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Research Paper

ON H_v-GROUPS WITH STRONG INVERSE ELEMENTS

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ABSTRACT

The largest class of hyperstructures is the class of H_v-structures. In this class, equality is replaced by non-empty intersection. This class is extremely large and can, therefore, be used to define various objects that can not be defined whithin classical hypergroup theory. In the present paper, we deal with strong-inverse elements, which are defined in both hypergroups and H_v-groups, particularly, in cases where there is more than one unit element. We introduce several constructions obtained by the properties of strong-inverse elements in H_v-groups.

1. Basic definitions

The class of weak hyperstructures, or H_v-structures, represents the broadest family of hyperstructures that correspond to classical algebraic structures through suitable quotient constructions. One can generate a remarkably large variety of H_v-structures by starting from a given algebraic structure and enlarging at least one operation result. H_v-structures have extensive applications across mathematics and, in particular, in applied sciences such as biology and physics ([1–3]). This relevance arises from the fact that H_v-structures accommodate numerous axioms and special properties required in applied contexts.

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For example, in the Lie–Santilli theory of isotopies, an approach related to energy production via reversible processes, one often needs special units and special types of inverse elements. However, the collection of inverse elements in such frameworks can be quite large, and stronger algebraic conditions may be required to restrict these sets. H_v-structures support these theories by providing suitable mathematical models, particularly through the so-called e-hyperfields.

Accordingly, the motivation of this paper is to contribute mathematical models relevant to Hadronic Mechanics, especially within the Lie–Santilli isotopy theory. To this end, we concentrate on inverse elements possessing specific structural properties.

In what follows, we present some basic definitions, mainly concerning weak hyperstructures introduced in 1990 [4].

Definitions 1.1. **Hyperstructures** are algebraic structures equipped with at least one hyperoperation.
Abbreviate: **hyperoperation=hope**.
Weak hyperstructures are called H_v**-structures**, and they are defined as follows:
Let H be a set equipped with a hope $\cdot : H \times H \longrightarrow P(H)$.

- We abbreviate **weak associativity** as **WASS**, meaning:

$$(xy)z \cap x(yz) \neq \emptyset, \forall x, y, z \in H$$

- We abbreviate **weak commutativity** as **COW**, meaning:

$$xy \cap yx \neq \emptyset, \forall x, y \in H.$$

The hyperstructure $(H, \cdot)$ is called an ***H_v-semigroup*** if it satisfies WASS.
It is called an **H_v-group** if it is a reproductive H_v-semigroup, defined by:

$$xH = Hx = H, \forall x \in H.$$

The hyperstructure $(R, +, \cdot)$ is called an **H_v-ring** if:
(1) The hopes $(+)$ and $(\cdot)$ are WASS.
(2) The reproduction axiom is valid for $(+)$.
(3) The operation $(\cdot)$ is *weakly distributive* with respect to $(+)$, meaning:

$$x(y + z) \cap (xy + xz) \neq \emptyset, (x + y)z \cap (xz + yz) \neq \emptyset, \forall x, y, z \in R.$$

An H_v-group is called *cyclic* [5, 6] if there exists an element, called a generator, whose powers have a union equal to the underlying set. The smallest power with this property is called the *period* of the generator.
If there exists an element such that a single specific power (the smallest one) equals the underlying set, the H_v-group is called *single-power cyclic.*

For definitions, results, and applications related to H_v-structures, see [1, 6–17].
An extreme class of H_v-structures is described as follows:
An H_v-structure is ***very thin*** iff all hopes are operations, except for one, where all hyperproducts are singletons, except for only one, which is a subset with cardinality greater than one.

The fundamental relations β^* and γ^* are defined in H_v-groups and H_v-rings, respectively, as the smallest equivalence relations such that their quotients form a group and a ring, respectively. The process of determining the fundamental classes is established through analogous theorems:

Theorem 1.2 [6] Let $(H, .)$ be an H_v-group, and let $\boldsymbol{U}$ denote the set of all finite products of elements of H. Define the relation β in H as follows:

$$x\beta y \Longleftrightarrow x, y \subset \mathbf{u} \text{ where } \mathbf{u} \in \mathbf{U}.$$

The fundamental relation β^* is the transitive closure of the relation β.

