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Research Paper

THE STEINER HOP DOMINATION NUMBER OF SPECIAL GRAPHS

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ABSTRACT

The main aim of this article is that the distance-based parameters, in particularly the Steiner concept and the hop domination combine to form interesting results in graph theory. A subset W of vertices in a graph G is called the Steiner hop dominating set of G if W is both a Steiner set and a hop dominating set of G. The minimum cardinality of a Steiner hop dominating set of G is its Steiner hop domination number $\gamma_{hs}(G)$. The Steiner hop domination number of some special graphs, such as the middle graph and degree splitting graph of standard graphs are determined. Also, it is determined for lollipop graph, Barbell graph and pop barbell graph.

1. Introduction

By a graph $G = (V, E)$, we mean a finite undirected connected graph without loops or multiple edges. The *order* and *size* of G are denoted by n and m respectively. For basic definitions and terminologies we refer to [5]. Two vertices u and v are said to be *adjacent* if uv is an edge of G. For any vertex v in a graph G, the number of vertices adjacent to v is called the *degree* of v in G, denoted by $deg_G(v)$. A vertex v is called a *universal vertex* if $deg_G(v) = n - 1$. For any set S of vertices of G, the *induced subgraph* $< S >$ is the maximal

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subgraph of G with vertex set S. The *distance* $d(u,v)$ between u and v is the length of a shortest $u-v$ paths in G and a $u-v$ path of length $d(u,v)$ is called a *u-v geodesic*.

Many authors have studied this concept with respect to several parameters like domination, Steiner, etc. In this paper, we study the concept of Steiner hop domination number of a graphs. For a nonempty set W of vertices in a connected graph G, the *Steiner distance* $d(W)$ of W is the minimum size of a connected subgraph of G containing W. Necessarily, each such subgraph is a tree and is called a *Steiner tree* with respect to W or a *Steiner W-tree*. For a given set $W \subseteq V(G)$, there may be more than one Steiner W-tree in G. Let $S(W)$ denote the set of vertices that lies in Steiner W-trees. Let G be a connected graph with at least 2 vertices. A set $W \subseteq V(G)$ is called a Steiner set of G if $S(W) = V(G)$. The Steiner number $s(G)$ is the minimum cardinality of its *Steiner sets* and any Steiner set of cardinality $s(G)$ is a minimum Steiner set of G. The Steiner number of a graph was studied in [6, 9, 11, 12, 16].

A set $S \subseteq V$ of a graph G is a *hop dominating set* of G if for every $v \in V - S$, there exists $u \in S$ such that $d(u,v) = 2$. The minimum cardinality of hop dominating set is called the *hop domination number* and is denoted by $\gamma_h(G)$, this concept was studied in [3, 4, 8, 13].

The *middle graph* $M(G)$ of a graph G is defined by the graph which the vertex set is $V(G) \cup E(G)$ and the two vertices a, b in the vertex set of $M(G)$ are adjacent in $M(G)$ in case one the following holds (1) $x, y \in E(G)$ and x, y are adjacent in G; (2) $x \in V(G)$, $y \in E(G)$, and x, y are incident in G, refer [7]. Let $G = (V, G)$ be a graph with $V = S_1 \cup S_2 \cup ...S_t \cup T$ where each S_i is a set of vertices having at least two vertices of the same degree and $T = V - \cup S_i$. The *degree splitting graph* of G denoted by $DS(G)$ is obtained from G by adding vertices $u_1, u_2, ..., u_t$ and joining u_i to each vertex of S_i for $1 \leq i \leq t$, refer [10, 15].

The *lollipop graph* is obtained by connecting a complete graph to the path graph with a bridge. It is denoted by $L_{p,q}$ or $L(p,q)$.
The vertices of a lollipop graph as two distinct sets.
(i) The vertices of the complete graph K_p as $\{u_1, u_2, ..., u_p\}$ and
(ii) The vertices of the path graph P_q as $\{v_1, v_2, ..., v_q\}$.
The vertices of $L_{p,q}$ are $V(L_{p,q}) = V(K_p) \cup V(P_q) = \{u_1, u_2, ..., u_p, v_1, v_2, ..., v_q\}$.
A $B(p,q)$ *Barbell graph* is obtained by joining $q-$*copies* of a complete graph K_p by a bridge. It is denoted by $B(p,q)$, we refer [2, 14].

A subset W of vertices in a graph G is called the *Steiner hop dominating set* of G if W is both a Steiner set and a hop dominating set of G. The minimum cardinality of a Steiner hop dominating set of G is its *Steiner hop domination number* $\gamma_{hs}(G)$. A Steiner hop domination set of size $\gamma_{hs}(G)$ is said to be a γ_{hs}-set of G, refer [1].

The Steiner and hop domination concepts merge to facilitate effective coordination and path planning for autonomous robots, sensor networks, and drone systems as they cover specific distances in designated areas. It also helps in transmitting data in network communications with certain distances and social networks, as well as addressing location allocation issues [1].

The following Theorems are used in the sequel.

Theorem 1.1. [6, 4]

(1) *Each extreme vertex of a connected graph G belongs to every Steiner set of G.*
(2) *No cut vertex of G belongs to any minimum Steiner set of G.*

(3) *For a connected graph G, $g(G) = 2$, if and only if every vertex of G lies in a $u-v$ diametral path of G.*
(4) *For a connected graph G, $g(G) = 2$ if and only if $s(G) = 2$.*

2. Steiner hop domination number of a middle graph of a graph

Proposition 2.1. (1) *Each extreme vertex of a connected graph G belongs to every Steiner hop dominating set of G.*
(2) *Each universal vertex of a connected graph G belongs to every Steiner hop dominating set of G.*
(3) *Let G be a connected graph with at least one universal vertex. Then $\gamma_{hs}(G) = n$.*

Theorem 2.2. *Let $G = M(P_n)$ be the middle graph of the path $P_n(n \geq 3)$. Then $\gamma_{hs}(G) = n$.*

Proof. Let us take the vertices of the path graph $P_n(n \geq 3)$ as $v_1, v_2, ..., v_n$. To obtain the graph $M(P_n)$, let $u_1, u_2, ..., u_{n-1}$ be the added vertices that correspond to the edges $e_1, e_2, ..., e_{n-1}$ of P_n. Then the vertex set $|V(M(P_n))| = 2n-1$ and the edge set $E(M(P_n)) = 3n-4$. The graph $M(P_n)$ are shown in Figure 1.
Let $Z = \{v_1, v_2, ..., v_n\}$ be the set of all extreme vertices of G. By Proposition 2.1 (1), Z is

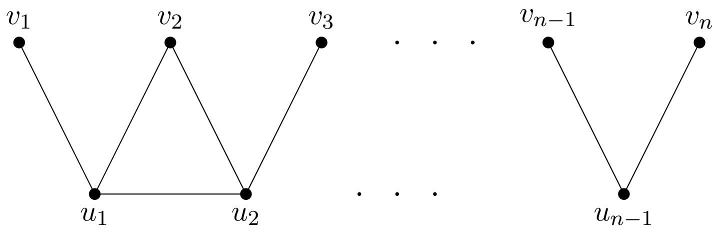

Figure 1. The graph $M(P_n)$

a subset of every Steiner hop dominating set of G as a result, $\gamma_{hs}(G) \geq n$. Since for every $x \in V(G)\backslash Z$ there exists $y \in Z$ such that $d(x, y) = 2$, it follows that Z is a Steiner hop dominating set of G. As a result, $\gamma_{hs}(G) = n$. □

Theorem 2.3. *Let $G = M(C_n)$ be the middle graph of the cycle $C_n(n \geq 3)$. Then $\gamma_{hs}(G) = n$.*

Proof. Let us take the vertices of the cycle graph $C_n(n \geq 3)$ as $v_1, v_2, ..., v_n, v_1$. To obtain the graph $M(C_n)$, let $u_1, u_2, ..., u_n, u_1$ be the added vertices that correspond to the edges $e_1, e_2, ..., e_n$ of C_n. Then the vertex set $|V(M(C_n))| = 2n$ and the edge set $E(M(C_n)) = 3n$. The graph $M(C_n)$ are shown in Figure 2.
Let $Z = \{v_1, v_2, ..., v_n\}$ be the set of all extreme vertices of G. By Proposition 2.1 (1), Z is a subset of every Steiner hop dominating set of G as a result, $\gamma_{hs}(G) \geq n$. Since for every $x \in V(G)\backslash Z$ there exists $y \in Z$ such that $d(x, y) = 2$, it follows that Z is a Steiner hop dominating set of G as a result, $\gamma_{hs}(G) = n$. □

