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Research Paper

ON THE COHERENT CONFIGURATIONS OF SOME TREES

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ABSTRACT

This paper investigates the relationship between coherent configurations of graphs and those of their automorphism groups, with a special focus on tree structures. For Schurian coherent configurations—those derived from permutation group orbitals—we establish an isomorphism between the coherent configuration of a graph and that of its automorphism group. We systematically classify the coherent configurations for three fundamental classes of trees: paths, star graphs, and spoke graphs, along with their generalized spoke variants. Through direct product and direct sum operations, we decompose these configurations into combinations of trivial and discrete coherent configurations. A detailed non-trivial example illustrates the coherent configuration of a spoke graph. This work advances the combinatorial analysis of tree symmetries and provides tools for broader graph family classifications, contributing to frameworks for studying graph isomorphism problems and quantum automorphism groups.

1. Introduction

The theory of coherent configurations, a powerful combinatorial tool for studying symmetry and regularity in graphs, was pioneered by Boris Weisfeiler and Evgeny Leman in their seminal work on the graph isomorphism problem [8]. Their approach, rooted in algebraic

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graph theory, sought to capture the essence of graph structure through matrix algebras derived from adjacency relations. Specifically, for a finite, undirected, simple graph Λ, the coherent configuration $\mathcal{C}(\Lambda)$ is defined as the smallest matrix algebra containing its adjacency matrix $A(\Lambda)$. This configuration, also viewed as the minimal coherent structure on the vertex set of Λ, is such that certain basis relations collectively represent the edge set of the graph [3].

The impetus behind this development lies in the inherent difficulty of determining whether two graphs are isomorphic—a problem of both theoretical and practical significance. By associating a coherent configuration with each graph, Weisfeiler and Leman provided a means to reduce the isomorphism problem to a comparison of algebraic structures. This approach has since found applications in diverse areas, including computational group theory, design theory, and network analysis.

The concept of coherent configurations generalizes several related structures. They can be seen as a natural extension of association schemes, which have long been used to study highly symmetric combinatorial objects. Additionally, coherent configurations provide a combinatorial framework for analyzing the orbital configurations of permutation groups, further highlighting their connection to group-theoretic methods in graph theory [3, 5].

Now, we prepare some definitions, notations and results for that will be used throughout the paper. We refer the reader to [3, 6, 9] for more details.

Given a finite set V, let $1_V = \{(u,u) \mid u \in V\}$ denote the diagonal relation. For $t \subset V^2$, define $t^* = \{(v,u) \in V^2 : (u,v) \in t\}$, and for $u \in V$, set $ut = \{v \in V : (u,v) \in t\}$ and $n_t(u) = |ut|$. For a partition T of V^2, $T^\cup$ denotes all unions of elements of T, and $T^* = \{t^* : t \in T\}$.

Definition 1.1. Let V be a finite set and T a partition of V^2. The pair $\mathcal{C} = (V,T)$ is called a *coherent configuration* if $1_V \in T^\cup$, $T^* = T$, and for any $r,s,t \in T$, the number $c_{rs}^t := |ur \cap vs^*|$ is independent of the choice of $(u,v) \in t$.

In a coherent configuration $\mathcal{C} = (V,T)$, $|V|$ is called the *degree* and $|T|$ the *rank*. A subset $\Delta \subseteq V$ is called a *fiber* if $1_\Delta \in T$. The set of all fibers is denoted by $\mathrm{Fib}(\mathcal{C})$. If $1_V \in T$, the configuration is called *homogeneous*. Each pair $(u,v) \in V^2$ belongs to a unique basis relation $t(u,v) \in T$. The numbers c_{rs}^t are the *intersection numbers*, T is the set of *basis relations*, and $T^\cup$ is the set of all *relations*.

Let Δ be a subset of the set V and $\mathcal{C} = (V,T)$ be a coherent configuration. Then, If $1_\Delta \in T$, the set Δ is called a *fiber* of $\mathcal{C}$. We denote the set of all fibers of the coherent configuration $\mathcal{C}$ by $\mathrm{Fib}(\mathcal{C})$.

