DOI: 10.22098/jhs.2026.16382.1062

Journal of Hyperstructures, Article in Press

Journal Homepage: https://jhs.uma.ac.ir/

Research Paper

APPROXIMATION RESULTS FOR GENERALIZED (α, β)-NONEXPANSIVE TYPE 1 MAPPING IN HYPERBOLIC METRIC SPACES

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ARTICLE INFO

Article history:
Received: 16 December 2024
Accepted: 15 June 2026
Communicated by Nasrin Eghbali

Keywords:
generalized (α, β)-nonexpansive type 1 mapping
hyperbolic metric space
Thakur iteration process
convergence theorems
data dependence
weak w^2-stable

MSC:

47H09, 47H10.

ABSTRACT

The aim of this paper is to modify the Thakur iteration process into hyperbolic metric space where the symmetry condition is satisfied and establish the weak w^2-stability and data dependency results for contraction mappings. Next, we establish Δ-convergence and strong convergence theorems for generalized (α, β)-nonexpansive type 1 mappings. We give an example of generalized (α, β)-nonexpansive type 1 mapping which is not nonexpansive mapping.

1. Introduction

In the field of analysis, fixed point theory has always played a significant role. Researchers are constantly motivated to obtain fixed points of various mappings over various spaces

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by the Banach contraction principle, which has become a landmark in fixed point theory. Banach contraction principle, which is the most celebrated result in fixed point theory uses Picard iteration process for approximating the fixed point. The Picard iteration process is useful for the approximation of the fixed point of the contraction mappings but when one is dealing with nonexpansive mappings then it may fail to converge to the fixed point even if the fixed point is unique.

In 1953, Mann[35] introduced new iteration scheme which was powerful iteration method for solving nonlinear operator equations involving nonexpansive mapping. It is known that Mann's iteration method is general not strongly convergent for non-expansive mappings

$$x_{n+1} = (1-\alpha_n)x_n + \alpha_n T x_n, \ n \in N. \tag{1.1}$$

In 1955, Krasnoselskii[13] showed that the Picard iteration scheme for nonexpansive mapping T may fail to converge to fixed point of T even if T has a unique fixed point, but the Mann sequence $\alpha_n = \frac{1}{2}$ for all $n \geq 1$, converges strongly to the fixed point of T.

In 1974, Ishikawa[22] introduced a two step iterative scheme to approximate fixed points of pseudo-contractive mappings,where the sequence $\{x_n\}$ is generated by $x_0 \in C$ as:

$$\begin{cases} y_n = (1-\beta_n)x_n + \beta_n T x_n, \\ x_{n+1} = (1-\alpha_n)x_n + \alpha_n T y_n \end{cases} \tag{1.2}$$

for all $n \geq 0$, where $\alpha_n, \beta_n \in (0,1)$.

In 2000, Noor [16], introduced a three-step iterative scheme for approximating the fixed point problem. Later on, several iterative schemes Agarwal [20], Abbas[15], SP[34], $S^{\star}$[8], CR[19], Normal-S[5], Picard Mann[21], Picard-S[7], etc., have been developed by many authors to study approximating fixed points.

The data-dependence result concerning Mann-Ishikawa iteration [25], where the data dependence of Ishikawa iteration was proved for contractions mappings. Soltuz et al.[24]proved data-dependence results for Ishikawa iteration for the contractive-like operators.

In 2016, Thakur et al.[3] introduced the following iteration scheme in frame work of a Banach space. Let X be a Banach space and C be any nonempty subset of X and for each $x_1 \in C$, the sequence $\{x_n\}$ in C is defined by

$$\begin{cases} z_n = (1-\beta_n)x_n + \beta_n T x_n, \\ y_n = T((1-\alpha_n)x_n + \alpha_n z_n), \\ x_{n+1} = T y_n, \end{cases} \tag{1.3}$$

for all $n \geq 1$, where $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in $(0,1)$. They proved that their iterative scheme is faster than Picard, Mann, Ishikawa, Agrawal, Noor, and Abbas iteration processes. They also proved some weak and strong convergence theorems for suzuki's generalized non-expansive mapping in a uniformly convex Banach space.

In this paper, we study some fixed-point results for generalized (α,β)-nonexpansive type 1 mappings in uniformly convex hyperbolic space and modify the Thakur iteration process into a hyperbolic metric space where the symmetry condition is satisfied and establish the weak w^2-stability and data dependency results for contraction mappings. Next, we establish Δ-convergence and strong convergence theorems for generalized (α,β)-nonexpansive type 1 mappings. We give an example of a generalized (α, β)-nonexpansive type 1 mapping which is not nonexpansive mapping.

2. Main Results

In this section, we recall some definitions and results to be used in establishing the main results.

In 1990, Reich and Shafrir [23] introduced hyperbolic metric spaces and studied an iteration process for nonexpansive mappings in these spaces. In 2004, Kohlenbach [32] introduced a more general hyperbolic metric space as follows;

Definition 2.1. A triple (X, d, W), where (X, d) is a metric space, is called a hyperbolic metric space if the function $W : X \times X \times [0, 1] \to X$ satisfies

(i) $d(u, W(x, y, \alpha)) \leq (1 - \alpha)d(u, y) + \alpha d(u, x)$
(ii) $d(W(x, y, \alpha), W(x, y, \beta)) = |\alpha - \beta|d(x, y)$
(iii) $W(x, y, \alpha) = W(y, x, 1 - \alpha)$
(iv) $d(W(x, z, \alpha), W(y, w, \alpha)) \leq (1 - \alpha)d(z, w) + \alpha d(x, y)$

for all $x, y, z, u \in X$ and $\alpha, \beta \in [0, 1]$.

