statistical manifolds, mixed 3-Sasakian structures, and introduces mixed 3-cosymplectic structures. Section 3 presents our main results on Einstein conditions for mixed 3-Sasakian manifolds. Section 4 focuses on mixed 3-cosymplectic statistical manifolds and their Ricci-flat property. Section 5 provides detailed examples, and Section 6 concludes with discussions and open problems.

2. Preliminaries

2.1. **Statistical Manifolds.** The theory of statistical manifolds provides a geometric framework for understanding statistical models. The fundamental definition, as formalized by Lauritzen [10], is as follows:

Definition 2.1. A *statistical structure* on a semi-Riemannian manifold $(\mathcal{M}, \mathbf{g})$ is a pair $(\nabla, \mathbf{g})$ where ∇ is a torsion-free affine connection satisfying the Codazzi condition:

$$(\nabla_{E_1}\mathbf{g})(E_2, E_3) = (\nabla_{E_2}\mathbf{g})(E_1, E_3) \quad \forall E_1, E_2, E_3 \in \Gamma(TM).$$

The *dual connection* ∇^* is defined through the relation:

$$E_1\mathbf{g}(E_2, E_3) = \mathbf{g}(\nabla_{E_1}E_2, E_3) + \mathbf{g}(E_2, \nabla^*_{E_1}E_3).$$

The Levi-Civita connection $\widehat{\nabla}$ associated with $\mathbf{g}$ satisfies $\widehat{\nabla} = \frac{1}{2}(\nabla + \nabla^*)$.

The *difference tensor* $\mathcal{K}$, defined by $\mathcal{K}_{E_1}E_2 = \nabla_{E_1}E_2 - \widehat{\nabla}_{E_1}E_2$, plays a crucial role in statistical geometry. It is symmetric and satisfies $\mathbf{g}(\mathcal{K}_{E_1}E_2, E_3) = \mathbf{g}(E_2, \mathcal{K}_{E_1}E_3)$ for all vector fields E_1, E_2, E_3.

The curvature tensors of the dual connections are related by:

$$\mathbf{g}(\mathcal{R}(E_1, E_2)E_3, E_4) = -\mathbf{g}(E_3, \mathcal{R}^*(E_1, E_2)E_4),$$

where $\mathcal{R}$ and $\mathcal{R}^*$ are the curvature tensors of ∇ and ∇^* respectively.

2.2. **Mixed 3-Sasakian Structures.** The concept of almost contact 3-structures was introduced by Kuo [9] and further developed by various authors [7]. A mixed 3-Sasakian structure represents a specialized case with rich geometric properties.

Definition 2.2. A *mixed 3-structure* on a semi-Riemannian manifold $\mathcal{M}$ consists of three almost contact structures (ψ_i, B_i, η_i), $i = 1, 2, 3$, satisfying:

$$\psi_i^2 = \tau_i(-I + \eta_i \otimes B_i),$$
$$\eta_i(B_i) = 1, \quad \eta_i(B_j) = 0 \quad (i \neq j),$$
$$\psi_i \circ \psi_j - \tau_i\eta_j \otimes B_i = -\psi_j \circ \psi_i + \tau_j\eta_i \otimes B_j = \tau_k\psi_k,$$

where (i, j, k) is a cyclic permutation of $(1, 2, 3)$, $\tau_1 = \tau_2 = 1$, $\tau_3 = -1$.

A semi-Riemannian metric $\mathbf{g}$ is *compatible* with the mixed 3-structure if:

$$\mathbf{g}(\psi_iE_1, \psi_iE_2) = \tau_i[\mathbf{g}(E_1, E_2) - \varepsilon_i\eta_i(E_1)\eta_i(E_2)],$$

where $\varepsilon_i = \mathbf{g}(B_i, B_i) = \pm 1$.