Theorem 2.4. *Let $G = M(K_{1,n-1})$ be the middle graph of the star graph $K_{1,n-1}(n \geq 3)$. Then $\gamma_{hs}(G) = n$.*

Proof. Let us take the vertices of the star graph $K_{1,n-1}(n \geq 3)$ as $x, v_1, v_2, ..., v_{n-1}$. To obtain the graph $M(K_{1,n-1})$, let $u_1, u_2, ..., u_{n-1}$ be the added vertices that correspond to the

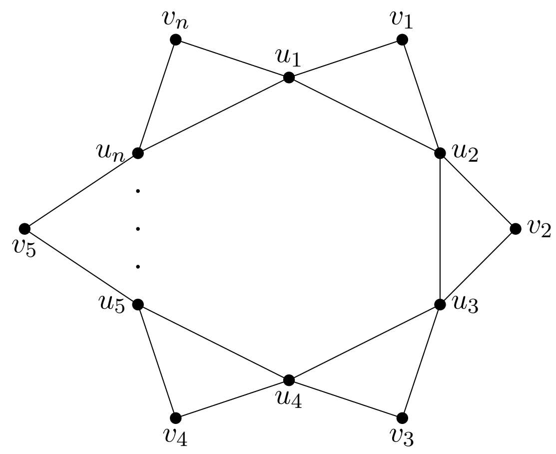

FIGURE 2. The graph $M(C_n)$

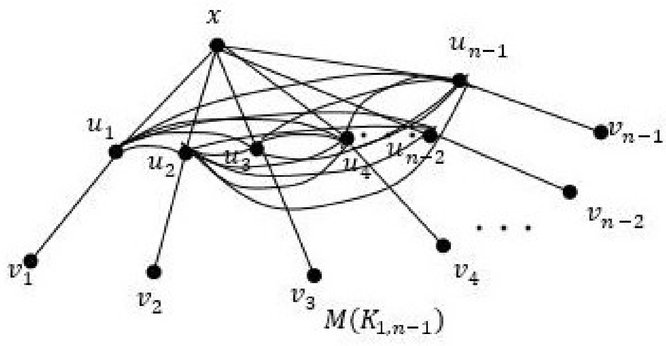

FIGURE 3. The graph $M(K_{1,n-1})$

edges $e_1, e_2, ..., e_{n-1}$ of $K_{1,n-1}$. Then the vertex set $|V(M(K_{1,n-1}))| = 2n - 1$ and the edge set $E(M(K_{1,n-1})) = \frac{(n+2)(n-1)}{2}$. The graph $M(K_{1,n-1})$ is shown in Figure 3.
Let $Z = \{v_1, v_2, ..., v_{n-1}\}$ be the set of extreme vertices of G. By Proposition 2.1 (1), Z is a subset of every Steiner hop dominating set of G as a result, $\gamma_{hs}(G) \geq n - 1$. Since $S(Z) \neq V(G)$, Z is not a Steiner hop dominating set of G as a result, $\gamma_{hs}(G) \geq n$. It is easily observed that x is a subset of every Steiner hop dominating set of G as a result, $\gamma_{hs}(G) \geq n$. Let $Z_1 = Z \cup \{x\}$. Since $S(Z_1) = V(G)$, Z_1 is a Steiner set of G. Since $d(u_i, v_j) = 2, (1 \leq i < j \leq n - 1)$ and $d(x, v_i) = 2, (1 \leq i \leq n - 1)$, Z_1 is a hop dominating set of G. Thereby, it is a minimum Steiner hop dominating set of G as a result, $\gamma_{hs}(G) = n$. □

3. THE STEINER HOP DOMINATING OF DEGREE SPLITTING GRAPH OF A GRAPH

Theorem 3.1. *Let $G = DS(P_n)$ be the degree splitting graph of the path $P_n (n \geq 3)$. Then*

$$\gamma_{hs}(G) = \begin{cases} 4 & \text{if } n = 3 \\ 2 & \text{if } n = 4 \\ n - 3 & \text{if } n \geq 5. \end{cases}$$

Proof. Let $P_n : v_1, v_2, ..., v_n$ be the path graph of order n. Since $deg(v_1) = deg(v_n) = 1$ and $deg(v_i) = 2$ where $(2 \leq i \leq n - 1)$, let the two partitions of G be $S_1 = \{v_2, v_3, ..., v_{n-1}\}$ and

$S_2 = \{v_1, v_n\}$. To obtain $DS(P_n)$ for P_3, we introduce a vertex u_1 that corresponds to S_2, since P_3 is isomorphic to C_4. Also, to obtain $DS(P_n)$ for $n \geq 4$, we introduce new vertices u_1 and u_2 which correspond to S_1 and S_2 respectively. Hence $V(G) = \{u_1, v_1, v_2, v_3\}$ and $V(G) = \{u_1, u_2, v_1, v_2, ..., v_n\}$, where $|V(G)| = n+2$ for $n \geq 4$. The graph $DS(P_n)$ are shown in Figure 4.

Case 1. Let $n = 3$. Then $G = C_4$. Therefore, any two antipodal vertices of G is a Steiner

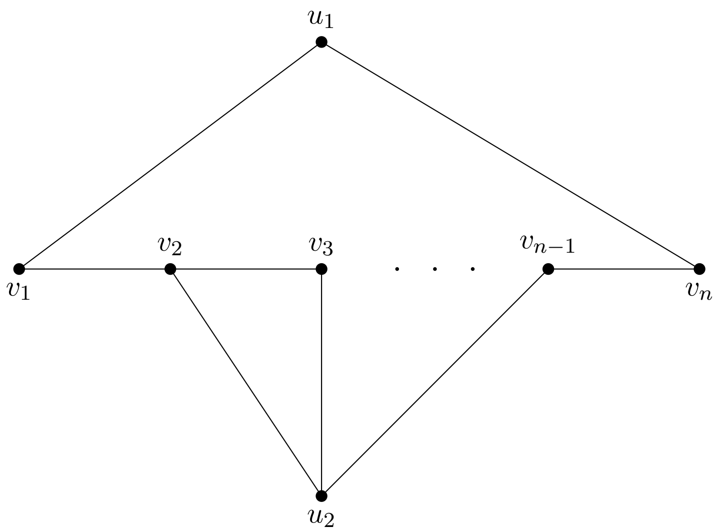

FIGURE 4. The graph $DS(P_n)$

set of G but not a hop dominating set of G. Hence it follows that $V(G)$ is the minimum Steiner hop dominating set of G, resulting in $\gamma_{hs}(G) = 4$.

Case 2. Consider $n = 4$. Let $W = \{u_1, u_2\}$. Since every vertex of G lies in the $u_1 - u_2$ diametral path, by Theorem 1.1 (3) and (4), $s(G) = 2$. Since for every $u \in V(G) \backslash W$, there exists $v \in W$ such that $d(u, v) = 2$, it follows that W is a hop dominating set of G. Therefore, W is a minimum Steiner hop dominating set of G, resulting in $\gamma_{hs}(G) = 2$.