Definition 2.2. [4] A hyperbolic space X is said to be uniformly convex if for any $r > 0$ and $\epsilon \in (0, 2]$, there exists a $\delta \in (0, 1]$ such that for all $x, y, z \in X$

$$d(W(x, y, \frac{1}{2}), z) \leq (1 - \delta)r$$

provided $d(x, z) \leq r$, $d(y, z) \leq r$ and $d(x, y) \geq \epsilon r$

Definition 2.3. [4] Let X be a hyperbolic space. A map $\eta : (0, \infty) \times (0, 2] \to (0, 1]$ which provides a $\delta = \eta(r, \epsilon)$ for a given $r > 0$ and $\epsilon \in (0, 2]$ is known as a modulus of uniform convexity of X. The mapping η is said to be monotone if it decreases with r.

Definition 2.4. Let $\{x_n\}$ be a bounded sequence in a nonempty subset of C of a metric space (X, d). Then, the mapping $r(\cdot, \{x_n\}) : X \to [0, \infty)$ is defined by

$$r(x, \{x_n\}) = \limsup_{n\to\infty} d(x, x_n), \ \forall x \in X.$$

The infimum of $r(\cdot, \{x_n\})$ over C is called the asymptotic radius of $\{x_n\}$ relative to C and is denoted by $r(C, \{x_\}})$. A point $u \in C$ is said to be an asymptotic center of the sequence $\{x_n\}$ relative to C if

$$r(u, \{x_n\}) = inf\{r(x, \{x_n\}) : x \in C\}$$

and the set of all asymptotic center of $\{x_n\}$ relative to C is denoted by $A(C, \{x_n\})$.

Definition 2.5. Let (X, d) be a metric space and C be a nonempty subset of X. A mapping $T : C \to C$ is said to be a contraction mapping if there exists a constant $\delta \in [0, 1)$ such that

$$d(Tx, Ty) \leq \delta d(x, y)$$

$\forall x, y \in C$.

Definition 2.6. [9] Let (X, d) be a metric space and C be a nonempty subset of X. A mapping $T : C \to C$ is said to be quasi-nonexpansive mapping if

$$d(Tx, e) \leq d(x, e)$$

for all $x \in C$ and $e \in F(T)$, where $F(T)$ is the set of all fixed points of T.

Definition 2.7. [6] Let C be a nonempty subset of a metric space (X, d). A mapping $T : C \to C$ is said to be generalized (α, β)-nonexpansive type 1 if there exist $\alpha, \beta, \lambda \in [0, 1)$, with $\alpha \leq \beta$ and $\alpha + \beta < 1$ such that

$$\lambda d(Tx, x) \leq d(x, y) \Rightarrow d(Tx, Ty) \leq \alpha d(Tx, y) + \beta d(x, Ty) + (1 - (\alpha + \beta))d(x, y)$$

for all $x, y \in C$

Proposition 2.1. [6]*If T is a generalized (α, β)-nonexpansive type 1 mapping and $F(T) \neq \phi$, then T is quasi nonexpansive mapping.*

Proposition 2.2. [6] *If T is a generalized (α, β)-nonexpansive type 1 mapping, then for all $x, y \in C$*

$$d(x, Ty) \leq \frac{2 + \alpha + \beta}{1 - \beta} d(x, Tx) + d(x, y)$$

Proposition 2.3. [6]*If T is a generalized (α, β)-nonexpansive type 1 mapping, then $F(T)$ is closed.*

Definition 2.8. [31] Let (X, d) be a metric space and $\{x_n\}$ and $\{y_n\}$ be two sequences in X. We say that these sequences are equivalent if $\lim_{n\to\infty} d(x_n, y_n) = 0$.

Definition 2.9. [26] Let (X, d) be a metric space, T be a self-mapping on X, and $\{x_n\} \subset X$ be an iterative sequences defined by

$$\begin{cases} x_1 \in X \\ x_{n+1} = f(T, x_n), \forall n \geq 1. \end{cases}$$

where f is a function. Suppose that $\{x_n\}$ converges strongly to $e \in F(T)$. If for any equivalent sequence $\{y_n\} \subset X$ of $\{x_n\}$

$$\lim_{n\to\infty} d(y_{n+1}, f(T, y_n)) = 0 \Rightarrow \lim_{n\to\infty} y_n = e$$

then the iterative process $\{x_n\}$is said to be weak w^2-stable with respect to T.

In 1976, Lim [30] introduced the concept of D-convergence, which is an analog of weak convergence, in metric spaces using the asymptotic center.

Definition 2.10. [30] A sequence $\{x_n\}$ in a metric space (X, d) is said to Δ-converges to a point $x \in X$ if x is the unique asymptotic center of $\{u_n\}$ for every subsequence $\{u_n\}$ of $\{x_n\}$. In this case, we write $\Delta - \lim_{n\to\infty} x_n = x$ and call x as $\Delta - limit$ of $\{x_n\}$.