Definition 2.3. A *mixed 3-Sasakian statistical structure* on $(\mathcal{M}, \mathbf{g})$ consists of a statistical structure $(\nabla, \nabla^*, \mathbf{g})$ and a mixed 3-structure $(\psi_i, B_i, \eta_i)_{i=1,2,3}$ satisfying the Sasakian statistical condition:

$$(\nabla_{E_1}\psi_i)E_2 - \psi_i(\nabla^*_{E_1}E_2) = \tau_i[\mathbf{g}(E_1, E_2)B_i - \varepsilon_i\eta_i(E_2)E_1], \quad i = 1, 2, 3.$$

Equally a 3-Sasakian statistical manifold is a statistical manifold (M, ∇, ∇^*, g) equipped with a 3-Sasakian structure (ψ_i, B_i, η_i) satisfying:

$$\mathcal{K}_V\psi_iW + \psi_i\mathcal{K}_VW = 0 \quad \forall V, W \in \Gamma(TM). \tag{2.1}$$

This condition generalizes the classical Sasakian condition to the statistical setting and imposes strong constraints on both the geometric structure and the statistical connection.

2.3. **Mixed 3-Cosymplectic Structures.** The cosymplectic analogue of Sasakian geometry provides a different class of almost contact metric manifolds with distinct curvature properties.

Definition 2.4. A *mixed 3-cosymplectic statistical structure* on $(\mathcal{M}, \mathbf{g})$ consists of a statistical structure $(\nabla, \nabla^*, \mathbf{g})$ and a mixed 3-structure $(\psi_i, B_i, \eta_i)_{i=1,2,3}$ satisfying the cosymplectic statistical condition:

$$\nabla_{E_1}\psi_i = \nabla^*_{E_1}\psi_i = 0, \quad \nabla_{E_1}B_i = \nabla^*_{E_1}B_i = 0, \quad \nabla_{E_1}\eta_i = \nabla^*_{E_1}\eta_i = 0, \quad i = 1, 2, 3.$$

This condition implies that all structure tensors are parallel with respect to the statistical connections, which imposes even stronger constraints than the Sasakian case.

3. Mixed 3-Sasakian Statistical Manifolds and Einstein Conditions

We now present our main theorems characterizing Einstein conditions on mixed 3-Sasakian statistical manifolds.

Theorem 3.1. *Let $(\mathcal{M}, \mathbf{g}, \nabla, (\psi_i, B_i, \eta_i)_{i=1,2,3})$ be a mixed 3-Sasakian statistical manifold of dimension $4n+3$. Then $\mathcal{M}$ is an Einstein statistical manifold with* $\mathbf{Ric} = \lambda\mathbf{g}$ *if and only if the following conditions hold:*

1. *The vertical distribution $\mathcal{V} = span\{B_i\}$ is Einstein with respect to the induced metric.*
2. *The horizontal distribution $\mathcal{H} = \mathcal{V}^\perp$ is Einstein with the same constant λ.*
3. *The difference tensor $\mathcal{K}$ satisfies:*

$$\mathbf{tr}(\mathcal{K}(E_1, \mathcal{K}(E_2, \cdot))) - \mathbf{g}(\mathcal{K}(E_1, \cdot), \mathcal{K}(E_2, \cdot)) = \mu\mathbf{g}(E_1, E_2) \quad \forall E_1, E_2 \in \mathcal{H},$$

for some function μ related to λ through the structure constants.

Proof. Let $\{e_a\}_{a=1}^{4n+3}$ be a local orthonormal frame adapted to the mixed 3-Sasakian structure, such that $e_{4n+i} = B_i$ for $i = 1, 2, 3$. The Ricci tensor for the statistical connection ∇ is:

$$\mathbf{Ric}(E_1, E_2) = \sum_{a=1}^{4n+3} \epsilon_a\mathbf{g}(\mathcal{R}(E_1, e_a)E_2, e_a), \quad \epsilon_a = \mathbf{g}(e_a, e_a).$$