A rank 2 homogeneous configuration on n points is the *trivial coherent configuration* $\mathcal{T}_n$, and the unique coherent configuration on one point is $\mathcal{T}_1$. The *discrete coherent configuration* $\mathcal{D}_n$ on n points has rank n^2 and consists of all singleton relations.

Two coherent configurations $\mathcal{C} = (V,T)$ and $\mathcal{C}' = (V',T')$ are *isomorphic* if there exists a bijection between V and V' that induces a bijection between T and T'. The group of automorphisms $\mathrm{Aut}(\mathcal{C})$ consists of all permutations $g \in \mathrm{Sym}(V)$ such that $t^g = t$ for all $t \in T$.

The *direct sum* $\mathcal{C} \boxplus \mathcal{S}$ and the *direct product* $\mathcal{C} \otimes \mathcal{S}$ of coherent configurations are defined analogously, allowing the construction of new configurations from existing ones [2]. For the coherent configurations $\mathcal{C} = (V, T)$ and $\mathcal{S} = (U, S)$, the direct sum of $\mathcal{C}$ and $\mathcal{S}$, denoted by $\mathcal{C} \boxplus \mathcal{S}$, is defined on the set $V \cup U$ as

$$\mathcal{C} \boxplus \mathcal{S} = (V \cup U, T \boxplus S),$$

where,

$$T \boxplus S = T \cup S \ \cup R_{T,S} \cup R^*_{T,S}$$

and

$$R_{T,S} = \{\Delta_1 \times \Delta_2 \ : \ (\Delta_1, \Delta_2) \in \mathrm{Fib}(\mathcal{C}) \times \mathrm{Fib}(\mathcal{S})\}.$$

The direct product, $\mathcal{C} \otimes \mathcal{S}$ of $\mathcal{C} = (V, T)$ and $\mathcal{S} = (U, S)$ is a coherent configuration on $V \times U$ as

$$\mathcal{C} \otimes \mathcal{S} = (V \times U, T \otimes S),$$

where,

$$T \otimes S = \{t \otimes s \ : \ t \in T, s \in S\}$$

and

$$t \otimes s = \{((a, c), (b, d)) \in (V \times U)^2 \ : \ (a, b) \in t, (c, d) \in s\}.$$

For the coherent configurations $\mathcal{C} = (V, T)$, let $M(\mathcal{C})$, be the matrix algebra defined by $\{A_t : t \in T\}$, where A_t is an $|V| \times |V|$, $\{0, 1\}$-matrix which is correspondence to the relation t. Define the relational matrix of the coherent configurations $\mathcal{C}$ as

$$A(\mathcal{C}) = \sum_{t \in T} a_t A_t.$$

Given a finite permutation group $G \leq \mathrm{Sym}(V)$, the configuration $\mathrm{Inv}(G) := (V, \mathrm{Orb}(G, V^2))$ is a coherent configuration, where $\mathrm{Orb}(G, V^2)$ is the set of orbits of G acting on V^2 [9]. A configuration $\mathcal{C}$ is called *Schurian* if

$$\mathcal{C} = \mathrm{Inv}(\mathrm{Aut}(\mathcal{C})). \tag{1.1}$$

Proposition 1.2 ([1]). *Let G be a permutation group on the set V and $\mathcal{C} = \mathrm{Inv}(G)$. Then,* $\mathrm{Fib}(\mathcal{C}) \ = \mathrm{Orb}(G, V)$.

Characterizing Schurian coherent configurations is a fundamental problem in the theory. For example, 1-regular coherent configurations and coherent configurations of algebraic forests are known to be Schurian.

For a graph $\Lambda = (V, R)$, the coherent configuration $\mathcal{C}(\Lambda)$ is the intersection of all configurations $\mathcal{C}$ on V such that R is a union of basis relations:

$$\mathcal{C}(\Lambda) = \bigcap_{\mathcal{C} \in \mathcal{S}(V,R)} \mathcal{C},$$

where

$$\mathcal{S}(V, R) = \{\mathcal{C} \leq \mathcal{D}_{|V|} : R \subseteq T(\mathcal{C})^{\cup}\}.$$

Let $\Lambda = (V, R)$ be a graph, where V is the vertex set of Λ and R is the edge set of Λ. The coherent configuration of Λ has been defined in [8] as follow.