Lemma 2.3.1. [12] *Let (X, d, W) be a complete uniformly convex hyperbolic metric space with the monotone modulus of uniform convexity η and C be a nonempty closed and convex subset of X. Then every bounded sequence $\{x_n\}$ in X has a unique asymptotic center relative to C.*

Lemma 2.3.2. [1]*Let (X, d, W) be a uniformly convex hyperbolic metric space with the monotone modulus of uniform convexity η. Let $x \in X$ and $\{\alpha_n\}$ be a sequence in $[p, q]$, for* *some* $p, q \in (0,1)$. *If* $\{x_n\}$ *and* $\{y_n\}$ *are sequences in* X *such that*

$$\limsup_{n\to\infty} d(x_n, x) \le r,$$
$$\limsup_{n\to\infty} d(y_n, x) \le r,$$
$$\lim_{n\to\infty} d(W(x_n, y_n, \alpha_n), x) = r$$

for some $r \ge 0$, *then*

$$\lim_{n\to\infty} d(x_n, y_n) = 0$$

Lemma 2.3.3. [24]*Let* $\{g_n\}$, $\{r_n\}$ *and* $\{t_n\}$ *be non-negative real sequences with* $r_n \in (0,1)$, $\forall n \ge 1$ *and* $\sum_{n=1}^{\infty} r_n = \infty$. *Suppose that there exists* $n_0 \in N$ *such that for all* $n \ge n_0$, *one has the inequality*

$$g_{n+1} \le (1 - r_n)g_n + r_n t_n$$

then the following inequality holds:

$$0 \le \limsup_{n\to\infty} g_n \le \limsup_{n\to\infty} t_n$$

3. Main Results

In 2023, Sahin et al.[2] modify the KF-iteration process into hyperbolic metric spaces,

$$\begin{cases} x_1 \in C \\ x_{n+1} = T(W(Tx_n, Ty_n, \alpha_n)) \\ y_n = Tz_n \\ z_n = T(W(x_n, Tx_n, \beta_n)). \end{cases}$$

where $\{\alpha_n\}$, $\{\beta_n\}$ are real sequences in $(0,1)$. They established the weak w^2-stability and data dependence results for contraction mappings. Also we proved some strong and Δ-convergence theorems for generalized (α, β)-nonexpansive type 1 mappings.

Motivated by the above results, we study some fixed points results for generalized (α, β) -nonexpansive type 1 mappings in uniformly convex hyperbolic space and establish strong and Δ-convergence theorems for approximating a fixed point of generalized (α, β) -nonexpansive type 1 mapping in hyperbolic spaces using the iterative scheme introduced by Thakur et al.[3].

First, we modify the Thakur[3] iteration process into a hyperbolic metric space. Let C be a nonempty closed and convex subset of a complete uniformly convex hyperbolic space X and $T : C \to C$ be generalized (α, β)- nonexpansive type 1 mapping. For $x_1 \in C$, we construct the sequence $\{x_n\}$ as follows:

$$\begin{cases} x_1 \in C \\ x_{n+1} = Ty_n \\ y_n = T(W(x_n, z_n, \alpha_n)) \\ z_n = W(x_n, Tx_n, \beta_n). \end{cases} \tag{3.1}$$

where $\{\alpha_n\}$, $\{\beta_n\}$ are real sequences in $(0,1)$.

3.1. **Convergence, Stability and Data Dependency Results.**

Theorem 3.1.1. Let C be a nonempty closed convex subset of a hyperbolic metric space X. $T : C \to C$ be a contraction mapping with the constant $\delta \in [0, 1)$ s.t. $F(T) \neq \phi$ and $\{x_n\}$ be the iterative scheme (3.1) with the real sequence $\{\alpha_n\}$, $\{\beta_n\}$ are in $(0, 1)$ and satisfying $\sum_{n=1}^{\infty} \alpha_n\beta_n = \infty$. Then the sequences $\{x_n\}$ converge strongly to a fixed point of T.

Proof. Let $e \in F(T)$ and using iteration (3.1), we have

$$\begin{aligned} d(z_n, e) &= d(W(x_n, Tx_n, \beta_n), e) \\ &\leq (1 - \beta_n)d(x_n, e) + \beta_n d(Tx_n, e) \\ &\leq (1 - \beta_n)d(x_n, e) + \beta_n \delta d(x_n, e) \\ &= (1 - (1 - \delta)\beta_n)d(x_n, e) \end{aligned} \tag{3.2}$$

Now

$$\begin{aligned} d(y_n, e) &= d(T(W(x_n, z_n, \alpha_n)), Te) \\ &\leq \delta d(W(x_n, z_n, \alpha_n), e) \\ &\leq \delta[(1 - \alpha_n)d(x_n, e) + \alpha_n d(z_n, e)] \end{aligned} \tag{3.3}$$

From equations (3.2) and (3.3), we get

$$\begin{aligned} d(y_n, e) &\leq \delta[(1 - \alpha_n)d(x_n, e) + \alpha_n(1 - (1 - \delta)\beta)d(x_n, e)] \\ &\leq \delta(1 - (1 - \delta)\alpha_n\beta_n)d(x_n, e) \end{aligned} \tag{3.4}$$

Again

$$\begin{aligned} d(x_{n+1}, e) &= d(Ty_n, Te) \\ &\leq \delta d(y_n, e) \end{aligned} \tag{3.5}$$