Using the relation between $\mathcal{R}$ and the Levi-Civita curvature $\mathcal{R}^{\widehat{\nabla}}$:

$$\begin{aligned}\mathcal{R}(E_1, E_2)E_3 = \mathcal{R}^{\widehat{\nabla}}(E_1, E_2)E_3 + (\widehat{\nabla}_{E_1}\mathcal{K})(E_2, E_3) - (\widehat{\nabla}_{E_2}\mathcal{K})(E_1, E_3) \\ + \mathcal{K}(E_1, \mathcal{K}(E_2, E_3)) - \mathcal{K}(E_2, \mathcal{K}(E_1, E_3)).\end{aligned}$$

Substituting and simplifying:

$$\begin{aligned}\mathbf{Ric}(E_1,E_2)=\mathbf{Ric}^{\widehat{\nabla}}(E_1,E_2)&-\sum_a \epsilon_a \mathbf{g}((\widehat{\nabla}_{e_a}\mathcal{K})(E_1,E_2)-(\widehat{\nabla}_{E_1}\mathcal{K})(e_a,E_2),e_a)\\&+\sum_a \epsilon_a \mathbf{g}(\mathcal{K}(e_a,\mathcal{K}(E_1,E_2))-\mathcal{K}(E_1,\mathcal{K}(e_a,E_2)),e_a).\end{aligned}$$

Now, we compute $\mathbf{Ric}^{\widehat{\nabla}}(E_1,E_2)$ using the mixed 3-Sasakian conditions. The Sasakian statistical condition implies:

$$(\widehat{\nabla}_{E_1}\psi_i)E_2=\frac{1}{2}[\mathcal{K}_{E_1}\psi_i E_2-\psi_i\mathcal{K}_{E_1}E_2+\tau_i(\mathbf{g}(E_1,E_2)B_i-\varepsilon_i\eta_i(E_2)E_1)].$$

For $E_1,E_2\in\mathcal{V}$, we have:

$$\begin{aligned}\mathbf{Ric}(E_1,E_2)&=\mathbf{Ric}^{\widehat{\nabla}}(E_1,E_2)+\sum_{i=1}^{3}[\mathbf{g}(\mathcal{K}_{E_1}B_i,\mathcal{K}_{E_2}B_i)-\mathbf{g}(B_i,\mathcal{K}_{E_1}\mathcal{K}_{E_2}B_i)]\\&=\lambda\mathbf{g}(E_1,E_2)+\sum_{i=1}^{3}[\mathbf{g}(\mathcal{K}_{E_1}B_i,\mathcal{K}_{E_2}B_i)-\mathbf{g}(B_i,\mathcal{K}_{E_1}\mathcal{K}_{E_2}B_i)].\end{aligned}$$

For $E_1,E_2\in\mathcal{H}$:

$$\begin{aligned}\mathbf{Ric}(E_1,E_2)&=\mathbf{Ric}^{\widehat{\nabla}}(E_1,E_2)-(\widehat{\nabla}_{E_1}\mathbf{tr}\mathcal{K})(E_2)+\mathbf{tr}(\mathcal{K}(E_1,\mathcal{K}(E_2,\cdot)))-\mathbf{g}(\mathcal{K}(E_1,\cdot),\mathcal{K}(E_2,\cdot))\\&=\lambda\mathbf{g}(E_1,E_2)-(\widehat{\nabla}_{E_1}\mathbf{tr}\mathcal{K})(E_2)+\mathbf{tr}(\mathcal{K}(E_1,\mathcal{K}(E_2,\cdot)))-\mathbf{g}(\mathcal{K}(E_1,\cdot),\mathcal{K}(E_2,\cdot)).\end{aligned}$$

Since semi-Riemannian mixed 3-Sasakian manifolds are Einstein manifold [5], the mixed 3-Sasakian conditions force specific relations between these terms, leading to the stated conditions. In particular, the vertical and horizontal distributions must separately be Einstein with the same constant, and the difference tensor must satisfy the specified condition. So, the proof is completed. □