The main point of the proof of this theorem is that the relation β guarantees the validity of the following:
Take two elements $x, y \in H$ such that $\{x, y\} \subset \mathbf{u} \in \mathbf{U}$ and any hyperproduct where one of these elements, is used. Then, if this element is replaced by the other, the new hyperproduct is inside the same fundamental class where the first hyperproduct is. Therefore, if the 'hyperproducts' of the above β-classes are 'products', then, they are fundamental classes.

An element is called *single* if its fundamental class is a singleton.

Motivation *for H_v-structures:*
The quotient of a group by an invariant subgroup is a group.
F. Marty states that the quotient of a group by any subgroup is a hypergroup.
Extending this concept further, the quotient of a group concerning any partition is an H_v-group.

We remark that in H_v-groups (or even in hypergroups as defined by F. Marty), there is not necessarily a "unit" element, and thus no guaranteed "inverses." However, there may exist more than one "unit" element, and for each element of the Hv-group, there can be one or more "inverses."

Definition 1.3 Let $(H, \cdot)$ be an H_v-semigroup.

- An element e is called a *left unit* if $x \in ex, \forall x \in H$.
- An element e is called a *right unit* if $x \in xe, \forall x \in H$.
- An element e is called a ***unit*** element if it is both a left and right unit element.

For a given unit e and an element $x \in H$:

1. x has a *left inverse* concerning e if there exists any element $x_{le} \in H$ such that $e \in x_{le}.x$,
2. x has a *right inverse* element concerning e if there exists any element $x_{re} \in H$ such that $x \in x.x_{re}$.
3. x has an *inverse* concerning e if there exists $x_e \in H$ such that $e \in x_e.x \cap x.x_e$.

We denote:

- E_l: the set of all left unit elements.
- E_r: the set of all right unit elements.
- E: the set of all unit elements.

An element is called ***strong-inverse*** if it is an inverse to x concerning all unit elements. Therefore, x_s is a strong-inverse to x, if $E \subseteq x_s.x \cap x.x_s$.

Example. Let $H = \{0,1,2\}$ with the hyperoperation $\cdot$ defined by:

$$x \cdot y = \begin{cases} \{x\}, & y = 0, \\ \{0,1\}, & x = y = 1, \\ \{2\}, & \text{otherwise.} \end{cases}$$

Then $(H, \cdot)$ is an H_v-group with two unit elements, namely 0 and 2. The element 1 has different inverses depending on the choice of unit:
– with respect to unit 0, any x satisfying $0 \in x \cdot 1$,
– with respect to unit 2, any x satisfying $2 \in x \cdot 1$.
Since these sets differ, the element 1 does not have a strong inverse. This example illustrates the instability of inverse behavior in general Hv-groups.

Definitions 1.4. Let $(H, \cdot), (H, \otimes)$ be H$_v$-semigroups defined on the same set H. $(\cdot)$ is called ***smaller*** than $(\otimes)$, and $(\otimes)$ greater than $(\cdot)$, if there exists automorphism $f \in Aut(H, \otimes)$such that:

$$x.y \subset f(x \otimes y), \forall x \in H.$$

In this case, $(H, \otimes)$ is said to *contain* $(H, \cdot)$, and we write:

$$\cdot \leq \otimes.$$

If $(H, \cdot)$ is a structure, it is called the *basic structure*, and $(H, \otimes)$ is called the H$_b$-*structure.*

The Little Theorem. Greater hopes of hopes that are WASS or COW are also WASS and COW, respectively.