From equations (3.4) and (3.5), we get

$$\begin{aligned} d(x_{n+1}, e) &\leq \delta^2(1 - (1 - \delta)\alpha_n\beta_n)d(x_n, e) \\ &\leq \delta^{2+2} \prod_{k=n-1}^{n} (1 - (1 - \delta)\alpha_k\beta_k)d(x_{n-1}, e) \\ &\vdots \\ &\leq \delta^{2n} \prod_{k=1}^{n}(1 - (1 - \delta)\alpha_k\beta_k)d(x_1, e) \end{aligned} \tag{3.6}$$

where $\delta \in (0, 1)$ and also $\{\alpha_n\}, \{\beta_n\}$ are in $(0, 1)$. Since $1 - x \leq e^{-x}$, $x \in [0, 1]$, so from equation (3.6), we obtain

$$d(x_{n+1}, e) \leq d(x_1, e)\delta^{2n}e^{-(1-\delta)\sum_{k=1}^{n} \alpha_k\beta_k} \tag{3.7}$$

Taking the limit of both sides of (3.7) and then using the hypothesis $\delta \in (0, 1)$ and $\sum_{n=1}^{\infty} \alpha_n\beta_n = \infty$, we obtain

$$\lim_{n\to\infty} d(x_n, e) = 0.$$

i.e. $x_n \to e$ as $n \to \infty$. □

Theorem 3.1.2. Let C be a nonempty closed convex subset of a hyperbolic metric space X. $T : C \to C$ be a contraction mapping with the constant $\delta \in [0, 1)$ s.t. $F(T) \neq \phi$ and $\{x_n\}$ be the iterative scheme (3.1) with the real sequence $\{\alpha_n\}$, $\{\beta_n\}$ are in $(0,1)$. Then the iteration process (3.1) is weak w^2 stable with respect to T.

Proof. Let $\{p_n\} \subset C$ be an equivalent sequence of $\{x_n\}$ and suppose that $\epsilon_n = d(p_{n+1}, Tq_n)$, where $q_n = T(W(p_n, r_n, \alpha_n))$ and $r_n = W(p_n, Tp_n, \beta_n)$. Suppose that $\lim_{n\to\infty} \epsilon_n 0$. It follows that using triangular inequality and (3.1), we have

$$\begin{aligned} d(p_{n+1}, e) &\leq d(p_{n+1}, x_{n+1}) + d(x_{n+1}, e) \\ &\leq d(p_{n+1}, Tq_n) + d(Tq_n, Ty_n) + d(x_{n+1}, e) \end{aligned} \tag{3.8}$$

Now, observe that

$$\begin{aligned} d(z_n, r_n) &= d(W(x_n, Tx_n, \beta_n), W(p_n, Tp_n, \beta_n)) \\ &\leq (1-\beta_n)d(x_n, p_n) + \beta_n d(Tx_n, Tp_n) \\ &\leq (1-\beta_n)d(x_n, p_n) + \beta_n \delta d(x_n, p_n) \\ &= (1-(1-\delta)\beta_n)d(x_n, p_n) \end{aligned} \tag{3.9}$$

Again

$$\begin{aligned} d(y_n, q_n) &= d(T(W(x_n, z_n, \alpha_n)), T(W(p_n, r_n, \alpha_n))) \\ &\leq \delta(d(x_n, z_n, \alpha_n), d(p_n, r_n, \alpha_n)) \\ &\leq \delta[(1-\alpha_n)d(x_n, p_n) + \alpha_n d(z_n, r_n)] \end{aligned} \tag{3.10}$$

From equations (3.8), (3.9), and (3.10), we get

$$d(p_{n=1}, e) \leq \epsilon_n + \delta^2(1-(1-\delta)\alpha_n\beta_n)d(x_n, p_n) + d(x_{n+1}, e) \tag{3.11}$$

From Theorem 3.1.1, it follows that $\lim_{n\to\infty} d(x_{n+1}, e) = 0$ and since $\{x_n\}$ and $\{p_n\}$ are equivalent, we have $\lim_{n\to\infty} d(x_n, p_n) = 0$. Now taking the limit of both sides of (3.11) and then using the assumption $\lim_{n\to\infty} \epsilon_n = 0$, yield to $\lim_{n\to\infty} d(p_{n+1}, e) = 0$. Thus, $\{x_n\}$ is weak w^2-stable with respect to T. $\square$

Theorem 3.1.3. Let C be a nonempty closed convex subset of a hyperbolic metric space X. $T : C \to C$ be a contraction mapping with the constant $\delta \in [0,1)$ s.t. $F(T) \neq \phi$ and $\{x_n\}$ be the iterative scheme (3.1) with the real sequence $\{\alpha_n\}$, $\{\beta_n\}$ are in $(0,1)$ and $\overline{T} : C \to C$ be an approximate operator of T for given ϵ and define an iterative sequence $\{\overline{x}_n\}$ as follows:

$$\begin{cases} \overline{x}_1 \in C \\ \overline{x}_{n+1} = \overline{T}\overline{y}_n \\ \overline{y}_n = \overline{T}(W(\overline{x}_n, \overline{z}_n, \alpha_n)); \\ \overline{z}_n = W(\overline{x}_n, \overline{T}\overline{x}_n, \beta_n) \end{cases} \tag{3.12}$$

with real sequence $\{\alpha_n\}$ and $\{\beta_n\}$ in $(0,1)$ satisfying $\alpha_n\beta_n \geq \frac{1}{2}$, $\forall n \geq 1$, and $\sum_{n=1}^{\infty} \alpha_n\beta_n = \infty$. If $Te = e$ and $\overline{T}\overline{e} = \overline{e}$ such that $\lim_{n\to\infty} \overline{x}_n = \overline{e}$, then one has

$$d(e, \overline{e}) \leq \frac{5\epsilon}{1-\delta}$$

where $\delta \in [0,1)$.