Case 3. Let $W = V(DS(P_n)) \backslash \{v_2, v_3, v_n, u_1, u_2\}$. Then W is a Steiner hop dominating set of G as a result, $\gamma_{hs}(G) \leq n - 3$. We prove $\gamma_{hs}(G) = n - 3$. On the contrary, suppose that $\gamma_{hs}(G) \leq n - 4$. Then there exists a γ_{hs}- set W' of G such that $\gamma_{hs}(G) \leq n - 4$. Let x be a vertex of G such that $x \in W$ and $x \notin W'$ if $x = v_1$, then $S(W') \neq V(G)$, which contradicts itself. If $x \in \{v_4, v_5, v_6, ..., v_{n-1}\}$. Then W' is not a Steiner hop dominating set of G, which contradicts itself. As a result, $\gamma_{hs}(G) = n - 3$. □

Theorem 3.2. *Let $G = DS(C_n)$ be the degree splitting graph of the cycle $C_n (n \geq 3)$. Then $\gamma_{hs}(G) = n + 1$.*

Proof. To obtain $DS(C_n)$ for $n \geq 3$, define a cycle as $v_1, v_2, ..., v_n$. We introduce a new vertex u_1, which is adjacent to each and every vertex in C_n. Hence $V(G) = \{u_1, v_1, v_2, ..., v_n\}$, where $|V(G)| = n + 1$ for $n \geq 3$. The graph $DS(C_n)$ are shown in Figure 5. It is easily observed that $DS(C_n)$ is isomorphic to the wheel graph W_n, since $DS(C_n)$ contains a universal vertex by Proposition 2.1 (3), $\gamma_{hs}(G) = n + 1$. □

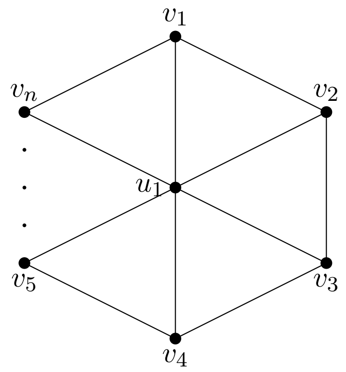

FIGURE 5. The graph $DS(C_n)$

Theorem 3.3. *Let $G = DS(K_{r,s})$ be the degree splitting graph of the bipartite graph $K_{r,s}(2 \leq r \leq s)$. Then*

$$\gamma_{hs}(G) = \begin{cases} 2 & \text{if } r \neq s \\ 2r+1 & \text{if } r = s. \end{cases}$$

Proof. Consider $K_{r,s}$ with $V(K_{r,s}) = \{x_a, y_b/1 \leq a \leq r, 1 \leq b \leq s\}$. Let the two bipartite sets of G be $X = \{x_1, x_2, ..., x_r\}$ and $Y = \{y_1, y_2, ..., y_s\}$. The following two cases are characterized.

Case 1. $r \neq s$. Let the added vertices be u_1 and u_2, where u_1 is adjacent to every x_a and u_2 is adjacent to every y_b. Each vertex x_a and y_b is of the same degree in this case, where $deg(x_a) \neq deg(y_b)$, $(1 \leq a \leq r)$ and $(1 \leq b \leq s)$. Thus, the graph $DS(K_{r,s})$ is obtained. Then $DS(K_{r,s}) = \{x_a, y_b, u_1, u_2/1 \leq a \leq r, 1 \leq b \leq s\}$ and $DS(K_{r,s}) = r + s + 2$. The graph $DS(K_{r,s})$ is shown in Figure 6.

Let $Z = \{u_1, u_2\}$. Then $S(G) = V(G)$ and Z is a subset of every Steiner hop dominating

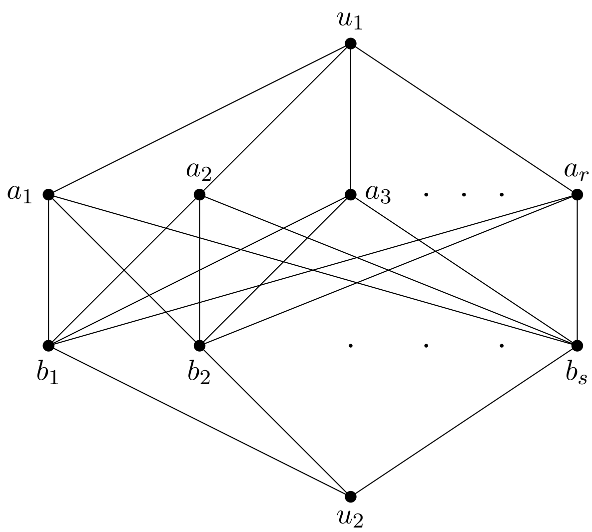

FIGURE 6. The graph $DS(K_{r,s})$

set of G. As a result, $\gamma_{hs}(G) = 2$. Since for every $x \in V(G) \backslash Z$ there exists $y \in Z$ such that $d(x, y) = 2$, it follows that Z is a Steiner hop dominating set of G resulting in, $\gamma_{hs}(G) = 2$.

Case 2. $r = s$. Let the added vertex be u_1 that is adjacent to x_a and y_b, where $1 \leq a \leq r$

and $1 \leq b \leq s$. In this case each vertex has the same degree. Thus, the graph $DS(K_{r,r})$ is obtained. Then $DS(K_{r,r}) = \{x_a, y_b, u_1 / 1 \leq a \leq r, 1 \leq b \leq s\}$ and $DS(K_{r,r}) = r + s + 1$. The graph $DS(K_{r,r})$ is shown in Figure 7.

Since u_1 is the universal vertex of G, by Proposition 2.1 (2), $\gamma_{hs}(G) = 2r + 1$.

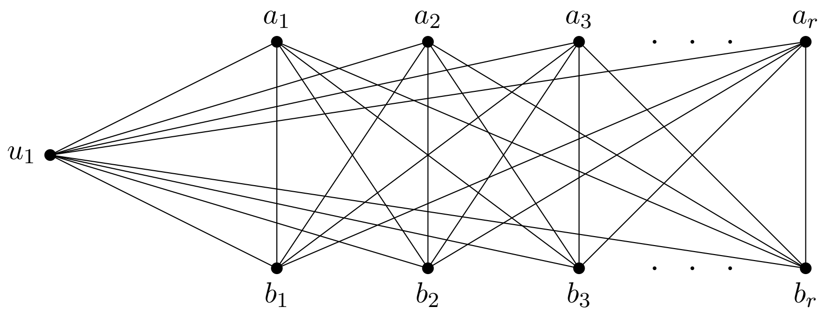

FIGURE 7. The graph $DS(K_{r,r})$

□

Theorem 3.4. *Let $G = DS(K_{1,n-1})$ be the degree splitting graph of the star graph $K_{1,n-1}$ $(n \geq 3)$. Then $\gamma_{hs}(G) = n + 1$.*

Proof. To obtain the graph $DS(K_{1,n-1})$, let x be the full vertex of the star graph $K_{1,n-1}$ and u_1 is the equivalent vertex that is introduced. Let $v_1, v_2, ..., v_{n-1}$ be the end vertices. Clearly, $V(DS(K_{1,n-1})) = \{x, v_1, v_2, v_3, \ldots, v_{n-1}, u_1\}$. Therefore, $|V(DS(K_{1,n-1}))| = n + 1$. The graph $DS(K_{1,n-1})$ are shown in Figure 8.

Let $W = V(G)$, since G is the complete bipartite graph $K_{2,n-1}$, by Theorem 3.3, W is the

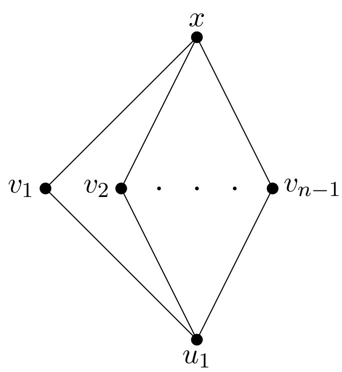

FIGURE 8. The graph $DS(K_{1,n-1})$

unique Steiner hop dominating set of G, resulting in $\gamma_{hs}(G) = n + 1$. □

Theorem 3.5. *Let $G = DS(B_{r,s})$ be the degree splitting graph of the bi-star graph $B_{r,s}(n \geq 3)$. Then*

$$\gamma_{hs}(G) = \begin{cases} 2 & \text{if } r \neq s \\ 3 & \text{if } r = s \end{cases}$$

Proof. The bi-star graph $B_{r,s}$ is characterized by the graph $V(B_{r,s}) = \{x, y, x_a, y_b / 1 \leq a \leq r, 1 \leq b \leq s\}$. In this cases, the vertices that are next to x and y are x_a and y_b, respectively.

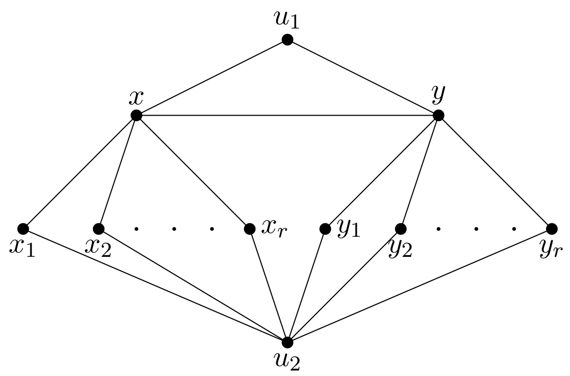

FIGURE 9. The graph $DS(B_{r,r})$

Now, the following two cases are characterized.