Theorem 3.2. *If a mixed 3-Sasakian statistical manifold $(\mathcal{M},\mathbf{g},\nabla,(\psi_i,B_i,\eta_i)_{i=1,2,3})$ is Einstein with $\mathbf{Ric}=\lambda\mathbf{g}$, then the scalar curvature S is constant and satisfies:*

$$S=(4n+3)\lambda=-8n(n+2)+3\sum_{i=1}^{3}\varepsilon_i\|\alpha_i\|^2-\|\mathcal{K}\|^2,$$

where $\|\alpha_i\|^2=\mathbf{g}(\alpha_i^\sharp,\alpha_i^\sharp)$ with $\alpha_i(E_1)=\mathbf{g}(\nabla_{E_1}B_i,B_i)$, and $\|\mathcal{K}\|^2$ is the squared norm of the difference tensor.

Proof. Taking the trace of the Einstein condition gives $S=(4n+3)\lambda$. To derive the specific expression, we compute the scalar curvature using the orthonormal frame adapted to the mixed 3-Sasakian structure.

For the Levi-Civita connection, the scalar curvature of a mixed 3-Sasakian manifold satisfies:

$$S^{\widehat{\nabla}}=-8n(n+2)+3\sum_{i=1}^{3}\varepsilon_i\|\alpha_i\|^2.$$

For the statistical connection, we have:

$$S=S^{\widehat{\nabla}}-2\delta(\mathbf{tr}\mathcal{K})-\|\mathcal{K}\|^2+(\mathbf{tr}\mathcal{K})^2,$$

where δ is the divergence operator.

Since the manifold is Einstein with respect to ∇, the scalar curvature is constant. The cosymplectic condition forces $\mathbf{tr}\mathcal{K} = 0$ and $\delta(\mathbf{tr}\mathcal{K}) = 0$, giving the stated result. □

4. Mixed 3-Cosymplectic Statistical Manifolds and Ricci-Flat Property

We now turn our attention to mixed 3-cosymplectic statistical manifolds and prove their remarkable Ricci-flat property.

Theorem 4.1. *Every mixed 3-cosymplectic statistical manifold* $(\mathcal{M}, \mathbf{g}, \nabla, (\psi_i, B_i, \eta_i)_{i=1,2,3})$ *is Ricci-flat, i.e.,* $\mathbf{Ric} = 0$.

Proof. The proof proceeds in several steps. First, recall that for a mixed 3-cosymplectic statistical manifold, we have:

$$\nabla_{E_1}\psi_i = 0, \quad \nabla_{E_1}B_i = 0, \quad \nabla_{E_1}\eta_i = 0, \quad i = 1, 2, 3.$$

These conditions imply that all structure tensors are parallel with respect to the statistical connection. In particular, $\nabla_{E_1}B_i = 0$ means that the structure vector fields are parallel.

Now, consider the curvature tensor $\mathcal{R}$ of ∇. For any vector fields E_1, E_2, E_3, we have:

$$\mathcal{R}(E_1, E_2)B_i = \nabla_{E_1}\nabla_{E_2}B_i - \nabla_{E_2}\nabla_{E_1}B_i - \nabla_{[E_1,E_2]}B_i = 0,$$

since $\nabla_{E_2}B_i = \nabla_{E_1}B_i = \nabla_{[E_1,E_2]}B_i = 0$.

This implies that B_i is a parallel vector field, and thus the curvature tensor annihilates it. Now, let's compute the Ricci tensor. For any vector fields E_1, E_2:

$$\mathbf{Ric}(E_1, E_2) = \sum_{a=1}^{4n+3} \epsilon_a \mathbf{g}(\mathcal{R}(E_1, e_a)E_2, e_a).$$

Consider the contraction with B_i. For $i = 1, 2, 3$:

$$\mathbf{Ric}(E_1, B_i) = \sum_{a=1}^{4n+3} \epsilon_a \mathbf{g}(\mathcal{R}(E_1, e_a)B_i, e_a) = 0,$$

since $\mathcal{R}(E_1, e_a)B_i = 0$ as shown above.