Proof. It follows from iteration (3.1) and (3.12) that

$$\begin{aligned} d(z_n, \overline{z}_n) &= d(W(x_n, Tx_n\beta_n), W(\overline{x}_n, \overline{T}\overline{x}_n, \beta_n)) \\ &\leq (1-\beta_n)d(x_n, \overline{x}_n) + \beta_n d(Tx_n, \overline{T}\overline{x}_n) \\ &\leq (1-\beta_n)d(x_n, \overline{x}_n) + \beta_n[d(Tx_n, T\overline{x}_n) + d(T\overline{x}_n, \overline{T}\overline{x}_n)] \\ &\leq (1-\beta_n)d(x_n, \overline{x}_n) + \beta_n[\delta d(x_n, \overline{x}_n) + \epsilon] \\ &= (1-(1-\delta)\beta_n)d(x_n, \overline{x}_n) + \beta_n\epsilon \end{aligned} \tag{3.13}$$

Now

$$\begin{aligned} d(y_n, \overline{y}_n) &= d(T(W(x_n, z_n, \alpha_n)), \overline{T}(W(\overline{x}_n, \overline{z}_n, \alpha_n))) \\ &\leq d(T(W(x_n, z_n, \alpha_n)), T(W(\overline{x}_n, \overline{z}_n, \alpha_n))) + d(T(W(\overline{x}_n, \overline{z}_n, \alpha_n)), \overline{T}(W(\overline{x}_n, \overline{z}_n, \alpha_n))) \\ &\leq \delta[d(W(x_n, z_n, \alpha_n), W(\overline{x}_n, \overline{z}_n, \alpha_n))] + \epsilon \\ &\leq \delta[(1-\alpha_n)d(x_n, \overline{x}_n) + \alpha_n d(z_n, \overline{z}_n)] + \epsilon \end{aligned} \tag{3.14}$$

From equations (3.13) and (3.14), we obtain

$$d(y_n, \overline{y}_n) \leq \delta(1-(1-\delta)\alpha_n\beta_n)d(x_n, \overline{x}_n) + \delta\alpha_n\beta_n\epsilon + \epsilon \tag{3.15}$$

Again

$$\begin{aligned} d(x_{n+1}, \overline{x}_{n+1}) &= d(Ty_n, \overline{T}\overline{y}_n) \\ &\leq d(Ty_n, T\overline{y}_n) + d(T\overline{y}_n, \overline{T}\overline{y}_n) \\ &\leq \delta d(y_n, \overline{y}_n) + \epsilon \end{aligned} \tag{3.16}$$

From equations (3.15) and (3.16), we obtain

$$d(x_{n+1}, \overline{x}_{n+1}) \leq \delta^2(1-(1-\delta)\alpha_n\beta_n)d(x_n, \overline{x}_n) + \delta^2\alpha_n\beta_n\epsilon + \delta\epsilon + \epsilon$$

Because $\{\alpha_n\}$ and $\{\beta_n\}$ are in $(0,1)$ and $\delta \in (0,1)$, we obtain

$$\begin{aligned} d(x_{n+1}, \overline{x}_{n+1}) &\leq (1-(1-\delta)\alpha_n\beta_n)d(x_n, \overline{x}_n) + \alpha_n\beta_n\epsilon + 2\epsilon \\ &= (1-(1-\delta)\alpha_n\beta_n)d(x_n, \overline{x}_n) + \alpha_n\beta_n\epsilon + 2(1-\alpha_n\beta_n + \alpha_n\beta_n)\epsilon \end{aligned} \tag{3.17}$$

By the assumption $\alpha_n\beta_n \geq \frac{1}{2}$, we have $1-\alpha_n\beta_n \leq \alpha_n\beta_n$. From equation (3.17), we get

$$\begin{aligned} d(x_{n+1}, \overline{x}_{n+1}) &\leq (1-(1-\delta)\alpha_n\beta_n) + 5\alpha_n\beta_n\epsilon \\ &=\leq (1-(1-\delta)\alpha_n\beta_n) + (1-\delta)\alpha_n\beta_n\frac{5\epsilon}{(1-\delta)} \end{aligned} \tag{3.18}$$

Let $g_n = d(x_n, \overline{x}_n)$, $r_n = \alpha_n\beta_n(1-\delta)$ and $t_n = \frac{5\epsilon}{(1-\delta)}$ and then from Lemma 2.3.3, together with (3.18), we have

$$0 \leq \limsup_{n\to\infty} g_n \leq \limsup_{n\to\infty} \frac{5\epsilon}{(1-\delta)} \tag{3.19}$$