Case 1. $r = s$. Let u_1 and u_2 be the corresponding vertices that are introduced to obtain $DS(B_{r,r})$. Hence, $V(DS(B_{r,r})) = \{x, y, x_a, y_b, u_1, u_2/1 \leq a \leq r, 1 \leq b \leq r\}$. Therefore, $|V(DS(B_{r,r}))| = 2r + 4$. The graph $DS(B_{r,r})$ is shown in Figure 9.

Let $Z = \{u_1\}$ be the extreme vertex of G. Then by Proposition 2.1 (1), Z is a subset of every Steiner hop dominating set of G. Let $Z_1 = \{u_1, u_2\}$. Since every vertex of G lies in the $u_1 - u_2$ diametral path, by Theorem 1.1 (3), $g(G) = 2$. Hence, by Theorem 1.1 (4), $s(G) = 2$. Therefore, Z_1 is a Steiner set of G. Since for every $u \in V(G) \backslash Z_1$, there exists $v \in Z_1$ such that $d(u, v) = 2$, it follows that Z_1 is a hop dominating set of G. Therefore, Z_1 is a Steiner hop dominating set of G resulting in, $\gamma_{hs}(G) = 2$.

Case 2. $r \neq s$. Let u_1 be the added vertex that is introduced to obtain $DS(B_{r,s})$. Hence, $V(DS(B_{r,s})) = \{x, y, x_a, y_b, u_1/1 \leq a \leq r, 1 \leq b \leq s\}$. Therefore, $|V(DS(B_{r,s}))| = r + s + 3$. The graph $DS(B_{r,s})$ is shown in Figure 10.

It is easily observed that no two element subsets of $V(G)$ is a Steiner hop dominating set

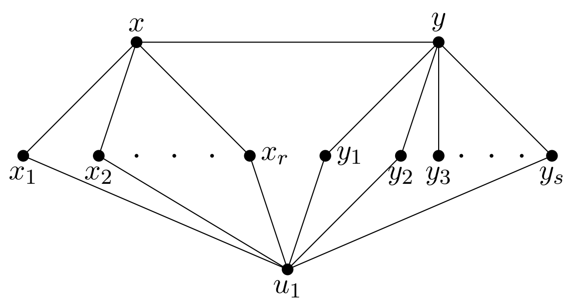

FIGURE 10. The graph $DS(B_{r,s})$

of G, as a result, $\gamma_{hs}(G) \geq 3$. Let $W = \{u_1, x, y\}$. Then every vertex of G lies in a Steiner W-tree of G. Hence it follows that $S(W) = V(G)$. Therefore W is a Steiner set of G. Since for every $u \in V(G) \backslash W$, there exists $v \in W$ such that $d(u, v) = 2$, it follows that W is a hop dominating set of G. Therefore W is a Steiner hop dominating set of G resulting in, $\gamma_{hs}(G) = 3$. □

Theorem 3.6. *Let $G = DS(K_1 + P_{n-1})$ be the degree splitting graph of the fan graph $K_1 + P_{n-1} (n \geq 4)$. Then*

$$\gamma_{hs}(G) = \begin{cases} 2 & \text{if } n = 4 \\ 3 & \text{if } n = 5 \\ n-3 & \text{if } n \geq 6. \end{cases}$$

Proof. The fan graph $F_n (n \geq 4)$ is characterized by $V(P_{n-1}) = \{v_1, v_2, .., v_{n-1}\}$ and $V(K_1) = x$. To obtain a graph G, we introduce the vertices u_1 and u_2 be the corresponding vertices. Then $V(G) = \{x, v_1, v_2, .., v_{n-1}, u_1, u_2\}$. Hence $|V(G)| = n+2$. The graph $DS(F_n)$ is shown in Figure 11.

Case 1. Let $n = 4$. Then $W = \{u_1, u_2\}$ is a minimum Steiner hop dominating set of G,

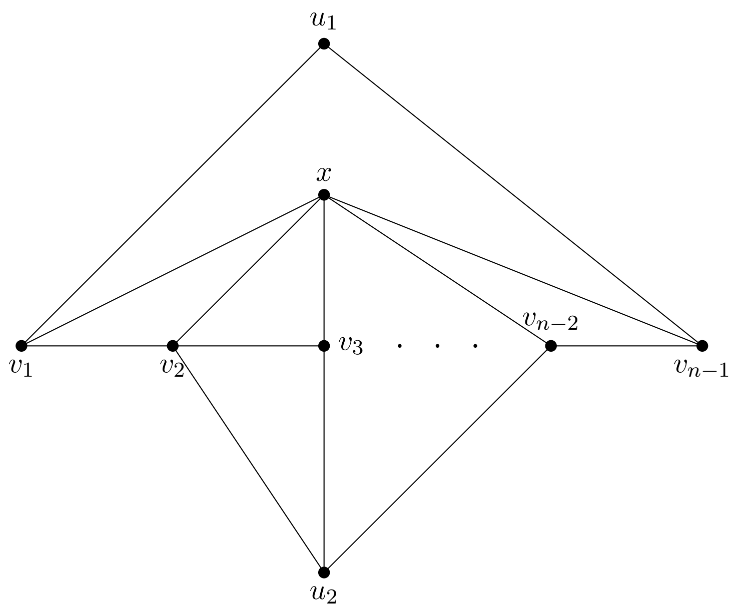

FIGURE 11. The graph $DS(F_n)$

resulting in $\gamma_{hs}(G) = 2$.

Case 2. Let $n = 5$. Then $W = \{x, u_1, u_2\}$ is a minimum Steiner hop dominating set of G, resulting in $\gamma_{hs}(G) = 3$.

Case 3. Let $W = V(DS(F_n)) \backslash \{x, v_1, v_2, v_{n-2}, v_{n-1}\}$. Then W is a Steiner hop dominating set of G as a result, $\gamma_{hs}(G) \leq n-3$. We demonstrate $\gamma_{hs}(G) = n-3$. On the contrary, suppose that $\gamma_{hs}(G) \leq n-4$. Then there exists a γ_{hs}-set W' of G such that $\gamma_{hs}(G) \leq n-4$. Let y be a vertex of G such that $y \in W$, $y \notin W'$ if $y = u_1$, then $S(W') \neq V(G)$, which contradicts itself. If $y \in \{u_2, v_3, v_4, ..., v_{n-3}\}$. Then W' is not a Steiner hop dominating set of G, which contradicts itself. Therefore, $\gamma_{hs}(G) = n-3$. $\square$

4. The Steiner hop domination number of Lollipop graph, Barbell graph and Pop Barbell graph

Theorem 4.1. *Let $G = L(p, q)$ be the lollipop graph $(p \geq 3)$. Then*

$$\gamma_{hs}(G) = \begin{cases} p & \text{if } q-1=1 \\ p+1 & \text{if } q-1=0 \text{ or } 2 \text{ or } 4 \\ p+2 & \text{if } q-1=3 \text{ or } 5 \\ 2s+p & \text{if } q-1=6s+1 \\ 2s+p+1 & \text{if } q-1=6s \text{ or } 6s+2 \text{ or } 6s+4 \\ 2s+p+2 & \text{if } q-1=6s+3 \text{ or } 6s+5. \end{cases}$$

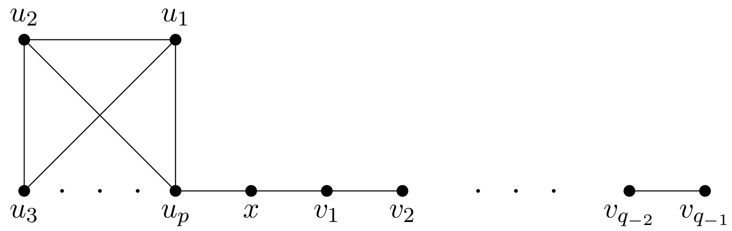

Figure 12. The graph $L(p, q)$

Proof. Let G can be splitted into two subgraphs namely $G[u_1, u_2, ..., u_p, x] = G_1$ and $G[v_1, v_2, ..., v_{q-1}] = P_{q-1}$. The graph $L(p, q)$ is shown in Figure 12. First we find the Steiner hop domination number of P_{q-1}. Let it be $\gamma_{hs}(P_{q-1})$. Since $Z = \{u_1, u_2, ..., u_{p-1}\}$ is the set of all extreme vertices of G by Proposition 2.1 (1), Z is a subset of every Steiner hop dominating set of G. Hence it follows that $\gamma_{hs}(G) = p - 1 + \gamma_{hs}(P_{q-1})$.