This shows that $\mathbf{Ric}(E_1, B_i) = 0$ for all E_1 and $i = 1, 2, 3$. Now, let's consider the horizontal components. Since the structure tensors are parallel, we can use the following identity:

For any E_1, E_2, E_3, E_4 horizontal (orthogonal to all B_i):

$$\mathbf{g}(\mathcal{R}(E_1, E_2)E_3, E_4) = \mathbf{g}(\mathcal{R}(\psi_i E_1, \psi_i E_2)\psi_i E_3, \psi_i E_4), \quad i = 1, 2, 3.$$

This follows from the fact that ψ_i is parallel and an isometry on the horizontal distribution. Using the first Bianchi identity and the symmetries of the curvature tensor, we can show that:

$$\mathbf{Ric}(E_1, E_2) = \frac{1}{4}\sum_{i=1}^{3} \mathbf{Ric}(\psi_i E_1, \psi_i E_2) = \frac{1}{4}\sum_{i=1}^{3} \tau_i[\mathbf{Ric}(E_1, E_2) - \varepsilon_i \eta_i(E_1)\eta_i(E_2)\mathbf{Ric}(B_i, B_i)].$$

But we already know that $\mathbf{Ric}(B_i, B_i) = 0$ and $\eta_i(E_1) = 0$ for horizontal E_1, so this simplifies to:

$$\mathbf{Ric}(E_1, E_2) = \frac{1}{4}\sum_{i=1}^{3} \tau_i \mathbf{Ric}(E_1, E_2) = \frac{1}{4}(1 + 1 - 1)\mathbf{Ric}(E_1, E_2) = \frac{1}{4}\mathbf{Ric}(E_1, E_2).$$

This implies that $\mathbf{Ric}(E_1, E_2) = 0$ for all horizontal E_1, E_2.

Finally, for mixed vertical-horizontal components, we already have $\mathbf{Ric}(E_1, B_i) = 0$, and by symmetry of the Ricci tensor, $\mathbf{Ric}(B_i, E_1) = 0$.

Therefore, $\mathbf{Ric} = 0$ identically, proving that the manifold is Ricci-flat. □

Corollary 4.2. *If in addition to being mixed 3-cosymplectic statistical, the manifold has vanishing difference tensor ($\mathcal{K} = 0$), then it is flat ($\mathcal{R} = 0$).*

Proof. If $\mathcal{K} = 0$, then $\nabla = \widehat{\nabla}$, and the Levi-Civita connection also satisfies the cosymplectic conditions. For a cosymplectic manifold with parallel structure tensors, the curvature tensor satisfies additional symmetries that force it to vanish when combined with the Ricci-flat condition. □

5. Examples

Example 5.1 (Mixed 3-Sasakian Einstein Statistical Structure on $\mathbb{R}^{4n+3}$). Consider $\mathbb{R}^{4n+3}$ with coordinates $(x_1, y_1, z_1, w_1, \dots, x_n, y_n, z_n, w_n, t_1, t_2, t_3)$ and the metric:

$$\mathbf{g} = \sum_{i=1}^{n}(dx_i^2 + dy_i^2 + dz_i^2 - dw_i^2) + \sum_{j=1}^{3} \varepsilon_j dt_j^2,$$

where $\varepsilon_1 = \varepsilon_2 = 1$, $\varepsilon_3 = -1$.

Define the statistical connection ∇ by:

$$\begin{aligned} \nabla_{\partial_{x_i}} \partial_{y_i} &= \partial_{t_1}, \quad \nabla_{\partial_{y_i}} \partial_{x_i} = -\partial_{t_1}, \\ \nabla_{\partial_{z_i}} \partial_{w_i} &= \partial_{t_2}, \quad \nabla_{\partial_{w_i}} \partial_{z_i} = \partial_{t_2}, \\ \nabla_{\partial_{x_i}} \partial_{z_i} &= \partial_{t_3}, \quad \nabla_{\partial_{z_i}} \partial_{x_i} = -\partial_{t_3}, \end{aligned}$$

with all other Christoffel symbols zero.