By Theorem 3.1.1, we have $\lim_{n\to\infty} x_n = e$ and by the assumption in the hypotheses, we have $\lim_{n\to\infty} \overline{x}_n = \overline{e}$. Using these together with (3.19), we obtain

$$d(e, \overline{e}) \leq \frac{5\epsilon}{(1-\delta)}$$

□

3.2. **Strong and Δ-Convergence Theorems for Generalized (α, β)- Nonexpansive Type 1 Mappings.**

Lemma 3.2.1. *Let T be a generalized (α, β)-nonexpansive type1 mapping defined on a nonempty convex subset C of a hyperbolic metric space X with $F(T) \neq \phi$. If $e \in F(T)$ and $\{x_n\}$ is the iterative process defined by* (3.1)*, then* $\lim_{n\to\infty} d(x_n, e)$ *exists.*

Proof. By Proposition 2.1, we have

$$\begin{aligned} d(z_n, e) &= d(W(x_n, Tx_n, \beta_n), e) \\ &\leq (1-\beta_n)d(x_n, e) + \beta_n d(Tx_n, e) \\ &\leq (1-\beta_n)d(x_n, e) + \beta_n d(x_n, e) \\ &= d(x_n, e) \end{aligned} \tag{3.20}$$

Now

$$\begin{aligned} d(y_n, e) &= d(T(W(x_n, z_n, \alpha_n)), e) \\ &\leq d(W(x_n, z_n, \alpha_n), e) \\ &\leq (1-\alpha_n)d(x_n, e) + \alpha_n d(z_n, e) \end{aligned} \tag{3.21}$$

From equations (3.20) and (3.21), we get

$$d(y_n, e) \leq d(x_n, e) \tag{3.22}$$

Again

$$\begin{aligned} d(x_{n+1}, e) &= d(Ty_n, e) \\ &\leq d(y_n, e) \end{aligned} \tag{3.23}$$

From (3.22) and (3.23), we obtain

$$d(x_{n+1}, e) \leq d(x_n, e)$$

This shows that $\{x_n\}$ is a non-increasing sequence and it is bounded from below for each $e \in F(T)$. So, we obtain that $\lim_{n\to\infty} d(x_n, e)$ exists for any $e \in F(T)$. □

Lemma 3.2.2. *Let C be a nonempty closed convex subset of a complete uniformly convex hyperbolic metric space X with the monotone modulus of uniform convexity η and $T : C \to C$ be a generalized (α, β)-nonexpansive type1 mapping. Let $\{x_n\}$ be the iterative scheme* (3.1) *with real sequences $\{\alpha_n\}$ and $\{\beta_n\}$ in $[p, q]$ for some $p, q \in (0, 1)$. Then $F(T) \neq \phi$ if and only if $\{x_n\}$ is bounded and* $\lim_{n\to\infty} d(x_n, Tx_n) = 0$.

Proof. Suppose $F(T) \neq \phi$ and choose $e \in F(T)$. Then by Lemma 3.2.1 $\lim_{n\to\infty} d(x_n, e)$ exists and $\{x_n\}$ is bounded. Put

$$\lim_{n\to\infty} d(x_n, e) = r \tag{3.24}$$

Case I: If $r = 0$ we are done.
Case II: If $r > 0$. From equation (3.20), we get

$$d(z_n, e) \leq d(x_n, e)$$

By taking limsup on both sides, we get

$$\limsup_{n\to\infty} d(z_n, e) \le \limsup_{n\to\infty} d(x_n, e) = r \tag{3.25}$$

By Proposition 2.1, we have

$$\limsup_{n\to\infty} d(Tx_n, e) \le \limsup_{n\to\infty} d(x_n, e) = r \tag{3.26}$$

On the other hand

$$\begin{aligned} d(x_{n+1}, e) &\le d(y_n, e) \\ &\le d(T(W(x_n, z_n, \alpha_n)), e) \\ &\le (1-\alpha_n)d(x_n, e) + \alpha_n d(z_n, e) \\ &= d(x_n, e) + \alpha_n(d(z_n, e) - d(x_n, e)) \end{aligned}$$

$$\frac{d(x_{n+1}, e) - d(x_n, e)}{\alpha_n} \le d(z_n, e) - d(x_n, e) \tag{3.27}$$

So

$$\begin{aligned} d(x_{n+1}, e) - d(x_n, e) &\le \frac{d(x_{n+1}, e) - d(x_n, e)}{\alpha_n} \le d(z_n, e) - d(x_n, e) \\ &\le d(z_n, e) - d(x_n, e) \end{aligned}$$

i.e.,

$$d(x_{n+1}, e) \le d(z_n, e) \tag{3.28}$$

Therefore

$$r \le \liminf_{n\to\infty} d(z_n, e) \tag{3.29}$$

From equations (3.25) and (3.29), we get

$$\lim_{n\to\infty} d(z_n, e) = r$$

$$\lim_{n\to\infty} d(W(x_n, Tx_n, \beta_n), e) = r \tag{3.30}$$

Finally, from (3.24), (3.25), (3.30), and Lemma 2.3.2, we deduce that

$$\lim_{n\to\infty} d(x_n, Tx_n) = 0$$

Converse: We assume that $\{x_n\}$ is bounded and $\lim_{n\to\infty} d(x_n, Tx_n) = 0$. Let $e \in A(C, \{x_n\})$. By Proposition 2.2, we have

$$\begin{aligned} r(Te, \{x_n\}) &= \limsup_{n\to\infty} d(x_n, Te) \\ &\le \frac{2+\alpha_n+\beta_n}{(1-\beta)} \limsup_{n\to\infty} d(x_n, Te) + \limsup_{n\to\infty} d(x_n, e) \\ &= \limsup_{n\to\infty} d(x_n, e) \qquad (3.31) \\ &= r(e, \{x_n\}) \qquad (3.32) \end{aligned}$$