Case 1. $q - 1 = 1$. Let $Z \cup \{v_1\}$ is the set of all extreme vertices of G. Then by Theorem 1.1 (1), $Z \cup \{v_1\}$ is a subset of every Steiner set of G. Since u_p and x are cut vertices of G, by Theorem 1.1 (2), $\{u_p, x\} \notin W$. Consequently, $W = Z \cup \{v_1\}$ is a minimum Steiner set of G. Since $d(u_p, v_1) = d(u_1, x) = 2$, W is a hop dominating set of G. Thus, W is a minimum Steiner hop dominating set of G resulting in, $\gamma_{hs}(G) = p$.

Case 2(a). $q - 1 = 0$. Let $Z \cup \{x\}$ is the set of all extreme vertices of G. Then by Theorem 1.1 (1), $Z \cup \{x\}$ is a subset of every Steiner set of G. Consequently, $Z \cup \{x\}$ is a minimum Steiner set of G. Since $d(x, u_p) \neq 2$, $\{x, u_p\}$ is the hop dominating set of G. Thus, $W_1 = \{u_1, u_2, ..., u_p, x\}$ is a minimum Steiner hop dominating set of G resulting in, $\gamma_{hs}(G) = p + 1$.

Case 2(b). $q - 1 = 2$. Let $Z \cup \{v_2\}$ is the set of all extreme vertices of G. Then by Theorem 1.1 (1), $Z \cup \{v_2\}$ is a subset of every Steiner set of G. As a result, $Z \cup \{v_2\}$ is a minimum Steiner set of G. Since $d(v_1, u_p) = d(v_2, x) = 2$, $\{v_1, v_2\}$ is the hop dominating set of G. Thus, $W_2 = Z \cup \{v_1, v_2\}$ is a minimum Steiner hop dominating set of G resulting in, $\gamma_{hs}(G) = p + 1$.

Case 2(c). $q-1 = 4$. Let $Z \cup \{v_4\}$ is the set of all extreme vertices of G. Then by Theorem 1.1 (1), $Z \cup \{v_4\}$ is a subset of every Steiner set of G. As a result, $Z \cup \{v_4\}$ is a minimum Steiner set of G. Since $d(v_4, v_2) = d(v_1, u_p) = 2$, $\{x, v_1, v_4\}$ is the hop dominating set of G. Thus, $W_3 = Z \cup \{v_1, v_4\}$ is a minimum Steiner hop dominating set of G resulting in, $\gamma_{hs}(G) = p + 1$.

Case 3(a). $q-1=3$. Let $Z\cup\{v_3\}$ is the set of all extreme vertices of G. Then by Theorem 1.1 (1), $Z\cup\{v_3\}$ is a subset of every Steiner set of G. Therefore, $Z\cup\{v_3\}$ is a minimum Steiner set of G. Since $d(x,v_2)=d(v_3,v_1)=d(v_1,u_p)=2$, $\{x,v_1\}$ is the hop dominating set of G. Thus, $W_4=Z\cup\{v_1,v_2,v_3\}$ is a minimum Steiner hop dominating set of G resulting in, $\gamma_{hs}(G)=p+2$.

Case 3(b). $q-1=5$. Let $Z\cup\{v_5\}$ is the set of all extreme vertices of G. Then by Theorem 1.1 (1), $Z\cup\{v_5\}$ is a subset of every Steiner set of G. As a result, $Z\cup\{v_5\}$ is a minimum Steiner set of G. Since $d(x,v_2)=d(v_1,u_p)=d(v_4,u_2)=2$, $\{x,v_1,v_4,v_5\}$ is the hop dominating set of G. Thus $W_5=Z\cup\{v_1,v_4,v_5\}$ is a minimum Steiner hop dominating set of G resulting in, $\gamma_{hs}(G)=p+2$.

Case 4. $q-1=6s+1,(s\geq 1)$. First, we demonstrate that $\gamma_{hs}(P_{q-1})=2s+1$. Let $W_6=\{v_1,v_4,v_7,...,v_{6s+1}\}$ is a Steiner hop dominating set of P_{q-1} resulting in, $\gamma_{hs}(P_{q-1})\leq |W_6|=2s+1$. We demonstrate that, $\gamma_{hs}(P_{q-1})=2s+1$. If $s=1$, then the outcome is clear. So, let $s\geq 2$. On the contrary, let us proceed by assuming that $\gamma_{hs}(P_{q-1})\leq 2s$. Then there exists a γ_{hs}-set W' of P_{q-1} such that $|W'|\leq 2s$. Hence, there exists $v\in W_6$ such that $v\notin W'$. Therefore, v is a cut vertex of P_{q-1}. By Theorem 1.1 (1), $\{v_1,v_{6s+1}\}\in W'$. Thereby it follows from Theorem 1.1 (2), W' is a Steiner set of P_{q-1}. Since $v\notin W'$, there does not exist $u\in W'$ such that $d(u,v)=2$. Hence, W' is not a hop dominating set of P_{q-1}, which contradicts itself. As a result, W' is not a Steiner hop dominating set of P_{q-1}. Thus $\gamma_{hs}(P_{q-1})=2s+1$. Now, it follows that $\gamma_{hs}(G)=2s+p$.

Case 5(a). $q-1=6s,(s\geq 1)$. First, we demonstrate that $\gamma_{hs}(P_{q-1})=2s+2$. Consider $W_7=\{v_1,v_4,v_7,...,v_{6s-2}\}\cup\{v_{6s-1},v_{6s}\}$ is a Steiner hop dominating set of P_{q-1} resulting in, $\gamma_{hs}(P_{q-1})\leq |W_7|=2s+2$. We demonstrate that, $\gamma_{hs}(P_{q-1})=2s+2$. If $s=1$, then the outcome is clear. So, let $s\geq 2$. On the contrary, let us proceed by assuming that $\gamma_{hs}(P_{q-1})\leq 2s+1$. Then there exists a γ_{hs}-set W_1' of P_{q-1} such that $|W_1'|\leq 2s+1$. Hence, there exists $v\in W_7$ such that $v\notin W_1'$. Therefore, v is a cut vertex of P_{q-1}. By Theorem 1.1 (1), $\{v_1,v_{6s}\}\in W_1'$. Hence it follows from Theorem 1.1 (2), W_1' is a Steiner set of P_{q-1}. Since $v\notin W_1'$, there does not exist $u\in W_1'$ such that $d(u,v)=2$. Thus, W_1' is not a hop dominating set of P_{q-1}, which contradicts itself. As a result, W_1' is not a Steiner hop dominating set of P_{q-1}. Hence $\gamma_{hs}(P_{q-1})=2s+2$. Now, it follows that $\gamma_{hs}(G)=2s+p+1$.

Case 5(b). $q-1=6s+2,(s\geq 1)$. First, we demonstrate that $\gamma_{hs}(P_{q-1})=2s+2$. Let $W_8=W_6\cup\{v_{6s+2}\}$ is a Steiner hop dominating set of P_{q-1} resulting in, $\gamma_{hs}(P_{q-1})\leq |W_8|=2s+2$. We demonstrate that, $\gamma_{hs}(P_{q-1})=2s+2$. If $s=1$, then the outcome is clear. So, let $s\geq 2$. On the contrary, let us proceed by assuming that $\gamma_{hs}(P_{q-1})\leq 2s+1$. Then there exists a γ_{hs}-set W_2' of P_{q-1} such that $|W_2'|\leq 2s+1$. Hence, there exists $v\in W_8$ such that $v\notin W_2'$. Therefore, v is a cut vertex of P_{q-1}. By Theorem 1.1 (1), $\{v_1,v_{6s+2}\}\in W_2'$. Hence it follows from Theorem 1.1 (2), W_2' is a Steiner set of P_{q-1}. Since $v\notin W_2'$, there does not exist $u\in W_2'$ such that $d(u,v)=2$. Hence, W_2' is not a hop dominating set of P_{q-1}, which contradicts itself. As a result, W_2' is not a Steiner hop dominating set of P_{q-1}. Hence $\gamma_{hs}(P_{q-1})=2s+2$. Now, it follows that $\gamma_{hs}(G)=2s+p+1$.