Finding the coherent configuration of a graph is generally challenging. Some known results are as follows [1, 4, 5, 8]: if $\mathcal{C}(\Lambda)$ is homogeneous, then Λ is regular; for vertex-transitive graphs, the configuration is homogeneous. For the cycle C_n, $\mathcal{C}(C_n)$ is Schurian and isomorphic to $\mathrm{Inv}(D_{2n})$, where D_{2n} is the dihedral group. For a strongly regular graph, $\mathcal{C}(\Lambda)$ is a rank 3 homogeneous configuration. The complete graph K_n yields $\mathcal{C}(K_n) \cong \mathcal{T}_n$.

Recent work [4] has shown that the coherent configurations of cographs, trees, interval graphs, and rooted directed path graphs are Schurian. The configurations of certain circular-arc graphs have also been studied [7].

Proposition 1.3 ([4])**.** *Let Γ be a tree. Then $\mathcal{C}(\Gamma)$ is a Schurian coherent configuration.*

While Proposition 1.3 establishes the Schurian nature of tree configurations, their explicit structural decomposition remains a compelling area of investigation. Trees are a fundamental class of graphs whose study is particularly worthwhile due to their unique position in graph theory; they often possess rich automorphism groups that yield non-trivial yet tractable coherent configurations. Analyzing these configurations provides deep insights into the interplay between graph symmetry and algebraic combinatorics. Furthermore, understanding the coherent configurations of trees serves as a critical stepping stone for developing classification frameworks for broader families of graphs, such as graphs with tree-like decompositions, thereby contributing to advanced problems in graph isomorphism and quantum automorphism groups.

The aim of this paper is to study the coherent configurations of several classes of trees, including paths, star graphs, and spoke graphs. Specifically, we provide explicit descriptions and classifications for these families, highlighting their structural properties and the interplay between graph structure and coherent configuration.

The path P_n is the graph with n vertices $v_1, v_2, \ldots, v_n$ and edges $\{v_i, v_{i+1}\}$ for $i = 1, \ldots, n-1$. The star graph S_n has one central vertex of degree $n-1$ and $n-1$ leaves of degree 1. The (n,k)-spoke graph $S_{(n,k)}$ generalizes the star by placing k vertices along each of the $n-1$ arms, all of degree 2.

2. Main Results

This section presents our main findings on the coherent configurations of paths, star graphs, and spoke graphs. We provide explicit classifications for each family, detailing their structural properties and the relationship between graph structure and coherent configuration.

We begin by restating key propositions that form the foundation of our analysis.

Proposition 2.1 ([1])**.** *Let Λ be a graph. Then,* $\mathrm{Aut}(\Lambda) = \mathrm{Aut}(\mathcal{C}(\Lambda))$.

Proposition 2.2 ([1])**.** *Let Λ be a graph. Then* $\mathcal{C}(\Lambda) \leq \mathrm{Inv}(\mathrm{Aut}(\Lambda))$.

Now, we explain the relation between the coherent configuration of a graph which is Schurian and the coherent configuration of its automorphism group.

Theorem 2.3. *Let Λ be a graph, such that $\mathcal{C}(\Lambda)$ is a Schurian coherent configuration. Then,* $\mathcal{C}(\Lambda) = \mathrm{Inv}(\mathrm{Aut}(\Lambda))$.

Proof. By Proposition 2.1, we have

$$\mathrm{Inv}(\mathrm{Aut}(\Lambda)) = \mathrm{Inv}(\mathrm{Aut}(\mathcal{C}(\Lambda))). \tag{2.1}$$

Now, by definition of Schurian coherent configuration,

$$\mathcal{C}(\Lambda) = \mathrm{Inv}(\mathrm{Aut}(\mathcal{C}(\Lambda))). \tag{2.2}$$

So, by equations 2.1 and 2.2, we have

$$\mathrm{Inv}(\mathrm{Aut}(\Lambda)) = \mathrm{Inv}(\mathrm{Aut}(\mathcal{C}(\Lambda))) = \mathcal{C}(\Lambda),$$

and we are done. □

2.1. **Paths.** In the following theorem we classify the coherent configuration of a path.