This implies that $Te \in A(C, \{x_n\})$. Because the sequence $\{x_n\}$ is bounded by Lemma 2.3.1, $A(C, \{x_n\})$ consists of exactly one point. Hence we have $Te = e$. Thus $F(T) \neq \phi$. □

Theorem 3.2.3. Let C be a nonempty closed convex subset of a complete uniformly convex hyperbolic metric space X with the monotone modulus of uniform convexity η and $T: C \to C$ be a generalized (α,β)-nonexpansive type1 mapping. Let $\{x_n\}$ be the iterative scheme (3.1) with real sequences $\{\alpha_n\}$ and $\{\beta_n\}$ in $[p,q]$ for some $p,q \in (0,1)$. Then the sequence $\{x_n\}$ Δ-convergence to a fixed point of T.

Proof. By Lemma 2.3.1, the sequence $\{x_n\}$ has a unique asymptotic center $A(C,\{x_n\}) = \{x\}$. Let $\{u_n\}$ be any subsequence of $\{x_n\}$ such that $A(C,\{u_n\}) = \{u\}$. Then by Lemma 3.2.2, we have

$$\lim_{n\to\infty} d(u_n, Tu_n) = 0$$

It follows similarly from the proof of Lemma 3.2.2, that u is a fixed point of S. Next, we have to show that the fixed point u is the unique asymptotic center for each subsequence $\{u_n\}$ of $\{x_n\}$. On the contrary, we assume that $x \neq u$. From Lemma 3.2.1, we deduce that $\lim_{n\to\infty} d(x_n,u)$ exists. Therefore, by the uniqueness of the asymptotic center, we can see that

$$\begin{aligned}
\limsup_{n\to\infty} d(u_n,u) &< \limsup_{n\to\infty} d(u_n,x) \\
&\leq \limsup_{n\to\infty} d(x_n,x) \\
&< \limsup_{n\to\infty} d(x_n,u) \\
&= \limsup_{n\to\infty} d(u_n,u)
\end{aligned}$$

which is contradiction. So $u \in F(T)$ is the unique asymptotic center for each subsequence $\{u_n\}$ of $\{x_n\}$. This prove that the sequence $\{x_n\}$ Δ-convergence to a fixed point of T. □

Theorem 3.2.4. Under the assumption of Theorem 3.2.3, if C is a compact subset of X, then the sequence $\{x_n\}$ converges strongly to a fixed point of T.

Proof. We consider an element $e \in C$. Because C is a compact set, we can say that there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $\lim_{k\to\infty} d(x_{n_k}, e) = 0$. By Proposition 2.2, we have

$$\lim_{k\to\infty} d(x_{n_k}, Te) \leq \frac{2+\alpha+\beta}{1-\beta} \lim_{k\to\infty} d(x_{n_k}, Tx_{n_k}) + \lim_{k\to\infty} d(x_{n_k}, e).$$

From Lemma 3.2.2, we obtain $\lim_{k\to\infty}(x_{n_k}, Tx_{n_k}) = 0$. Then we obtain $Te = e$, that is $e \in F(T)$. Using Lemma 3.2.1, $\lim_{n\to\infty} d(x_n, e)$ exists and hence $\{x_n\}$ converges strongly to e. □

Theorem 3.2.5. Let C be a nonempty closed convex subset of a complete uniformly convex hyperbolic metric space X with the monotone modulus of uniform convexity η and $T: C \to C$ be a generalized (α,β)-nonexpansive type1 mapping. Let $\{x_n\}$ be the iterative scheme (3.1) with real sequences $\{\alpha_n\}$ and $\{\beta_n\}$ in $[p,q]$ for some $p,q \in (0,1)$. Then the sequence $\{x_n\}$ converges strongly to a fixed point of T if and only if

$$\liminf_{n\to\infty} d(x_n, F(T)) = 0.$$

where $d(x,F(T)) = inf\{d(x,e) : e \in F(T)\}$.

Proof. If the sequence $\{x_n\}$ converges strongly to a point $e \in F(T)$, then $\lim_{n\to\infty} d(x_n,e) = 0$. Because $0 \leq d(x_n,F(T)) \leq d(x_n,e)$, we have $\lim_{n\to\infty} d(x_n,F(T)) = 0$.