Case 5(c). $q-1=6s+4,(s\geq 1)$. First, we demonstrate that $\gamma_{hs}(P_{q-1})=2s+2$. Let $W_9=\{v_1,v_4,v_7,...,v_{6s+4}\}$ is a Steiner hop dominating set of P_{q-1} resulting in, $\gamma_{hs}(P_{q-1})\leq |W_9|=2s+2$. We demonstrate that, $\gamma_{hs}(P_{q-1})=2s+2$. If $s=1$, then the outcome is clear. So, let $s\geq 2$. On the contrary, let us proceed by assuming that $\gamma_{hs}(P_{q-1})\leq 2s+1$. Then

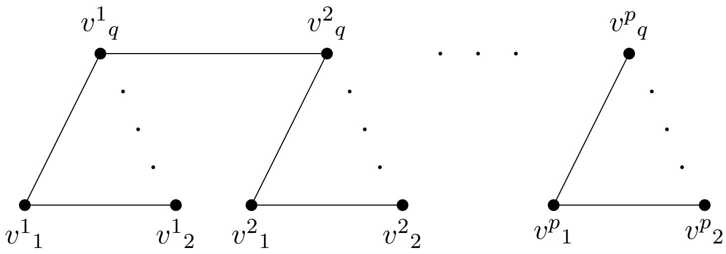

FIGURE 13. The graph $B(p, q)$

there exists a γ_{hs}-set W_3' of P_{q-1} such that $|W_3'| \leq 2s+1$. Hence, there exists $v \in W_9$ such that $v \notin W_3'$. Therefore, v is a cut vertex of P_{q-1}. By Theorem 1.1 (1), $\{v_1, v_{6s+4}\} \in W_3'$. Hence it follows from Theorem 1.1 (2), W_3' is a Steiner set of P_{q-1}. Since $v \notin W_3'$, there does not exist $u \in W_3'$ such that $d(u, v) = 2$. Thus, W_3' is not a hop dominating set of P_{q-1}, which contradicts itself. As a result, W_3' is not a Steiner hop dominating set of P_{q-1}. Hence $\gamma_{hs}(P_{q-1}) = 2s+2$. Now, it follows that $\gamma_{hs}(G) = 2s+p+1$.

Case 6(a). $q-1 = 6s+3, (s \geq 1)$. First, we demonstrate that $\gamma_{hs}(P_{q-1}) = 2s+3$. Let $W_{10} = W_8 \cup \{v_{6s+3}\}$ is a Steiner hop dominating set of P_{q-1} resulting in, $\gamma_{hs}(P_{q-1}) \leq |W_{10}| = 2s+3$. We demonstrate that, $\gamma_{hs}(P_{q-1}) = 2s+3$. If $s = 1$, then the outcome is obvious. So, let $s \geq 2$. On the contrary, let us proceed by assuming that $\gamma_{hs}(P_{q-1}) \leq 2s+2$. Then there exists a γ_{hs}-set W_4' of P_{q-1} such that $|W_4'| \leq 2s+2$. Hence, there exists $v \in W_{10}$ such that $v \notin W_4'$. Therefore, v is a cut vertex of P_{q-1}. By Theorem 1.1 (1), $\{v_1, v_{6s+3}\} \in W_4'$. Hence it follows from Theorem 1.1 (2), W_4' is a Steiner set of P_{q-1}. Since $v \notin W_4'$, there does not exist $u \in W_4'$ such that $d(u, v) = 2$. Hence, W_4' is not a hop dominating set of P_{q-1}, which contradicts itself. As a result, W_4' is not a Steiner hop dominating set of P_{q-1}. Thus $\gamma_{hs}(P_{q-1}) = 2s+3$. Now, it follows that $\gamma_{hs}(G) = 2s+p+2$.

Case 6(b). $q-1 = 6s+5, (s \geq 1)$. First we demonstrate that $\gamma_{hs}(P_{q-1}) = 2s+3$. Let $W_{11} = W_9 \cup \{v_{6s+5}\}$ is a Steiner hop dominating set of P_{q-1} resulting in $\gamma_{hs}(P_{q-1}) \leq |W_{11}| = 2s+3$. We demonstrate that, $\gamma_{hs}(P_{q-1}) = 2s+3$. If $s = 1$, then the outcome is obvious. So, let $s \geq 2$. On the contrary, let us proceed by assuming that $\gamma_{hs}(P_{q-1}) \leq 2s+2$. Then there exists a γ_{hs}-set W_5' of P_{q-1} such that $|W_5'| \leq 2s+2$. Hence, there exists $v \in W_{11}$ such that $v \notin W_5'$. Therefore, v is a cut vertex of P_{q-1}. By Theorem 1.1 (1), $\{v_1, v_{6s+5}\} \in W_5'$. Hence it follows from Theorem 1.1 (2), W_5' is a Steiner set of P_{q-1}. Since $v \notin W_5'$, there does not exist $u \in W_5'$ such that $d(u, v) = 2$. Hence, W_5' is not a hop dominating set of P_{q-1}, which contradicts itself. As a result, W_5' is not a Steiner hop dominating set of P_{q-1}. Thus $\gamma_{hs}(P_{q-1}) = 2s+3$. Now, it follows that $\gamma_{hs}(G) = 2s+p+2$.

□

Theorem 4.2. *Let $G = B(p, q)$, $(p \geq 2)$ and $(q \geq 3)$ be the Barbell graph. Then $\gamma_{hs}(G) = p(q-1)$.*

Proof. Let $B(p, q)$ be a barbell graph. Let the vertices of $B(p, q)$ be $V(G) = \{v^i{}_1, v^i{}_2, ..., v^i{}_q\}$ where $(i = 1, 2, ..., p)$. Then $|V(G)| = pq$ and $|E(G)| = p(\frac{q(q-1)}{2}) + 1$. The graph $B(p, q)$ is shown in Figure 13.

Let $W = \{\{v^1{}_1, v^1{}_2, ..., v^1{}_{q-1}\} \cup \{v^2{}_1, v^2{}_2, ..., v^2{}_{q-1}\} \cup ... \cup \{v^p{}_1, v^p{}_2, ..., v^p{}_{q-1}\}\}$ be the set of all extreme vertices of G, Then by Proposition 2.1 (1), W is a subset of every Steiner hop dominating set of G and so $\gamma_{hs}(G) \geq p(q-1)$. Since $S(W) = V(G)$ and $d(u, v) = 2$ for every $u \in W$ and $v \notin W$, W is a Steiner hop dominating set of G resulting in, $\gamma_{hs}(G) = p(q-1)$. □

Definition 4.3. The pop barbell graph is the graph obtained by joining two copies of a complete graph to a path graph by a bridge. It is denoted by $PB(p, q)$.
The vertices of a pop barbell graph as:
(a) The vertices of two copies of a complete graph K_p as $\{u_1, u_2, ..., u_p\}$ and K'_p as $\{u_1', u_2', ..., u_p'\}$.
(b) The vertices of the path graph P_q as $\{v_1, v_2, ..., v_q\}$.
(c) The vertices of $PB(p, q)$ are $V(PB(p, q)) = V(K_p) \cup V(K_p') \cup V(P_q) = \{u_1, u_2, ..., u_p, v_1, v_2, ..., v_q, u_1', u_2', ..., u_p'\}$
(d) The first vertex v_1 of the path graph P_q is connected to the last vertex u_p of the complete graph K_p and the last vertex v_q of the path graph P_q is connected to the last vertex u_p' of the complete graph K'_p, as shown in the Figure 14.