Theorem 2.4. *Let P_n be the path graph on n vertices. Then its coherent configuration $\mathcal{C}(P_n)$ is isomorphic to:*

$$\mathcal{C}(P_n) \simeq \begin{cases} \mathcal{D}_k \otimes \mathcal{T}_2, \ \ for \ n = 2k \\ \\ (\mathcal{D}_k \otimes \mathcal{T}_2) \boxplus \mathcal{T}_1, \ \ for \ n = 2k+1. \end{cases}$$

Proof. Any path is a tree, so by Proposition 1.3, $\mathcal{C}(P_n)$ is a Schurian coherent configuration. Then, by Theorem 2.3, we have, $\mathcal{C}(P_n) = \mathrm{Inv}(\mathrm{Aut}(P_n))$. Thus, it is enough to find the coherent configuration of the automorphism group of P_n. For $n = 2k$, suppose that the vertex sequence of P_n is equal to $v_1, v_2, \ldots, v_k, u_k, u_{k-1}, \ldots, u_1$. Also, for $n = 2k+1$, suppose that the vertex sequence of P_n is equal to $v_1, v_2, \ldots, v_k, w, u_k, u_{k-1}, \ldots, u_1$.

Then, $\mathrm{Aut}(P_n)$ consists of exactly two permutations, the identity and the permutation σ, where

$$\sigma = (v_1, u_1)(v_2, u_2)...(v_k, u_k). \tag{2.3}$$

From Proposition 1.2, we have $\mathrm{Fib}(\mathcal{C}(P_n)) = \mathrm{Orb}(\mathrm{Aut}(P_n), V)$, where V is the vertex set of P_n. So,

$$\mathrm{Fib}(\mathcal{C}(P_n)) = \begin{cases} \{\{v_i, u_i\} \mid 1 \leq i \leq k\}, for \ n = 2k, \\ \\ \{\{v_i, u_i\} \mid 1 \leq i \leq k\} \cup \{\{w\}\}, \ \ for \ n = 2k+1. \end{cases} \tag{2.4}$$

Also, from definition of Schurian coherent configuration, the set $\mathrm{Orb}_2(\mathrm{Aut}(P_n), V)$ is equal to the set of basis relations of $\mathrm{Inv}(\mathrm{Aut}(P_n))$. So, by considering the automorphism group of P_n and the permutation σ in equation 2.3, for $n = 2k$ we have

$$\mathrm{Orb}_2(\mathrm{Aut}(P_n), V) =$$

$$\{\{(v_i, v_j), (u_i, u_j)\} \mid 1 \leq i, j \leq k\} \cup \{\{(v_i, u_j), (u_i, v_j)\} \mid 1 \leq i, j \leq k\}. \tag{2.5}$$

Also, for $n = 2k+1$,

$$\mathrm{Orb}_2(\mathrm{Aut}(P_n)) =$$

$$\begin{aligned} &\{\{(v_i, v_j), (u_i, u_j)\} \mid 1 \leq i, j \leq k\} \cup \{\{(v_i, u_j), (u_i, v_j)\} \mid 1 \leq i, j \leq k\} \cup \\ &\{\{(w, v_i), (w, u_i)\} \mid 1 \leq i \leq k\} \cup \{\{(v_i, w), (u_i, w)\} \mid 1 \leq i \leq k\} \cup \{(w, w)\}. \end{aligned} \tag{2.6}$$

Now, we use the fiber of the coherent configuration $\mathrm{Inv}(\mathrm{Aut}(P_n))$ in equation 2.4 for reordering the set V. In the next step, using 2-orbits of the group $\mathrm{Aut}(P_n)$ in equation 2.5, for $n = 2k$ the relational matrix of the coherent configuration $\mathrm{Inv}(\mathrm{Aut}(P_n))$ is isomorphic to Figure 1.