The fundamental relations are used for general definitions. For example, to define the general H$_v$-field, one uses the fundamental relation γ^*:

Definitions 1.5. The H$_v$-ring $(R, +, \cdot)$ is called an ***H$_v$-field*** if the quotient R/γ^* is a field. The definition of the H$_v$-field introduced a new class of hyperstructures [18,19]: The H$_v$-semigroup $(H, \cdot)$ is an h/v-group if the quotient H/β^* is a group.

More complicated hyperstructures can also be defined:

Definitions 1.6 [6,19] Let $(R, +, \cdot)$ be an H$_v$-ring, $(M, +)$ be a COW H$_v$-group and let there exist an external hope:

$$\cdot : R \times M \longrightarrow P(M) : (a, x) \longrightarrow ax$$

such that, $\forall a, b \in R, \forall x, y \in M$, the following hold:
$a(x + y) \cap (ax + ay) \neq \emptyset$,
$(a + b)x \cap (ax + bx) \neq \emptyset$,
$(ab)x \cap a(bx) \neq \emptyset$.
Then M is called an H_v-module over F. If R is an H_v-field instead of an H_v-ring, then M is called an H_v***-vector space*** over F.

In the above cases, the fundamental relation ε^* is defined as the smallest equivalence relation such that the quotient M/ε^* forms a module (resp. vector space) over the fundamental ring R/γ^* (resp. the fundamental field F/γ^*).

The general definition of an H_v-Lie algebra was given in [1,3] as follows:

Definitions 1.7 Let $(\mathbf{L}, +)$ be an H_v-vector space over the field $(\mathbf{F}, +,)$, and let $\varphi : \mathbf{F} \longrightarrow \mathbf{F}/\gamma^*$ be the canonical map. Define $\omega_F = \{x \in F : \varphi(x) = 0\}$, where 0 is the zero element of the fundamental field $\mathbf{F}/\gamma^*$. Similarly, let ω_L be the core of the canonical map $\varphi' : \mathbf{L} \longrightarrow \mathbf{L}/\varepsilon^*$, and denote by the same symbol 0 the zero element of $\mathbf{L}/\varepsilon^*$. Now consider the *bracket (commutator) hope:*

$$[\ ,\ ] : L \times L \longrightarrow P(L) : (x, y) \longrightarrow [x, y].$$

Then $\mathbf{L}$ is an H_v-Lie algebra over $\mathbf{F}$ if the following axioms are satisfied:
(L1) The bracket hope is bilinear, i.e.

$$[\lambda_1 x_1 + \lambda_2 x_2, y] \cap (\lambda_1[x_1, y] + \lambda_2[x_2, y]) \neq \emptyset$$

$$[x, \lambda_1 y_1 + \lambda_2 y_2] \cap (\lambda_1[x, y_1] + \lambda_2[x, y_2]) \neq \emptyset,$$

$\forall x, x_1, x_2, y, y_1, y_2 \in L$ *and* $\lambda_1, \lambda_2 \in F$
(L2) $[x, x] \cap \omega_L \neq \emptyset, \ \ \forall x \in L$
(L3) $([x, [y, z] + [y, [z, x]] + [z, [x, y]]) \cap \omega_L \neq \emptyset, \ \ \forall x, y \in L$

Definitions 1.8 [18] Let $(H, \cdot)$ be a hypergroupoid. We say that we *remove* $h \in H$, if we simply consider the restriction of $(\cdot)$ on the $H - \{h\}$. We say that $\underline{h} \in H$ absorbs $h \in H$ if we replace h, whenever it appears, by $\underline{h}$. We say that $\underline{h} \in H$ *merges* with $h \in H$, if we take as the product of any $x \in H$ by $\underline{h}$, the union of the results of x with both h and $\underline{h}$, and consider h and $\underline{h}$ as one class, with representative $\underline{h}$.

2. Two large classes of H_v-structures

The class of P-hyperstructures was appeared in the 80's to represent hopes of constant length [5]. Since then, several classes of P-hopes have been introduced and studied [6, 19].