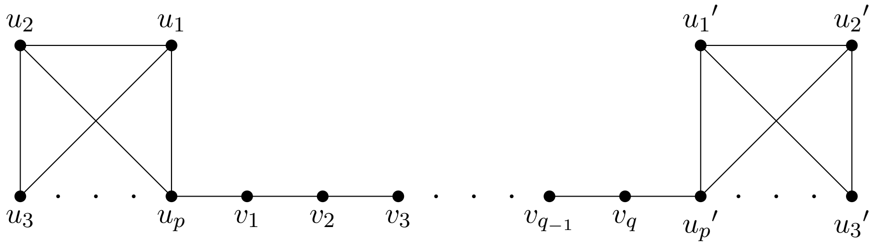

FIGURE 14. The graph $BP(p, q)$

Theorem 4.4. *Let $G = PB(p, q)$ be the pop barbell graph $(p \geq 3)$. Then*

$$\gamma_{hs}(G) = \begin{cases} 2p-1 & \text{if } q-2=1 \\ 2p & \text{if } q-2=0 \text{ or } 2 \text{ or } 4 \\ 2p+1 & \text{if } q-2=3 \text{ or } 5 \\ 2s+2p-1 & \text{if } q-2=6s+1 \\ 2s+2p & \text{if } q-2=6s \text{ or } 6s+2 \text{ or } 6s+4 \\ 2s+2p+1 & \text{if } q-2=6s+3 \text{ or } 6s+5. \end{cases}$$

Proof. Let G can be splitted into three subgraphs, namely $G[u_1, u_2, ..., u_p, x] = G_1$, $G[u_1', u_2', ..., u_p', y] = G_2$ and $G[v_1, v_2, ..., v_{q-2}] = P_{q-2}$. The graph $BP(p, q)$ is shown in Figure 15.

First, we find the Steiner hop domination number of P_{q-2}. Let it be $\gamma_{hs}(P_{q-2})$. Since $Z = \{u_1, u_2, ..., u_{p-1}, u_1', u_2', ..., u_{p-1}'\}$ is the set of all extreme vertices of G by Proposition 2.1 (1), Z is a subset of every Steiner hop dominating set of G. Hence, it follows that $\gamma_{hs}(G) = 2p - 2 + \gamma_{hs}(P_{q-2})$.

Case 1. $q - 2 = 1$. Since $d(v_1, u_p) = d(v_1, u_p') = 2$, $\{v_1\}$ is a subset of Steiner hop dominating set of G as a result, $\gamma_{hs}(G) \geq 2p - 2 + 1 = 2p - 1$. Let $W = Z \cup \{v_1\}$. Then W is a minimum Steiner hop dominating set of G resulting in, $\gamma_{hs}(G) = 2p - 1$.

Case 2(a). $q - 2 = 0$. Since $d(y, u_p) = d(x, u_p') = 2$, either $\{u_p, x\}$ or $\{x, y\}$ or $\{y, u_p'\}$ or $\{u_p, u_p'\}$ is a subset of Steiner hop dominating set of G as a result, $\gamma_{hs}(G) \geq 2p - 2 + 2 = 2p$. Let $W_1 = Z \cup \{x, y\}$. Then W_1 is a minimum Steiner hop dominating set of G resulting in,

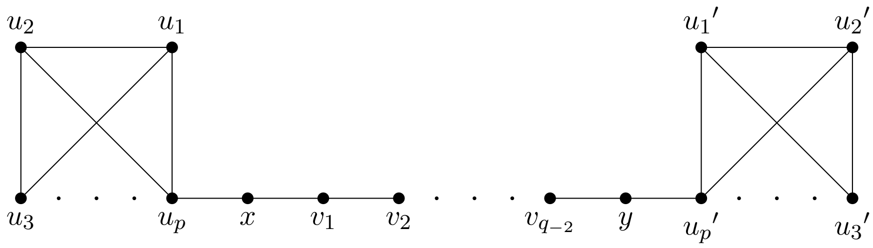

FIGURE 15. The graph $BP(p, q)$

$\gamma_{hs}(G) = 2p$.

Case 2(b). $q-2 = 2$. Since $d(v_1, u_p) = d(v_2, u_p{}') = 2$, either $\{u_p, v_2\}$ or $\{v_1, v_2\}$ or $\{v_1, u_p{}'\}$ is a subset of Steiner hop dominating set of G with the result that, $\gamma_{hs}(G) \geq 2p-2+2 = 2p$. Let $W_2 = Z \cup \{v_1, v_2\}$. Then W_2 is a minimum Steiner hop dominating set of G resulting in, $\gamma_{hs}(G) = 2p$.

Case 2(c). $q-2 = 4$. Since $d(v_1, u_p) = d(v_4, u_p{}') = 2$, $\{v_1, v_4\}$ is a subset of Steiner hop dominating set of G with the result that, $\gamma_{hs}(G) \geq 2p-2+2 = 2p$. Let $W_3 = Z \cup \{v_1, v_4\}$. Then W_3 is a minimum Steiner hop dominating set of G resulting in, $\gamma_{hs}(G) = 2p$.

Case 3(a). $q-2 = 3$. Since $d(v_1, u_p) = d(v_3, u_p{}') = d(v_2, x) = 2$, either $\{u_p, x, v_3\}$ or $\{u_p, x, u_p{}'\}$ or $\{v_1, v_2, v_3\}$ or $\{v_1, y, u_p{}'\}$ or $\{u_p, y, u_p{}'\}$ or $\{x, v_1, u_p{}'\}$ or $\{x, v_1, v_3\}$ or $\{v_1, v_3, y\}$ or $\{v_1, v_2, u_p{}'\}$ or $\{u_p, v_3, y\}$ or $\{u_p, v_2, v_3\}$ is a subset of Steiner hop dominating set of G with the result that $\gamma_{hs}(G) \geq 2p-2+3 = 2p+1$. Let $W_4 = Z \cup \{v_1, v_2, v_3\}$. Then W_4 is a minimum Steiner hop dominating set of G resulting in, $\gamma_{hs}(G) = 2p+1$.

Case 3(b). $q-2 = 5$. Since $d(v_1, u_p) = d(v_5, u_p{}') = d(v_4, v_2) = 2$, either $\{u_p, v_4, v_5\}$ or $\{v_1, v_2, v_5\}$ or $\{v_1, v_2, u_p{}'\}$ or $\{v_1, v_4, u_p{}'\}$ or $\{v_1, v_4, v_5\}$ or $\{u_p, v_2, v_5\}$ is a subset of Steiner hop dominating set of G as a result, $\gamma_{hs}(G) \geq 2p-2+3 = 2p+1$. Let $W_5 = Z \cup \{v_1, v_4, v_5\}$. Then W_5 is a minimum Steiner hop dominating set of G resulting in, $\gamma_{hs}(G) = 2p+1$.

Case 4. $q-2 = 6s+1, (s \geq 1)$. First, we demonstrate that $\gamma_{hs}(P_{q-2}) = 2s+1$. Let $W_6 = \{v_1, v_4, v_7, ..., v_{6s+1}\}$ is a Steiner hop dominating set of P_{q-2} resulting in $\gamma_{hs}(P_{q-2}) \leq |W_6| = 2s+1$. We demonstrate that, $\gamma_{hs}(P_{q-2}) = 2s+1$. If $s = 1$, then the outcome is clear. So, let $s \geq 2$. On the contrary, let us proceed by assuming that $\gamma_{hs}(P_{q-2}) \leq 2s$. Then there exists a γ_{hs}-set W' of P_{q-2} such that $|W'| \leq 2s$. Hence, there exists $v \in W_6$ such that $v \notin W'$. Therefore, v is a cut vertex of P_{q-2}. By Theorem 1.1 (1), $\{v_1, v_{6s+1}\} \in W'$. Hence it follows from Theorem 1.1 (2), W' is a Steiner set of P_{q-2}. Since $v \notin W'$, there does not exist $u \in W'$ such that $d(u, v) = 2$. Hence, W' is not a hop dominating set of P_{q-2}, which contradicts itself. As a result, W' is not a Steiner hop dominating set of P_{q-2}. Thus $\gamma_{hs}(P_{q-2}) = 2s+1$. Now, it follows that $\gamma_{hs}(G) = 2s+2p-1$.