$$
\begin{array}{cc}
 & \begin{matrix} v_1 & u_1 & v_2 & u_2 & v_3 & u_3 & \cdots & v_k & u_k \end{matrix} \\
\begin{matrix} v_1 \\ u_1 \\ v_2 \\ u_2 \\ v_3 \\ u_3 \\ \vdots \\ v_k \\ u_k \end{matrix} &
\begin{bmatrix}
a_{11} & b_{11} & a_{12} & b_{12} & a_{13} & b_{13} & & a_{1k} & b_{1k} \\
b_{11} & a_{11} & b_{12} & a_{12} & b_{13} & a_{13} & \cdots & b_{1k} & a_{1k} \\
a_{21} & b_{21} & a_{22} & b_{22} & a_{23} & b_{23} & & a_{2k} & b_{2k} \\
b_{21} & a_{21} & b_{22} & a_{22} & b_{23} & a_{23} & \cdots & b_{2k} & a_{2k} \\
a_{31} & b_{31} & a_{32} & b_{32} & a_{33} & b_{33} & & a_{3k} & b_{3k} \\
b_{31} & a_{31} & b_{32} & a_{32} & b_{33} & a_{33} & \cdots & b_{3k} & a_{3k} \\
\vdots & & \vdots & & \vdots & & \ddots & \vdots & \\
a_{k1} & b_{k1} & a_{k2} & b_{k2} & a_{k3} & b_{k3} & & a_{kk} & b_{kk} \\
b_{k1} & a_{k1} & b_{k2} & a_{k2} & b_{k3} & a_{k3} & \cdots & b_{kk} & a_{kk}
\end{bmatrix}
\end{array}
$$

FIGURE 1. The relational matrix of $\mathrm{Inv}(\mathrm{Aut}(P_n))$, for $n = 2k$

Moreover, by equation 2.6, for $n = 2k+1$ the relational matrix of the coherent configuration $\mathrm{Inv}(\mathrm{Aut}(P_n))$ is isomorphic to Figure 2.

$$
\begin{array}{cc}
 & \begin{matrix} v_1 & u_1 & v_2 & u_2 & v_3 & u_3 & \cdots & v_k & u_k & w \end{matrix} \\
\begin{matrix} v_1 \\ u_1 \\ v_2 \\ u_2 \\ v_3 \\ u_3 \\ \vdots \\ v_k \\ u_k \\ w \end{matrix} &
\begin{bmatrix}
a_{11} & b_{11} & a_{12} & b_{12} & a_{13} & b_{13} & & a_{1k} & b_{1k} & d_1 \\
b_{11} & a_{11} & b_{12} & a_{12} & b_{13} & a_{13} & \cdots & b_{1k} & a_{1k} & d_1 \\
a_{21} & b_{21} & a_{22} & b_{22} & a_{23} & b_{23} & & a_{2k} & b_{2k} & d_2 \\
b_{21} & a_{21} & b_{22} & a_{22} & b_{23} & a_{23} & \cdots & b_{2k} & a_{2k} & d_2 \\
a_{31} & b_{31} & a_{32} & b_{32} & a_{33} & b_{33} & & a_{3k} & b_{3k} & d_3 \\
b_{31} & a_{31} & b_{32} & a_{32} & b_{33} & a_{33} & \cdots & b_{3k} & a_{3k} & d_3 \\
\vdots & & \vdots & & \vdots & & \ddots & \vdots & & \vdots \\
a_{k1} & b_{k1} & a_{k2} & b_{k2} & a_{k3} & b_{k3} & & a_{kk} & b_{kk} & d_k \\
b_{k1} & a_{k1} & b_{k2} & a_{k2} & b_{k3} & a_{k3} & \cdots & b_{kk} & a_{kk} & d_k \\
c_1 & c_1 & c_2 & c_2 & c_3 & c_3 & \cdots & c_k & c_k & e
\end{bmatrix}
\end{array}
$$

FIGURE 2. The relational matrix of $\mathrm{Inv}(\mathrm{Aut}(P_n))$, for $n = 2k+1$

Subsequently, using definition of direct sum and tensor product of coherent configurations, it is straight to check that, for $n = 2k$, $\mathcal{C}(P_n)$ is isomorphic to $\mathcal{D}_k \otimes \mathcal{T}_2$, and for $n = 2k+1$, $\mathcal{C}(P_n)$ is isomorphic to $(\mathcal{D}_k \otimes \mathcal{T}_2) \boxplus \mathcal{T}_1$, and we are done. $\square$

2.2. **Spoke Graphs.** In the following theorem we classify the coherent configuration of an (n,k)-spoke graph. We recall that the (n,k)-*spoke graph*, namely $S_{(n,k)}$, is a generalization of the star graph in which k vertices are placed along each of the n arms of the star graph, and all of this points are of degree 2. For example, see the $(4,3)$-spoke graph in Figure 5.