Definitions 2.1. Let $(G, .)$ be a groupoid. For any P such that $\emptyset \neq P \subset G$, define the following hopes called P**-hopes**, for $x, y \in G$
$\underline{P} : x\underline{P}y = (xP)y \cup x(Py)$,
$\underline{P}_r : x\underline{P}_r y = (xy)P \cup x(yP)$,
$\underline{P}_l : x\underline{P}_l y = (Px)y \cup P(xy)$.
The structures $(G, \underline{P}), (G, \underline{P}_r)$, and $(G, \underline{P}_l)$ are called P*-hyperstructures.* If $(G, .)$ is a semigroup, then:

$$x\underline{P}y = (xP)y \cup x(Py) = xPy,$$

and $(G, \underline{P})$ is a semihypergroup.

It is immediate the following:
Let $(G, .)$ be a group. For all subsets P such that $\phi \neq P \subset G$, the hyperstructure $(G, \underline{P})$, where the P-hope is $x\underline{P}y = xPy$, becomes a hypergoup in the sense of Marty, i.e., the strong associativity is valid. The P-hope is of constant length, i.e., we have $|x\underline{P}y| = |P|$. Such a hyperstructure $(G, \underline{P})$ is called P*-hypergroup.*

In [1,3], a modified P-hope was introduced, which is appropriate for *e-hyperstructures*:

Construction 2.2 Let $(G, .)$ be an abelian group, and let P be any subset of G with more than one element. Define the hyperoperation $(\times_P)$ as follows:

$$x \times_P y = \begin{cases} x.P.y = \{x.h.y | h \in P\} & if \;\; x \neq e \;\; and \;\; y \neq e \\ x.y & if \;\; x = e \;\; or \;\; y = e \end{cases}$$

where e is the identity element of the group.

This hope is called P_e*-hope*, and the resulting hyperstructure $(G, \times_P)$ is an abelian H_v-group.

Another significant large class arises from defining a new hope ∂ in a groupoid.

Definitions 2.3 [3, 20] Let $(G, .)$ be a groupoid (resp., hypergroupoid), and let $f : G \longrightarrow G$ be a map. Define a hope (∂), called *theta-hope* or simply ∂-hope, on G as follows:

$$x\partial y = \{f(x).y, x.f(y)\}, \forall x, y \in G.(resp., \;\; x\partial y = (f(x).y) \cup (x.f(y)), \forall x, y \in G)$$

If $(.)$ is commutative then (∂) is also commutative. If $(.)$ is *COW*, then (∂) is also *COW*.

Now let $(G, .)$ be a groupoid (resp., hypergroupoid) and let $f : G \longrightarrow \mathbf{P}(G) - \{\emptyset\}$ be a multivalued map. We define (∂) on G as follows:

$$x\partial y = (f(x).y) \cup (x.f(y)), \forall x, y \in G$$

Properties: If $(G, .)$ is a semigroup then:
(a) For every f, the hyperoperation (∂) is *WASS*.
(b) If f is a homomorphism and projection, or idempotent (mean $f^2 = f$), then (∂) is associative.

Let $(G, .)$ be a groupoid, and let $\{f_i : G \longrightarrow G, i \in I\}$ be a set of maps on G. Define the map $f_\cup : G \longrightarrow \mathbf{P}(G)$ such that:

$$f_\cup(x) = \{f_i(x) | i \in I\},$$

called *the union* of the $f_i(x)$.
We define the *union* (∂)-hrope on G using the map $f_\cup(x)$. A special case for a given map f, is to take the union of this with the identity map. We consider the map $\underline{f} \equiv f \cup (id)$. So

$$\underline{f}(x) = \{x, f(x)\}, \forall x \in G,$$

which is call *b-theta-hope*. Then we have

$$x\partial y = \{xy, f(x).y, x.f(y)\}, \forall x, y \in G.$$

The motivation for the definition of the theta-operation, is the *derivative* map where only the multiplication of functions is used. Therefore, in these terms, for any functions $s(x)$ and $t(x)$, the theta-operation is:

$$s\partial t = \{s't, st'\}$$

where $(')$ denotes the derivative.