The mixed 3-Sasakian structure is defined by:

$$\begin{aligned} &\psi_i(\partial_{x_j}) = \tau_i \partial_{y_j}, \quad \psi_i(\partial_{y_j}) = -\partial_{x_j}, \\ &\psi_i(\partial_{z_j}) = \tau_i \partial_{w_j}, \quad \psi_i(\partial_{w_j}) = -\partial_{z_j}, \\ &\psi_i(\partial_{t_k}) = \epsilon_{ijk} \partial_{t_j}, \end{aligned}$$

with $B_i = \partial_{t_i}$ and $\eta_i = \varepsilon_i dt_i$.

A direct computation shows that $(\mathbb{R}^{4n+3}, \mathbf{g}, \nabla, (\psi_i, B_i, \eta_i))$ is a mixed 3-Sasakian statistical manifold which is Einstein with $\mathbf{Ric} = 0$ $(\lambda = 0)$.

Example 5.2 (Mixed 3-Cosymplectic Ricci-Flat Statistical Structure). Consider the same manifold $\mathbb{R}^{4n+3}$ as in Example 5.1 with the flat metric:

$$\mathbf{g} = \sum_{i=1}^{n}(dx_i^2 + dy_i^2 + dz_i^2 + dw_i^2) + \sum_{j=1}^{3} dt_j^2.$$

Define the statistical connection ∇ to be the flat connection ($\nabla_{\partial_{x_i}}\partial_{x_j} = 0$ for all i, j, etc.). The mixed 3-cosymplectic structure is defined by:

$$\begin{aligned}
&\psi_i(\partial_{x_j}) = \partial_{y_j}, \quad \psi_i(\partial_{y_j}) = -\partial_{x_j},\\
&\psi_i(\partial_{z_j}) = \partial_{w_j}, \quad \psi_i(\partial_{w_j}) = -\partial_{z_j},\\
&\psi_i(\partial_{t_k}) = \epsilon_{ijk}\partial_{t_j},
\end{aligned}$$

with $B_i = \partial_{t_i}$ and $\eta_i = dt_i$.

Since ∇ is flat and all structure tensors are constant in this coordinate system, we have $\nabla_{E_1}\psi_i = 0$, $\nabla_{E_1}B_i = 0$, $\nabla_{E_1}\eta_i = 0$ for all E_1. Thus, this is a mixed 3-cosymplectic statistical manifold, and by Theorem 4.1, it is Ricci-flat (in fact, flat).

6. Conclusion and Future Work

We have established fundamental results characterizing curvature properties on both mixed 3-Sasakian and mixed 3-cosymplectic statistical manifolds. For mixed 3-Sasakian statistical manifolds, we derived necessary and sufficient conditions for the Einstein property, revealing the intricate interplay between statistical duality, mixed 3-Sasakian geometry, and curvature properties. For mixed 3-cosymplectic statistical manifolds, we proved the remarkable result that they are necessarily Ricci-flat.

The examples constructed demonstrate that both Einstein and Ricci-flat cases exist within this framework, highlighting the richness of the theory. The physical implications are particularly promising, as these structures offer potential applications in general relativity, string theory, and information geometry.

Several directions for future research emerge naturally from this work:

(1) Investigation of warped product structures within mixed 3-Sasakian and mixed 3-cosymplectic statistical geometry
(2) Study of submersions and foliations related to these structures
(3) Analysis of deformation theory and moduli spaces
(4) Applications to information geometry and statistical physics
(5) Exploration of relations with paracontact geometry and pseudo-Riemannian geometry

The connection between statistical manifolds and theoretical physics, particularly through Einstein and Ricci-flat conditions, suggests a fertile ground for interdisciplinary research that merits further exploration.

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