Theorem 2.5. *Let $S_{(n,k)}$ be an (n,k)-spoke graph. Then $\mathcal{C}(S_{(n,k)})$ is isomorphic to $(\mathcal{D}_k \otimes \mathcal{T}_n) \boxplus \mathcal{T}_1$.*

Proof. Since an (n,k)-spoke graph is a tree, by Proposition 1.3, $\mathcal{C}(S_{(n,k)})$ is a Schurian coherent configuration. So, by Theorem 2.3, we have,

$$\mathcal{C}(S_{(n,k)}) = \mathrm{Inv}(\mathrm{Aut}(S_{(n,k)})).$$

Thus, first we investigate the automorphism group of the graph $S_{(n,k)}$. Let the vertex sequence of arms of $S_{(n,k)}$ be as follow:

$$\begin{cases} v_{11}, v_{12}, \dots, v_{1k}, v_0, \\ v_{21}, v_{22}, \dots, v_{2k}, v_0, \\ \vdots \\ v_{n1}, v_{n2}, \dots, v_{nk}, v_0. \end{cases}$$

For $2 \leq t \leq n$, $1 \leq i_j \leq n$ and $i_j \neq i_{j'}$ define a permutation $\sigma_{i_1 i_2 \dots i_t} \in \mathrm{Aut}(S_{(n,k)})$, as follow:

$$\sigma_{i_1 i_2 \dots i_t} = (v_{i_1 1}, v_{i_2 1}, \dots, v_{i_n 1})(v_{i_1 2}, v_{i_2 2}, \dots, v_{i_n 2}) \dots (v_{i_1 k}, v_{i_2 k}, \dots, v_{i_t k}).$$

Now, each σ in $\mathrm{Aut}(S_{(n,k)})$ is equal to product of some permutations like $\sigma_{i_1 i_2 \dots i_t}$ such that their indices do not have any intersection.

From Proposition 1.2, we have $\mathrm{Fib}(\mathcal{C}(S_{(n,k)})) = \mathrm{Orb}(\mathrm{Aut}(S_{(n,k)}), V)$, where V is the vertex set of the graph $S_{(n,k)}$. Now, using the permutation

$$\sigma_{12\dots n} = (v_{11}, v_{21}, \dots, v_{n1})(v_{12}, v_{22}, \dots, v_{n2}) \dots (v_{1k}, v_{2k}, \dots, v_{nk})$$

in $\mathrm{Aut}(S_{(n,k)})$, and since any other element of $\mathrm{Aut}(S_{(n,k)})$ are equal to the product of some permutations like $\sigma_{i_1 i_2 \dots i_t}$, hence,

$$\mathrm{Fib}(\mathcal{C}(S_{(n,k)})) = \{\{v_{1j}, v_{2j}, \dots, v_{nj}\} \mid 1 \leq j \leq k\} \cup \{\{v_0\}\}. \tag{2.7}$$

Also from the definition of Schurian coherent configuration, $\mathrm{Orb}_2(\mathrm{Aut}(S_{(n,k)}), V)$ are the basis relations of $\mathrm{Inv}(\mathrm{Aut}(S_{(n,k)}))$. And, using the permutations of $\mathrm{Aut}(S_{(n,k)})$, we observe that

$$\mathrm{Orb}_2(\mathrm{Aut}(S_{(n,k)}), V) =$$
$$\{\{(v_{ij}, v_{ij'}) \mid 1 \leq i \leq n\} \mid 1 \leq j, j' \leq k\} \cup \{\{(v_{ij}, v_{i'j'}) \mid 1 \leq i, i' \leq n,\ i \neq i'\} \mid 1 \leq j, j' \leq k\} \cup$$
$$\{\{(v_{ij}, v_0) \mid 1 \leq i \leq n\} \mid 1 \leq j \leq k\} \cup \{\{(v_0, v_{ij}) \mid 1 \leq i \leq n\} \mid 1 \leq j \leq k\} \cup \{(v_0, v_0)\}.$$