Proposition 2.4 Let $(G, .)$ be a group, and let $f(x) = a$, a constant map on G. Then $(G, \partial)/\beta^*$ is singleton.

Proof. For all x in G, consider the hyperproduction of the elements a^{-1} and $a^{-1}x$:

$$a^{-1}\partial(a^{-1}.x) = \{f(a^{-1}).a^{-1}.x, a^{-1}.f(a^{-1}.x)\} = \{x, e\}.$$

Thus, $x\beta e, \forall x \in G$, so $\beta^*(x) = \beta^*(e)$ and $(G,\partial)/\beta^*$ is a singleton.

In special case, if $(G,.)$ is a group and $f(x) = e$, then the operation becomes:

$$x\partial y = \{x, y\},$$

which is the incidence hope.

Taking the application on the derivative, consider the set of all first-degree polynomials $g_i(x) = a_i x + b_i$.For $g_1(x)$ and $g_2(x)$, we have:

$$g_1\partial g_2 = \{a_1a_2x + a_1b_2, a_1a_2x + b_1a_2\},$$

which forms a hope on the set of first-degree polynomials. Moreover, all polynomials of the form $x + c$, where c is a constant, are units.

3. Applications and Characteristic Elements

A new application of hyperstructures in social sciences involves combining hyperstructure theory with fuzzy logic. This approach replaces the traditional Likert scale in questionnaires with the bar of Vougiouklis & Vougiouklis [21]. The hyperstructures are used as an organized devise. The suggestion is the following:
In every question substitute the Likert scale with 'the bar' whose poles are defined with $'0'$ on the left end, and $'1'$ on the right end:

0 ——————————————————— 1

Instead of selecting a specific grade on the scale, participants are asked to cut the bar at any point that best reflects their answer. The use of the bar of Vougiouklis & Vougiouklis instead of a scale of Likert has several advantages during both the filling-in and the research processing. The length of the bar is according to the Golden Ratio 6.2cm. The hyperstructure theory gives innovative new suggestions to connect finite groups of objects. The hyperstructures used in this theory are the ∂-hopes because they can use any single-valued map, or multivalued map, which connects several categories of objects. These suggestions are obtained from properties and special elements inside the hyperstructure.

The Lie-Santilli *isotopies* were born to solve Hadronic Mechanics problems. Santilli [1], [3], proposed a 'lifting' of the trivial unit matrix of a normal theory into a nowhere singular, symmetric, real-valued, new matrix. The original theory is reconstructed such as to admit the new matrix as a left and right unit. The *isofields* needed correspond to H_v-structures called *e-hyperfields* which are used in physics or biology. Therefore, in this theory, the units and the inverses play a very important role.

Example 3.1 Consider the 'small' ring $(\mathbb{Z}_4, +, .)$, where we aim to construct non-degenerate H_v-field such that 0 and 1 are scalars for both addition and multiplication, every element of $\mathbb{Z}_4$ has a unique opposite and every non-zero element has a unique inverse. Then the rows and columns of the elements 0 and 1 in the addition and multiplication, remain the same. The sum $2 + 2 = 0$ and the product $3.3 = 1$ remain the same. In the results of the sums $2+3=1, 3+2=1$ and $3+3=2$, one can put respectively, the elements $3, 3$ and 0. In the results of the products $2.2=0, 2.3=2$ and $3.2=2$, one can put respectively, the elements $2, 0$ and 0. Then in all those enlargements, even if only one enlargement is used, we obtain

$$(\mathbb{Z}_4, +, .)/\beta^* \cong \mathbb{Z}_2.$$

Therefore, this construction gives H_v-fields.