Converse: Suppose that $\liminf_{n\to\infty} d(x_n, F(T)) = 0$. It follows from Lemma 3.2.1 that $\lim_{n\to\infty} d(x_n, F(T))$ exists and hence $\lim_{n\to\infty} d(x_n, F(T)) = 0$. Therefore, there exist a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ and a sequence $\{e_k\}$ in $F(T)$ such that

$$d(x_{n_k}, e_k) < \frac{1}{2^k}\ \forall k \geq 1.$$

By the proof of Lemma 3.2.1, we have

$$d(x_{n_{k+1}}, e_{k+1}) \leq d(x_{n_k}, e_k) < \frac{1}{2^k}$$

which implies that

$$\begin{aligned} d(e_{k+1}, e_k) &\leq d(e_{k+1}, x_{n_{k+1}}) + d(x_{n_{k+1}}, e_k) \\ &< \frac{1}{2^{k+1}} + \frac{1}{2^k} \\ &< \frac{1}{2^{k-1}} \\ &\to 0 \ as\ k \to \infty. \end{aligned} \tag{3.33}$$

Hence $\{e_k\}$ is a Cauchy sequence in $F(T)$. By Proposition 2.3, $F(T)$ is closed and so $\{e_k\}$ converges strongly to $e \in F(T)$. On the other hand, we have

$$d(x_{n_k}, e) \leq d(x_{n_k}, e_k) + d(e_k, e)$$

By taking the limit of both sides of this inequality, we obtain that $\{x_{n_k}\}$ converges stronglye $\in F(T)$. Because $\lim_{n\to\infty} d(x_n, e)$ exists by Lemma 3.2.1, e is the strong limit of $\{x_n\}$. □

3.3. **Numerical Examples.** In this subsection, we see an example of Generalized (α, β)-nonexpansive type 1 mapping, which is not nonexpansive mapping.

Example 3.3.1. *Let $X = R$ with the usual metric $C = [0, \infty)$. Define a mapping $T : C \to C$ by*

$$Tx = \begin{cases} 0 & x \in [0, \frac{7}{6}) \\ \frac{6x}{13} & x \in [\frac{7}{6}, \infty) \end{cases}$$

Clearly $x = 0$ is the fixed point of T. Then the following

(1) *Because T is not continuous at the point $x = \frac{7}{6}$, then T is not nonexpansive mapping.*
(2) *Now, we prove that T is a generalized (α, β)-nonexpansive mapping type 1 mapping. For this purpose, let $\lambda = \frac{1}{3}$, $\alpha = \frac{5}{13}$, $\beta = \frac{7}{13}$ and the consider the following cases:*
  (I) **Case A***: $x \in [0, \frac{7}{6})$. Then $\lambda|x - Tx| = \frac{1}{3}x \leq |x - y|$, which gives two possibilities;*
  (1) *Let $x < y$. Then*

$$\begin{aligned} \frac{1}{3}x \leq y - x &\Rightarrow \frac{4}{3}x < y \\ &\Rightarrow y \in [0, \frac{14}{9}) \end{aligned} \tag{3.34}$$

  (a) *If $y \in [0, \frac{7}{6})$, then we have*

$$|Tx - Ty| = 0 \leq \frac{5}{13}|y| + \frac{7}{13}|x| + \frac{1}{13}|x - y|$$

(b) *If* $y \in [\frac{7}{6}, \frac{14}{9})$, *then we have*

$$|Tx - Ty| = |\frac{6}{13}y| \leq \frac{5}{13}|y| + \frac{7}{13}|x - \frac{6}{13}y| + \frac{1}{13}|x - y|$$

(2) *Let* $x > y$. *Then*

$$\frac{1}{3}x \leq x - y \Rightarrow y \leq \frac{2x}{3}$$
$$\Rightarrow y \in [0, \frac{7}{9}) \subset [0, \frac{7}{6})$$

which is already included in case (1)(a).

(II) **Case B**: $x \in [\frac{7}{6}, \infty)$. *Then*

$$\lambda|x - Tx| = \frac{1}{3}|x - \frac{6x}{13}|$$
$$= \frac{7x}{39} \leq |x - y|$$
$$= \frac{7x}{39} \leq x - y$$

which gives two possibilities:

(1) *Let* $x < y$. *Then* $\frac{7x}{39} \leq y - x \Rightarrow y \geq \frac{46x}{39} \Rightarrow y \in [\frac{161}{117}, \infty) \subset [\frac{7}{6}, \infty)$. *So*

$$|Tx - Ty| = \frac{6}{13}|x - y|$$
$$\leq \frac{5}{13}|\frac{6x}{13} - y| + \frac{7}{13}|x - \frac{6y}{13}| + \frac{1}{13}|x - y|$$

(2) *Let* $x > y$. *Then* $\frac{7x}{39} \leq x - y \Rightarrow x \geq \frac{39y}{32} \Rightarrow y \in [\frac{112}{117}, \infty)$ *which gives two possibilities;*

(a) *If* $y \in [\frac{112}{117}, \frac{7}{6})$, *then we have*

$$|Tx - Ty| = \frac{5}{13}|x| \leq \frac{5}{13}|\frac{6}{13}x| + \frac{7}{13}|x| + \frac{1}{13}|x - y|$$

(b) $y \in [\frac{7}{6}, \infty)$ *is already included in Case (1).*

Hence, T *is a generalized* (α,β)*-nonexpansive type 1 mapping with* $F(T) \neq \phi$, *where* $\alpha = \frac{5}{13}$, $\beta = \frac{7}{13}$.

4. Conclusions

The main contributions of our work are:

- We modify the Thakur iteration process into hyperbolic metric space where the symmetry condition is satisfied.
- We established the weak w^2-stability and data dependency results for contraction mappings.
- We established Δ-convergence and strong convergence theorems for generalized (α,β)-nonexpansive type 1 mappings.
- Numerical Examples.

Acknowledgments

The authors thank the anonymous referee for his/her comments that have significantly improved the quality of the paper.

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