Now, we use the fiber of the coherent configuration $\mathrm{Inv}(\mathrm{Aut}((S_{(n,k)}))$ in equation 2.7 for reordering the set V. In the next step, using 2-orbits of the group $\mathrm{Aut}(S_{(n,k)})$ in equation 2.5, for $n = 2k$ the relational matrix of the coherent configuration $\mathrm{Inv}(\mathrm{Aut}(S_{(n,k)}))$ is isomorphic to Figure 3, where A_{ij} is equal to the matrix in Figure 4.

$$
\begin{array}{cc}
 & \begin{array}{ccccccccccccc} v_{11} & \cdots & v_{n1} & v_{12} & \cdots & v_{n2} & \cdots & v_{1k} & \cdots & v_{nk} & v_0 \end{array} \\
\begin{array}{c} v_{11} \\ \vdots \\ v_{n1} \\ v_{12} \\ \vdots \\ v_{n2} \\ \vdots \\ v_{1k} \\ \vdots \\ v_{nk} \\ v_0 \end{array} &
\begin{bmatrix}
 & & & & d_1 \\
A_{11} & A_{12} & \cdots & A_{1k} & \vdots \\
 & & & & d_1 \\
 & & & & d_2 \\
A_{21} & A_{22} & \cdots & A_{2k} & \vdots \\
 & & & & d_2 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
 & & & & d_k \\
A_{k1} & A_{k2} & \cdots & A_{kk} & \vdots \\
 & & & & d_k \\
c_1 \ \cdots \ c_1 & c_2 \ \cdots \ c_2 & \cdots & c_k \ \cdots \ c_k & e
\end{bmatrix}
\end{array}
$$

FIGURE 3. The relational matrix of $\mathrm{Inv}(\mathrm{Aut}(S_{(n,k)}))$

$$
\begin{array}{cc}
 & \begin{array}{ccccc} v_{1j} & v_{2j} & \cdots & v_{(n-1)j} & v_{nj} \end{array} \\
\begin{array}{c} v_{1i} \\ v_{2i} \\ \vdots \\ v_{(n-1)i} \\ v_{ni} \end{array} &
\begin{bmatrix}
a_{ij} & b_{ij} & \cdots & b_{ij} & b_{ij} \\
b_{ij} & a_{ij} & \cdots & \vdots & \vdots \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
b_{ij} & \cdots & \cdots & a_{ij} & b_{ij} \\
b_{ij} & \cdots & \cdots & b_{ij} & a_{ij}
\end{bmatrix}
\end{array}
$$

FIGURE 4. The relational matrix of A_{ij}

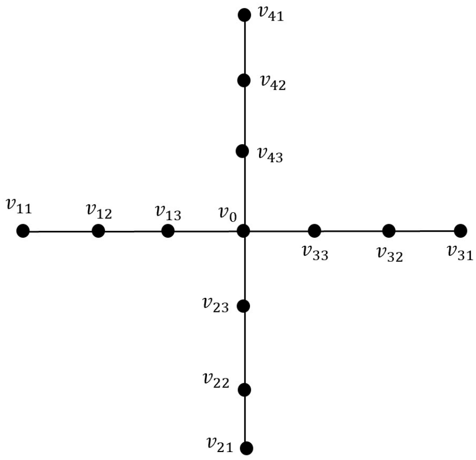
Now, by definition of direct sum and tensor product of coherent configurations, it is straight to check that, $\mathcal{C}(S_{(n,k)})$ is isomorphic to the coherent configuration $(\mathcal{D}_k \otimes \mathcal{T}_n) \boxplus \mathcal{T}_1$, and we are done. $\square$

Example 2.6. Let $S_{(4,3)}$ be a spoke graph. Then $\mathcal{C}(S_{(4,3)})$ is isomorphic to the coherent configuration whose relational matrix has been shown in the Figure 5.

2.3. **Star Graphs.** Since each star graph with $n+1$ vertices is an (n,k)-spoke graph, for $k=1$. Thus, we have the following corollary.