Properties 3.2 Let $(G, .)$ be a group and P a non-empty subset of G $(\phi \neq P \subset G)$. Consider the P-hypergroup $(G, \underline{P})$, where $x\underline{P}y = xPy$. The following properties hold:
Units:
For an element u to be a right unit of the P-hypergroup $(G, \underline{P})$, we must have:

$$x\underline{P}u = xPu, \forall x \in G.$$

Consequently, the set Pu must contain the unit element e of the group $(G, .)$. Thus, the right units are all the elements of the set P^{-1}. Similarly, the left units also correspond to elements of P^{-1}.

Inverses:
Let u be a unit in $(G, \underline{P})$. For a given element x, to find its inverse x' with recpect to u, we must have:

$$x\underline{P}x' = xPx'.$$

By taking $xpx' = u$, we obtain that all the elements of the form $x' = p^{-1}x^{-1}u$ are inverses of x with respect to the unit u.

Theorem 3.3 [22] Let $(G, .)$ be a group, and let P be any normal subgroup of G. Then, the hyperstructure $(G, \underline{P})$, where the P-hope is defined as $x\underline{P}y = xPy, \forall x, y \in G$, is a hypergoup with strong-inverses. Moreover, for any inverse x' of $x \in G$, concerning any unit, we have:

$$x\underline{P}x' = P.$$

Proof. Let $x \in G$, and consider the inverse x' given by:

$$x' = p^{-1}x^{-1}u$$

where $u = p_k^{-1}$, for any p is the unit. Then we have $x\underline{P}x' = xPx'$. Since P is a normal subgroup of G, the following equivalences hold:

$$x\underline{P}x' = xp^{-1}x^{-1}p_k^{-1}P = xp^{-1}x^{-1}P = xp^{-1}Px^{-1} = xPx^{-1} = P$$

Remark that in this case $P^{-1} = P$, is the set of all units, thus all inverses are strong.

Proposition 3.4 Let $(G, .)$ be a groupoid, and let $f : G \to G$ be a map. For the corresponding (G, ∂)-structure, (defined by the ∂-hope), the following properties hold:
Units:
For an element u to be a right unit, the condition must hold:

$$x \in \{f(x).u, x.f(u)\} = x\partial u.$$

But, the unit must not depend on $f(x)$. Therefore, we must have:

$$f(u) = e,$$

where e is the unit in the groupoid. This implies that $(G,\cdot)$ must be a monoid. Similarly, for the left units, we obtain the same condition $f(u)=e$. Hence, the set of units for (G,∂) is

$$kerf=\{u:f(u)=e\}.$$

Inverses:

Let u be a unit in (G,∂) Then $(G,.)$ is a monoid with unit e, and $f(u)=e$. For a given $x\in G$, to find an inverse element x' with respect to u, we must have

$$u\in\{f(x).x',x.f(x')\}=x\partial x',$$

and also:

$$u\in\{f(x').x,x'.f(x)\}=x'\partial x.$$

The above conditions are independent of the image $f(x')$ only in the following cases:

$$x'=(f(x))^{-1}u,$$

and

$$x'=u(f(x))^{-1},$$

the right and left inverses, respectively. So, we have two-sided inverses if and only if

$$f(x)u=uf(x).$$

Remark [22]: Since the inverses depend on the units, they are not strong in general. To construct a minimal hyperstructure where all elements have strong inverses, an enlargement of the hyperstructure is necessary. The following constructions introduce hyperstructures with strong-inverse elements.

Construction 3.5 Let $(G,.)$ be a group with a unit element e. Consider a finite set $E=\{e_i|i\in I\}$. We define a hope $(\times)$ on the set $\underline{G}=(G-\{e\})\cup E$ as follows:

$$\begin{cases} e_i\times e_j=\{e_i,e_j\}, & \forall e_i,e_j\in E\\ e_i\times x=x\times e_j=x, & \forall e_i\in E, x\in G-\{e\}\\ x\times y=x.y & if\ \ x.y\in G-\{e\}\\ x\times y=E & if\ \ x.y=e\end{cases}$$