Case 5(a). $q-2 = 6s, (s \geq 1)$. First we demonstrate that $\gamma_{hs}(P_{q-2}) = 2s+2$. Let $W_7 = \{v_1, v_4, v_7, ..., v_{6s-2}\} \cup \{v_{6s-1}, v_{6s}\}$ is a Steiner hop dominating set of P_{q-2} resulting in $\gamma_{hs}(P_{q-2}) \leq |W_7| = 2s+2$. We demonstrate that, $\gamma_{hs}(P_{q-2}) = 2s+2$. If $s = 1$, then the outcome is obvious. So, let $s \geq 2$. On the contrary, let us proceed by assuming that $\gamma_{hs}(P_{q-2}) \leq 2s+1$. Then there exists a γ_{hs}-set W_1' of P_{q-2} such that $|W_1'| \leq 2s+1$. Hence, there exists $v \in W_7$ such that $v \notin W_1'$. Therefore, v is a cut vertex of P_{q-2}. By Theorem 1.1 (1), $\{v_1, v_{6s}\} \in W_1'$. Hence it follows from Theorem 1.1 (2), W_1' is a Steiner set of P_{q-2}. Since $v \notin W_1'$, there does not exist $u \in W_1'$ such that $d(u, v) = 2$. Hence, W_1' is not a hop dominating set of P_{q-2}, which contradicts itself. Consequently, W_1' is not a Steiner hop dominating set of P_{q-2}. Thus $\gamma_{hs}(P_{q-2}) = 2s + 2$. Now, it follows that $\gamma_{hs}(G) = 2s + 2p$.

Case 5(b). $q - 2 = 6s + 2, (s \geq 1)$. First we demonstrate that $\gamma_{hs}(P_{q-2}) = 2s + 2$. Let $W_8 = W_6 \cup \{v_{6s+2}\}$ is a Steiner hop dominating set of P_{q-2} resulting in $\gamma_{hs}(P_{q-2}) \leq |W_8| = 2s + 2$. We demonstrate that, $\gamma_{hs}(P_{q-2}) = 2s + 2$. If $s = 1$, then the outcome is obvious. So, let $s \geq 2$. On the contrary, let us proceed by assuming that $\gamma_{hs}(P_{q-2}) \leq 2s + 1$. Then there exists a γ_{hs}-set W_2' of P_{q-2} such that $|W_2'| \leq 2s + 1$. Hence, there exists $v \in W_8$ such that $v \notin W_2'$. Therefore, v is a cut vertex of P_{q-2}. By Theorem 1.1 (1), $\{v_1, v_{6s+2}\} \in W_2'$. Hence it follows from Theorem 1.1 (2), W_2' is a Steiner set of P_{q-2}. Since $v \notin W_2'$, there does not exist $u \in W_2'$ such that $d(u, v) = 2$. Hence, W_2' is not a hop dominating set of P_{q-2}, which contradicts itself. As a result, W_2' is not a Steiner hop dominating set of P_{q-2}. Thus $\gamma_{hs}(P_{q-2}) = 2s + 2$. Now, it follows that $\gamma_{hs}(G) = 2s + 2p$.

Case 5(c). $q - 2 = 6s + 4, (s \geq 1)$. First we demonstrate that the Steiner hop dominating set of $\gamma_{hs}(P_{q-2}) = 2s + 2$. Let $W_9 = \{v_1, v_4, v_7, ..., v_{6s+4}\}$ is a Steiner hop dominating set of P_{q-2} resulting in $\gamma_{hs}(P_{q-2}) \leq |W_9| = 2s + 2$. We demonstrate that, $\gamma_{hs}(P_{q-2}) = 2s + 2$. If $s = 1$, then the outcome is clear. So, let $s \geq 2$. On the contrary, let us proceed by assuming that $\gamma_{hs}(P_{q-2}) \leq 2s + 1$. Then there exists a γ_{hs}-set W_3' of P_{q-2} such that $|W_3'| \leq 2s + 1$. Hence, there exists $v \in W_9$ such that $v \notin W_3'$. Therefore, v is a cut vertex of P_{q-2}. By Theorem 1.1 (1), $\{v_1, v_{6s+4}\} \in W_3'$. Hence it follows from Theorem 1.1 (2), W_3' is a Steiner set of P_{q-2}. Since $v \notin W_3'$, there does not exist $u \in W_3'$ such that $d(u, v) = 2$. Hence, W_3' is not a hop dominating set of P_{q-2}, which contradicts itself. As a result, W_3' is not a Steiner hop dominating set of P_{q-2}. Thus, $\gamma_{hs}(P_{q-2}) = 2s+2$. Now, it follows that $\gamma_{hs}(G) = 2s + 2p$.

Case 6(a). $q - 2 = 6s + 3, (s \geq 1)$. First we demonstrate that the Steiner hop dominating set of $\gamma_{hs}(P_{q-2}) = 2s + 3$. Let $W_{10} = W_8 \cup \{v_{6s+3}\}$ is a Steiner hop dominating set of P_{q-2} resulting in $\gamma_{hs}(P_{q-2}) \leq |W_{10}| = 2s + 3$. We demonstrate that, $\gamma_{hs}(P_{q-2}) = 2s + 3$. If $s = 1$, then the outcome is clear. So, let $s \geq 2$. On the contrary, let us proceed by assuming that $\gamma_{hs}(P_{q-2}) \leq 2s + 2$. Then there exists a γ_{hs}-set W_4' of P_{q-2} such that $|W_4'| \leq 2s + 2$. Hence, there exists $v \in W_{10}$ such that $v \notin W_4'$. Therefore, v is a cut vertex of P_{q-2}. By Theorem 1.1 (1), $\{v_1, v_{6s+3}\} \in W_4'$. Hence it follows from Theorem 1.1 (2), W_4' is a Steiner set of P_{q-2}. Since $v \notin W_4'$, there does not exist $u \in W_4'$ such that $d(u, v) = 2$. Hence, W_4' is not a hop dominating set of P_{q-2}, which contradicts itself. As a result, W_4' is not a Steiner hop dominating set of P_{q-2}. Thus, $\gamma_{hs}(P_{q-2}) = 2s+3$. Now, it follows that $\gamma_{hs}(G) = 2s + 2p + 1$.

Case 6(b). $q - 2 = 6s + 5, (s \geq 1)$. First we demonstrate that the Steiner hop dominating set of $\gamma_{hs}(P_{q-2}) = 2s + 3$. Let $W_{11} = W_9 \cup \{v_{6s+5}\}$ is a Steiner hop dominating set of P_{q-2} resulting in, $\gamma_{hs}(P_{q-2}) \leq |W_{11}| = 2s + 3$. We demonstrate that, $\gamma_{hs}(P_{q-2}) = 2s + 3$. If $s = 1$, then the outcome is clear. So, let $s \geq 2$. On the contrary, suppose that $\gamma_{hs}(P_{q-2}) \leq 2s + 2$. Then there exists a γ_{hs}-set W_5' of P_{q-2} such that $|W_5'| \leq 2s + 2$. Hence, there exists $v \in W_{11}$ such that $v \notin W_5'$. Therefore, v is a cut vertex of P_{q-2}. By Theorem 1.1 (1), $\{v_1, v_{6s+5}\} \in W_5'$. Hence it follows from Theorem 1.1 (2), W_5' is a Steiner set of P_{q-2}. Since $v \notin W_5'$, there does not exist $u \in W_5'$ such that $d(u, v) = 2$. Hence, W_5' is not a hop dominating set of P_{q-2}, which contradicts itself. As a result, W_5' is not a Steiner hop dominating set of P_{q-2}. Thus, $\gamma_{hs}(P_{q-2}) = 2s + 3$. Now, it follows that $\gamma_{hs}(G) = 2s + 2p + 1$.

$\square$

5. Conclusions

In this article, the Steiner hop domination number for some special graphs was generalized. This concept is also derived for the lollipop graph, the barbell graph. Also, here a new graph formed by combining them, as the pop barbell graph, which is defined and generalized.

Real world Applications

In CCTV placement, Steiner concept helps to calculate the minimum number of cameras needed to cover all areas this results in minimum costs and resource usage and hop domination concept helps in area coverage with certain distance (each within a distance 2). Also the cameras are liked together to maintain a functional, connected network.

In wireless networks, Steiner concept helps in forming network routes and hop domination concept helps in transmitting the data for communication with certain distance. Thus, this concept helps in real world solving problems in various fields.

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