Corollary 2.7. *Let Λ be a star graph with $n+1$ vertices. Then $\mathcal{C}(\Lambda)$ is isomorphic to $\mathcal{T}_n \boxplus \mathcal{T}_1$.*

2.4. **Generalized Spoke Graphs.** Here, we present potential extensions of our results. Consider a generalization of spoke graphs where the lengths of the "spokes" can vary. Let T be a tree with a central vertex v_0 and n_i branches, each of length k_i, where $1 \leq i \leq t$, called $(n_1, ..., n_t, k_1, ..., k_t)$-spoke graph.



$$\begin{bmatrix}
0 & 7 & 7 & 7 & 7 & 8 & 8 & 8 & 8 & 9 & 9 & 9 & 9 \\
7' & 1 & 2 & 2 & 2 & 10 & 11 & 11 & 11 & 14 & 15 & 15 & 15 \\
7' & 2 & 1 & 2 & 2 & 11 & 10 & 11 & 11 & 15 & 14 & 15 & 15 \\
7' & 2 & 2 & 1 & 2 & 11 & 11 & 10 & 11 & 15 & 15 & 14 & 15 \\
7' & 2 & 2 & 2 & 1 & 11 & 11 & 11 & 10 & 15 & 15 & 15 & 14 \\
8' & 16 & 17 & 17 & 17 & 3 & 4 & 4 & 4 & 12 & 13 & 13 & 13 \\
8' & 17 & 16 & 17 & 17 & 4 & 3 & 4 & 4 & 13 & 12 & 13 & 13 \\
8' & 17 & 17 & 16 & 17 & 4 & 4 & 3 & 4 & 13 & 13 & 12 & 13 \\
8' & 17 & 17 & 17 & 16 & 4 & 4 & 4 & 3 & 13 & 13 & 13 & 12 \\
9' & 18 & 19 & 19 & 19 & 20 & 21 & 21 & 21 & 5 & 6 & 6 & 6 \\
9' & 19 & 18 & 19 & 19 & 21 & 20 & 21 & 21 & 6 & 5 & 6 & 6 \\
9' & 19 & 19 & 18 & 19 & 21 & 21 & 20 & 21 & 6 & 6 & 5 & 6 \\
9' & 19 & 18 & 19 & 18 & 21 & 21 & 21 & 20 & 6 & 6 & 6 & 5
\end{bmatrix}$$

FIGURE 5. The $(4, 3)$-spoke graph and its relational matrix

Theorem 2.8. *Let $S_{(n_1,\ldots,n_t,k_1,\ldots,k_t)}$ be an $(n_1, \ldots, n_t, k_1, \ldots, k_t)$-spoke graph. Then*

$$\mathcal{C}(S_{(n_1,\ldots,n_t,k_1,\ldots,k_t)}) \cong \boxplus_{i=1}^{t}(\mathcal{D}_{k_i} \otimes \mathcal{T}_{n_i}) \boxplus \mathcal{T}_1.$$

Proof. The proof is similar to the proof of Theorem 2.5. The central vertex is fixed by the automorphism group. Take branches of equal length together like an (n_i, k_i)-spoke graph. Since the branches are disjoint, their contributions are combined via a direct sum. □

3. CONCLUSIONS

In this paper, we have explored the intricate relationship between the coherent configurations of graphs and those of their automorphism groups, with a particular emphasis on tree structures. Our main result establishes that for Schurian coherent configurations, the coherent configuration of a graph is isomorphic to that of its automorphism group. This foundational insight bridges the combinatorial properties of graphs with group-theoretic symmetries.

We provided a comprehensive classification of the coherent configurations for three important classes of trees: paths, star graphs, and spoke graphs, including their generalized spoke variants. By employing direct product and direct sum operations, we successfully decomposed these configurations into combinations of trivial and discrete coherent configurations, revealing the underlying structural simplicity within seemingly complex symmetries.

The detailed example of a spoke graph demonstrated the practical application of our theoretical framework and highlighted the nuanced behavior of coherent configurations in spoke graphs.

Overall, our results contribute to a deeper understanding of tree symmetries and offer valuable tools for the classification of broader families of graphs. This work also aligns with ongoing efforts in graph isomorphism studies and has potential implications for the analysis of quantum automorphism groups. Future research may extend these methods to more complex graph classes and explore algorithmic applications in symmetry detection and graph classification.

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