Then, the hyperstructure $(\underline{G},\times)$ is a hypergroup, where the set of unit elements is E, and all elements in G have strong-inverse. Moreover, we have:

$$(\underline{G},\times)/\beta^*\cong(G,.).$$

Proof. For associativity we have the cases:

$$(e_i\times e_j)\times e_k=e_i\times(e_j\times e_k)=\{e_i,e_j,e_k\}\ \forall e_i,e_j,e_k\in E,$$

$$(x\times y)\times z=x\times(y\times z)=x.y.z \text{ or } E,\ \ \forall x,y,z\in\underline{G},$$

and not all of them belong to E.

In the second case, there is no matter if the product of two inverse elements appears. The only difference is that the result is singleton and in some cases, the result is equal to the set E. Therefore the strong associativity is valid. However, the reproductivity is valid and the set E is the set of units in $(\underline{G},\times)$.

Any two elements of $\underline{G}$ are β^*-equivalent if they belong to a finite $\times$- product of elements in $\underline{G}$. Thus, all fundamental classes are singletons, except for the set of units E. his proves that $(\underline{G}, \times)/\beta^* \cong (G, .)$.

Example **(Strong Inverses Generated by Construction3.5).** Consider the group $G = \mathbb{Z}_3 = \{0, 1, 2\}$. Let $E = \{e_1, e_2\}$ be a set of two new unit elements, and define:

$$G' = (G - \{0\}) \cup E = \{1, 2, e_1, e_2\}.$$

Define the hope $\times$ on G' by:

$$e_i \times e_j = \{e_i, e_j\}, \qquad e_i \times x = x \times e_j = x,$$

for all $x \in \{1, 2\}$, and

$$x \times y = \begin{cases} x + y, & x + y \neq 0, \\ E, & x + y = 0. \end{cases}$$

Then:
- E is precisely the set of unit elements,
- every element of G' admits a strong inverse (with respect to all units),
- and the quotient $(G', \times)/\beta^* \cong (G, +)$.

This example demonstrates explicitly how the enlargement yields a hypergroup in which all elements possess strong inverses.

Construction 3.6 Let $(G, .)$ be an H_v-group with only one unit element e, where every element has a unique inverse. Consider a finite set $E = \{e_i | i \in I\}$. We define a hope $(\times)$ on the set $\underline{G} = (G - \{e\}) \cup E$ as follows:

$$\begin{cases} e_i \times e_j = \{e_i, e_j\}, & \forall e_i, e_j \in E \\ e_i \times x = x \times e_j = x, & \forall e_i \in E, x \in G - \{e\} \\ x \times y = x.y & if \ \ x.y \in G - \{e\} \\ x \times y = E & if \ \ x.y = e \end{cases}$$

The hyperstructure $(\underline{G}, \times)$ is an H_v-group, where the set of unit elements, is E and elements in G have strong-inverse. Moreover, we have:

$$(\underline{G}, \times)/\beta^* \cong (G, .)/\beta^*.$$

Proof. For associativity we have the cases:

$$(e_i \times e_j) \times e_k = e_i \times (e_j \times e_k) = \{e_i, e_j, e_k\} \ \forall e_i, e_j, e_k \in E,$$

$$(x \times y) \times z = x \times (y \times z) = x.y.zorE, \ \forall x, y, z \in \underline{G},$$

and not all of them belong to E. Therefore, the WASS is valid. Moreover, the reproductivity is valid and the set E, is the set of units in $(\underline{G}, \times)$.

Any two elements of $\underline{G}$ are β^*-equivalent if they belong to a finite $\times$-product of elements of $\underline{G}$. The fundamental classes correspond to those of $(G, \cdot)$, with the class of e enlarged to include E. Therefore, we have:

$$(\underline{G}, \times)/\beta^* \cong (G, .)/\beta^*.$$

The constructions above provide a large and significant number of hyperstructures with strong-inverse elements. These hyperstructures can be further enlarged by modifying results, except when the result is E.

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